









# CALCULUS



THE MACMILLAN COMPANY  
NEW YORK • BOSTON • CHICAGO • DALLAS  
ATLANTA • SAN FRANCISCO

MACMILLAN AND CO., LIMITED  
LONDON • BOMBAY • CALCUTTA • MADRAS  
MELBOURNE

THE MACMILLAN COMPANY  
OF CANADA, LIMITED  
TORONTO

# CALCULUS

BY

HENRY BURCHARD FINE

DOD PROFESSOR OF MATHEMATICS  
IN PRINCETON UNIVERSITY

New York

THE MACMILLAN COMPANY

COPYRIGHT, 1926, 1927,  
By THE MACMILLAN COMPANY.

---

All rights reserved — no part of this book may be reproduced in any form without permission in writing from the publisher, except by a reviewer who wishes to quote brief passages in connection with a review written for inclusion in magazine or newspaper.

---

Set up and electrotyped. Complete edition published October, 1927.

48-7-30

## PREFACE

THE first part of this book is a text-book of elementary calculus and contains the material usually given in a one year course in the calculus. It aims to present the underlying notions of the subject in a manner intelligible to beginners but with precision, to aid the student to master the technique of the calculus, and by a variety of applications to give him a realization of its usefulness and power.

Except for brief chapters on determinants and solid geometry, the remainder of the book is devoted to topics of advanced calculus, including the theory of partial differentiation and some of its more important and interesting applications, differential equations, line and surface integrals, the development of implicit functions in series by the method of successive approximations with applications to curve tracing, and Fourier series. It closes with a chapter on functions of a complex variable. In the treatment of this material I have endeavored not to assume a higher degree of mathematical maturity than a student in his second year of calculus is likely to possess. I have supplemented the theoretical discussion by numerous exercises, some merely illustrative, others requiring independent thought.

Throughout the book I have sought to present the theory with as little formality as possible. The proofs are based on the fundamental theorems respecting the existence of limits and the properties of continuous functions. These theorems are formulated and illustrated geometrically early in the book, but their arithmetical proofs are postponed to a chapter on numbers and continuous functions near the end.

The book has taken shape gradually in pamphlets which for several years past it has been my practice to prepare for the use of classes at Princeton. In writing it I have profited by suggestions from other books on the subject, especially the *Cours d'Analyse* of de la Vallée Poussin. I am also indebted to my colleague Professor Wedderburn for many valuable suggestions and criticisms and for aid in seeing the book through the press.

HENRY B. FINE

PRINCETON, September, 1927



# CONTENTS

| CHAPTER                                                   | PAGE |
|-----------------------------------------------------------|------|
| I. VARIABLES. LIMITS. FUNCTIONS . . . . .                 | 1    |
| II. THE DERIVATIVE . . . . .                              | 16   |
| III. DERIVATIVES OF ALGEBRAIC FUNCTIONS . . . . .         | 30   |
| IV. MAXIMA AND MINIMA . . . . .                           | 42   |
| V. INVERSE FUNCTIONS . . . . .                            | 55   |
| VI. THE TRIGONOMETRIC FUNCTIONS . . . . .                 | 59   |
| VII. THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS . . . . .  | 69   |
| VIII. CURVES IN POLAR COORDINATES . . . . .               | 78   |
| IX. CURVATURE . . . . .                                   | 83   |
| X. CURVILINEAR MOTION . . . . .                           | 88   |
| XI. CURVE TRACING . . . . .                               | 96   |
| XII. MEAN VALUE THEOREM. INDETERMINATE<br>FORMS . . . . . | 105  |
| XIII. NEWTON'S METHOD OF APPROXIMATION . . . . .          | 113  |
| XIV. INTEGRATION . . . . .                                | 115  |
| XV. DEFINITE INTEGRALS . . . . .                          | 137  |
| XVI. DOUBLE INTEGRALS . . . . .                           | 168  |
| XVII. MEAN VALUES . . . . .                               | 176  |
| XVIII. MOMENTS OF INERTIA . . . . .                       | 183  |
| XIX. ATTRACTION OF GRAVITATION . . . . .                  | 189  |
| XX. INFINITE SERIES . . . . .                             | 192  |
| XXI. DETERMINANTS . . . . .                               | 221  |
| XXII. SOLID GEOMETRY . . . . .                            | 230  |
| XXIII. PARTIAL DIFFERENTIATION . . . . .                  | 238  |
| XXIV. APPLICATIONS OF PARTIAL DIFFERENTIATION . . . . .   | 260  |
| XXV. DIFFERENTIAL EQUATIONS . . . . .                     | 283  |

| CHAPTER                                      | PAGE |
|----------------------------------------------|------|
| XXVI. LINE AND SURFACE INTEGRALS . . .       | 331  |
| XXVII. DOUBLE SERIES. IMPLICIT FUNCTIONS . . | 359  |
| XXVIII. FOURIER SERIES . . . . .             | 378  |
| XXIX. PROPERTIES OF CONTINUOUS FUNCTIONS . . | 386  |
| XXX. FUNCTIONS OF A COMPLEX VARIABLE . .     | 398  |
| INDEX . . . . .                              | 419  |

# CALCULUS



# CALCULUS

## I. VARIABLES. LIMITS. FUNCTIONS

**1. The real numbers and the points of a line.** Take a horizontal line and on it a fixed point  $O$  as origin ; also a unit segment  $OI$  for measuring lengths or distances. We may represent any given positive number,  $a$ , rational or irrational, by the point  $A$  which is at the distance  $a$  to the right of  $O$  ; and  $-a$  by the corresponding point  $A'$  to the left of  $O$ .



FIG. 1.

We thus obtain a picture of the relation of order which exists among the real numbers. To two numbers  $a$  and  $b$  such that  $a < b$  correspond two points  $A$  and  $B$  such that  $A$  is to the left of  $B$ .

**2. Number intervals.** The set of all real numbers between  $a$  and  $b$ ,  $a$  and  $b$  themselves included, is called the *number interval*  $a, b$ . It is represented by the symbol  $(a, b)$ . We call  $b - a$  the *length* of  $(a, b)$ .

**3. Variables.** A constant, as  $c$ , is a symbol for a single number. A *variable*, as  $x$ , is a symbol for any one of a set of numbers. If the set consists of all the real numbers, or of all in an interval  $(a, b)$ , then  $x$  is called a *continuous variable*.

We often think of a variable as running through the numbers of its set in some definite order. We may call the set as thus arranged, the *range* of the variable. In particular,

(1) The range may be a never-ending sequence of isolated numbers, as when  $x$  is supposed equal to  $1/2, 2/3, 3/4, 4/5, \dots$  successively.

(2) The range may be a number interval. Thus in Fig. 1 let  $X$  denote any point between  $A$  and  $B$ , and  $x$  the corresponding number. If  $X$  move along the line from  $A$  to  $B$ ,  $x$  will equal successively all numbers between  $a$  and  $b$ . A variable which is supposed thus to change from one number to another through all intermediate numbers is said to *vary continuously*.

**4. Limit of a variable.** As  $x$  runs through the sequence  $1/2, 2/3, 3/4, \dots$ , it approaches 1 in such a manner that the difference  $1 - x = 1/2, 1/3, 1/4, \dots$  will ultimately become and remain less than .01, or .001, or any other positive number that may be assigned, however small. To indicate this, we say: " $x$  approaches 1 as limit." In general

*The variable  $x$  is said to approach the constant  $c$  as limit, if it be true for every positive number  $\delta$  that may be assigned, however small, that, as  $x$  runs through its range, the difference  $x - c$  will ultimately become and remain numerically less than that number  $\delta$ .*

To indicate that  $x$  approaches the limit  $c$ , we write

$$x \rightarrow c \quad \text{or} \quad c = \lim x \quad (1)$$

The symbol  $|a|$  means the numerical value of  $a$ . Thus  $|-3| = |3| = 3$ . Hence, using  $\delta$  to denote any given positive number whatsoever, however small, we may write:

$$x \rightarrow c \quad \text{if} \quad |x - c| \text{ ultimately remains} < \delta \quad (2)$$

**EXAMPLE.** What limit, if any, does  $x$  approach as it runs through each of the following ranges?

1.  $3, 2\frac{1}{2}, 2\frac{1}{4}, 2\frac{1}{8}, \dots$

2.  $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots$

3.  $.3, .33, .333, \dots$

4.  $1, 1\frac{1}{2}, 1, 1\frac{1}{4}, 1, 1\frac{1}{8}, \dots$

5.  $1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{8}, \dots$

6.  $1, 2, 3, 4, \dots$

Respecting the *numerical values of sums and products*, observe that

$$|a + b| \leq |a| + |b| \quad |ab| = |a| |b| \quad (3)$$

Thus,  $|-2 + (-3)| = |-2| + |-3|$ ,  $|-2 + 3| < |-2| + |3|$ ,  
 $|-3(-2)| = |-3| \cdot |-2|$



**5. On the existence of a limit.** Not every variable approaches a limit. Thus  $x = 1, 2, 3, 4, \dots$  does not. Also  $x = 1, 2, \frac{1}{2}, \frac{1}{3}, \dots$  does not. But, as will be shown later, in the following two cases we may always conclude that a limit *exists*, though the actual determination of its value may be difficult:

1. *If  $x$  continually increases but remains less than some finite number  $c$ , it approaches a limit; and this limit is either  $c$  or some lesser number.*

2. *If  $x$  continually decreases but remains greater than some finite number  $c$ , it approaches a limit; and this limit is either  $c$  or some greater number.*

**EXAMPLE.** A circle of radius  $r$  being given, let  $x$  denote the area of an inscribed regular polygon of  $n$  sides. As  $n$  increases,  $x$  increases but remains always less than the area of any circumscribed polygon; hence, by 1, it approaches a limit  $l$ . Similarly, by 2, the area  $y$  of the corresponding circumscribed polygon approaches a limit  $l'$ . It can be proved geometrically that  $l = l' = r^2\pi$  ( $\pi = 3.14159 \dots$ ). This number is called the area of the circle.

**6. Infinitesimals.** A variable which approaches 0 as limit is called an *infinitesimal*. Thus  $x = 1, 1/2, 1/3, 1/4, \dots$  is an infinitesimal.

Evidently *the difference between any variable which approaches a limit and that limit is an infinitesimal*. If  $x \rightarrow c$ , we may set  $x - c = h$  and then write  $x = c + h$ ,  $h \rightarrow 0$ .

Consider the following examples:

**EXAMPLE 1.** Prove that  $3h + 2k \rightarrow 0$  when  $h \rightarrow 0$  and  $k \rightarrow 0$ .

Let  $\delta$  denote any given positive number, however small. It is to be proved that as  $h$  and  $k$  approach 0,  $|3h + 2k|$  ultimately becomes and remains  $< \delta$ . But by § 4 (3),  $|3h + 2k| \leq |3h| + |2k|$ . Hence we shall have  $|3h + 2k| < \delta$  when  $|3h| < \delta/2$  and  $|2k| < \delta/2$   $\therefore$  when  $|h| < \delta/6$  and  $|k| < \delta/4$ .

Thus  $|3h + 2k| < .01$  when  $|h| < .0016$  and  $|k| < .0025$ ; and so on. Hence  $3h + 2k \rightarrow 0$ .

**EXAMPLE 2.** Prove that  $(3 - k)h + (4 + h^3)k \rightarrow 0$  when  $h \rightarrow 0$  and  $k \rightarrow 0$ .

Since  $h, k \rightarrow 0$ , we may begin by restricting  $h, k$  to values such that  $|h|, |k| < 1$ , and therefore  $|3 - k|, |4 + h^3| < 5$ . We then have  $|(3 - k)h + (4 + h^3)k| < \delta$  when  $5|h| + 5|k| < \delta$   $\therefore$  when  $|h|, |k| < \delta/10$ . But  $h, k \rightarrow 0$ . Hence however small  $\delta$  be taken,  $|h|, |k|$  ultimately remain  $< \delta/10$ .

Therefore  $(3 - k)h + (4 + h^3)k \rightarrow 0$  when  $h \rightarrow 0$  and  $k \rightarrow 0$ .

These examples illustrate the following theorem:

*Let  $h$  and  $k$  be infinitesimals, and let  $P$  and  $Q$  denote constants, or variables for which positive numbers  $c$  and  $C$  can be found such that  $|P|, |Q| < C$  when  $|h|, |k| < c$ . Then*

$$Ph + Qk \rightarrow 0 \text{ when } h, k \rightarrow 0 \quad (1)$$

For, restricting  $h, k$  to values such that  $|h|, |k| < c$ , we have  $|Ph + Qk| < \delta$  when  $C|h| + C|k| < \delta$  and therefore when  $|h|, |k| < \delta/2C$ . But  $h, k \rightarrow 0$ . Hence, however small  $\delta$  is taken, we have ultimately  $|h|, |k| < \delta/2C$ .

Therefore  $Ph + Qk \rightarrow 0$  when  $h, k \rightarrow 0$ .

**7. Limits of sums, products, quotients.** *The limit of the sum of two variables which approach limits is the sum of those limits.*

*The limit of the product of two variables which approach limits is the product of those limits.*

*The limit of the quotient of two variables which approach limits is the quotient of those limits, unless the limit of the divisor is 0.*

Let  $u \rightarrow a$  and  $v \rightarrow b$ . It is to be proved that

$$1. (u + v) - (a + b) \quad 2. uv - ab \quad 3. \frac{u}{v} - \frac{a}{b} \quad (b \neq 0)$$

all  $\rightarrow 0$  when  $u, v \rightarrow a, b$ . But, setting  $u = a + h, v = b + k$ , 1., 2., 3., become

$$1. [(a + h) + (b + k)] - (a + b) = h + k$$

$$2. (a + h)(b + k) - ab = bh + (a + h)k$$

$$3. \frac{a + h}{b + k} - \frac{a}{b} = \frac{bh - ak}{b(b + k)} = \frac{1}{b + k}h - \frac{a}{b(b + k)}k$$

The last members of 1., 2., 3. are all of the form<sup>1</sup>  $Ph + Qk$  of § 6, and therefore  $\rightarrow 0$  when  $h, k \rightarrow 0$ , that is, when  $u, v \rightarrow a, b$ .

By repeated applications of the first two theorems they may be extended to sums and products of any finite number of variables. In particular,

4. If  $u \rightarrow a$ , then  $u^n \rightarrow a^n$

**EXAMPLE.** Let  $R(x)$  denote any expression which, like  $x^2 + 8x$  or  $(2x + 1)/x^2$ , involves  $x$  rationally; and let  $c$  be any value of  $x$  for which no denominator in  $R(x)$  vanishes. It follows from the theorems just proved that

when  $x \rightarrow c$  then  $R(x) \rightarrow R(c)$

Thus, when  $x \rightarrow c (\neq 0)$ , we have

$$\lim \frac{2x + 1}{x^2} = \frac{\lim (2x + 1)}{\lim x^2} = \frac{2 \lim x + 1}{(\lim x)^2} = \frac{2c + 1}{c^2}.$$

**8. Limits of roots.** *The limit of the  $n$ th root of a positive variable which approaches a limit is the  $n$ th root of that limit.*

First, let  $x$  decrease toward  $c (\geq 0)$  as limit. Then  $x^{1/n}$  decreases but remains  $> c^{1/n}$  and therefore approaches some limit  $l$  (§ 5, 2.).

But when  $x^{1/n} \rightarrow l$ , then  $x \rightarrow l^n$ , (§ 7, 4.). Hence  $l^n = c$  and therefore  $l = c^{1/n}$ .

Similarly (by § 5, 1.) when  $x$  increases toward  $c (> 0)$  as limit,  $x^{1/n} \rightarrow c^{1/n}$ .

Hence if  $x \rightarrow c$  through any positive values, then  $x^{1/n} \rightarrow c^{1/n}$ .

**9. The symbol  $\lim_{x \rightarrow c} E(x)$ .** Let  $E(x)$  be an expression in  $x$  which has a single real value for each value of  $x$  under consideration.

If for all modes of approach of  $x$  to  $c$  as limit ( $x$  remaining  $\neq c$ ),  $E(x)$  approaches one and the same number  $l$  as limit, this limit  $l$  is represented by the symbol  $\lim_{x \rightarrow c} E(x)$ .

<sup>1</sup> In 3., when  $|k| < |b|/2$  we have  $|P| < 2/|b|$  and  $|Q| < 2|a|/b^2$ .

<sup>2</sup> It is here assumed that  $x$  cannot approach two different limits simultaneously. Prove this.

If  $x$  is restricted to values  $< c$ , or to values  $> c$ , the symbol  $\lim_{x \rightarrow c} E(x)$  may be replaced by  $\lim_{x \rightarrow c^-} E(x)$ , or by  $\lim_{x \rightarrow c^+} E(x)$ .

$$\text{Thus, } \lim_{x \rightarrow 1} (x^2 + 3x) = 4, \quad \lim_{x \rightarrow 1} \sqrt{1-x} = 0, \quad \lim_{x \rightarrow 1} \sqrt{x-1} = 0.$$

When  $E(x)$  is rational with respect to  $x$  and has no denominator which vanishes when  $x = c$ , then, (§ 7, Example)

$$\lim_{x \rightarrow c} E(x) = E(c) \quad (1)$$

that is, *the limit approached by  $E(x)$  when  $x \rightarrow c$  is the value which  $E(x)$  has when  $x = c$ .*

**10. The limit  $\infty$ .** If a variable  $x$  will ultimately become and remain numerically greater than every given positive number  $C$ , however large, we say that it *approaches infinity as limit*,<sup>1</sup> and write  $|x| \rightarrow \infty$ . If  $x$  ultimately remains positive, we write  $x \rightarrow \infty$ ; if  $x$  ultimately remains negative, we write  $x \rightarrow -\infty$ .

Thus if  $x = 1, 2, 3, \dots$ , then  $x \rightarrow \infty$ ; if  $x = -1, -2, -3, \dots$  then  $x \rightarrow -\infty$ .

**11. Theorem 1.** *Let  $a$  be any constant; then  $\lim_{|x| \rightarrow \infty} a/x = 0$*

For, however small a positive value is assigned to  $\delta$ , we have

$$|a/x| < \delta \quad \text{when} \quad |x| > |a|/\delta$$

Thus we make  $2/|x| < .001$  when we take  $|x| > 2/.001 = 2000$ .

For  $\lim_{x \rightarrow \infty} E(x)$  we often write  $E(\infty)$ ; thus  $a/\infty = 0$ .

**EXAMPLE 1.** *When  $x \rightarrow \pm \infty$ , then  $a_0 x^n + a_1 x^{n-1} + \dots + a_n$  approaches infinity with the sign of  $a_0 x^n$ , its term of highest degree.*

$$a_0 x^n + a_1 x^{n-1} + \dots + a_n = x^n (a_0 + a_1/x + \dots + a_n/x^n),$$

and

$$a_1/x, \dots, a_n/x^n \rightarrow 0 \text{ when } x \rightarrow \pm \infty.$$

<sup>1</sup> The word "limit" in the phrase "approaches infinity as limit" has a meaning wholly distinct from that in § 4.  $x \rightarrow \infty$  means merely that no number can be assigned which  $x$  will not ultimately continually exceed.

EXAMPLE 2. If  $E(x) = (3x^2 + 4x)/(2x^2 + 1)$ , find  $\lim_{x \rightarrow \infty} E(x)$ .

$$\frac{3x^2 + 4x}{2x^2 + 1} = \frac{x^2(3 + 4/x)}{x^2(2 + 1/x^2)} = \frac{3 + 4/x}{2 + 1/x^2} \quad \therefore \lim_{x \rightarrow \infty} E(x) = \frac{3}{2}.$$

EXAMPLE 3. Prove that 1.  $\lim_{x \rightarrow \infty} \frac{2x^4 + x}{x^3 + 5} = \infty$  2.  $\lim_{x \rightarrow \infty} \frac{x^2 + 3}{x^3 + 2x} = 0$

$$3. \lim_{x \rightarrow \infty} \frac{(1-x)(3+x^2)}{(2x^2 + 5)x} = -\frac{1}{2}.$$

**12. Theorem 2.** Let  $a$  be any constant except 0; then  $\lim_{x \rightarrow 0} |a/x| = \infty$

For, however great a number  $C$  is assigned, we have

$$|a/x| > C \quad \text{when} \quad |x| < |a|/C$$

Thus we make  $|2/x| > 1000$  when we take  $|x| < 2/1000 = .002$ .

If  $\lim_{x \rightarrow c} |E(x)| = \infty$ , we say that  $E(x)$  becomes infinite when  $x = c$ , and we write  $|E(c)| = \infty$ . Thus  $|a|/0 = \infty$ .

A rational fraction in  $x$ , reduced to its lowest terms, becomes infinite for values of  $x$  for which its denominator vanishes.

Thus  $x^2/(x-2)$  becomes  $\infty$  when  $x = 2$ . For when  $x \rightarrow 2$ ,  $x^2$  ultimately becomes and remains  $> 2^2 - 1 = 3 \quad \therefore x^2/|x-2| > 3/|x-2|$ . But  $3/|x-2| \rightarrow \infty$ . Hence  $x^2/|x-2| \rightarrow \infty$ .

**13. Indeterminate forms.** 1. When  $x = 1$ , the fraction  $(x^2 - 1)/(x - 1)$  takes the meaningless form  $0/0$ . This is because both numerator and denominator contain the factor  $x - 1$  which vanishes when  $x = 1$ . We cannot remove this common factor when  $x = 1$  since that would involve division by 0, which is never admissible. But, for  $x \neq 1$ , we have  $(x^2 - 1)/(x - 1) = x + 1$  and therefore

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} (x + 1) = 2$$

We therefore assign to  $(x^2 - 1)/(x - 1)$  for  $x = 1$  the value 2.

2. In general, suppose that when  $x = c$  an expression  $E(x)$  takes an indeterminate form, that is, a form, such as  $0/0$ ,

$\infty \cdot 0$ ,  $\infty - \infty$ , which is arithmetically meaningless. If  $\lim_{x \rightarrow c} E(x)$  exists, we assign to  $E(c)$  the value  $\lim_{x \rightarrow c} E(x)$ .

EXAMPLE 1. The expression  $E(x) = (2x^2 - 5x + 2)/(x^2 - 3x + 2)$  becomes  $0/0$  when  $x = 2$ . This means that both terms of the fraction contain the factor  $x - 2$ . Canceling this common factor, we find

$$E(2) = \lim_{x \rightarrow 2} \frac{2x^2 - 5x + 2}{x^2 - 3x + 2} = \lim_{x \rightarrow 2} \frac{2x - 1}{x - 1} = 3$$

EXAMPLE 2. Find the following limiting values:

1.  $\lim_{x \rightarrow 3} \frac{x^2 - 4x + 3}{x^2 - 7x + 12}$

2.  $\lim_{x \rightarrow 1/2} \frac{2x^3 + x^2 + 5x - 3}{8x^3 - 8x^2 + 4x - 1}$

3.  $\lim_{x \rightarrow 4} \frac{2x^3 - 7x^2 - 3x - 4}{x^3 - 5x^2 - 8x + 48}$

4.  $\lim_{x \rightarrow 1} \left[ \frac{1}{2x^2 + x - 3} - \frac{1}{3x^2 - x - 2} \right]$

**14. Functions.** 1. If  $y$  is so related to  $x$  that to each value of  $x$  in the interval  $(a, b)$  corresponds a single real value of  $y$ , then  $y$  is said to be a *function* of  $x$  in  $(a, b)$ .

Thus, the equation  $y = x^2$  defines  $y$  as a function of  $x$  in the interval  $(-\infty, \infty)$ . Again  $y = \sqrt{1 - x^2}$  defines  $y$  as a function of  $x$  in the interval  $(-1, 1)$ .

2. The statement that  $y$  is a function of  $x$  does not necessarily imply that we can express  $y$  in terms of  $x$ .

Thus, suppose that  $t$  hours after noon on a certain day a particular thermometer records  $T$  degrees of temperature. To each value of  $t$  corresponds a definite value of  $T$ . Hence  $T$  is a function of  $t$ . But we cannot express  $T$  in terms of  $t$ .

3. To indicate that  $y$  is a function of  $x$ , we write  $y = f(x)$  and then represent the value which  $y$  has when  $x = c$  by  $f(c)$ . When  $y$  is given as equal to some expression in  $x$ , we use  $f(x)$  to represent that expression. Thus, if  $y = x^2 + 3x$ , then  $f(x) = x^2 + 3x$  and  $f(2) = 2^2 + 3 \cdot 2 = 10$ .

**15. Graphs of functions.** Let  $y = f(x)$  denote some given function of  $x$  in the interval  $(a, b)$ . Taking rectangular axes  $Ox, Oy$ , plot the point  $P$  whose abscissa is any value of  $x$  in  $(a, b)$  and whose ordinate is the corresponding value of  $f(x)$ .



The set of all such points  $P$  is called the *graph* of  $y = f(x)$  in  $(a, b)$ . The definition of function, § 14, says nothing as to the arrangement of these points  $P$  except that one of them, and but one, lies on each parallel to  $Oy$  between  $aA$  and  $bB$ . But in Fig. 2 it is supposed that they form a smooth unbroken curve arc,  $APB$ , such as may be traced

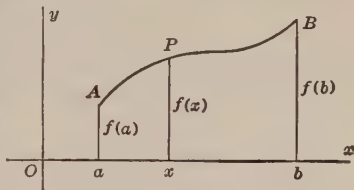


FIG. 2.

by a point moving continuously from  $A$  to  $B$ . It will be found that the functions  $y = f(x)$  which we meet in the Calculus all have graphs of this type — except when  $f(x)$  becomes infinite at some point in  $(a, b)$ , a case illustrated in Fig. 4.

**16. Continuous functions.** Obviously the function  $y = f(x)$  pictured in Fig. 2 has the property that small changes in the value of  $x$  cause but small changes in the value of  $f(x)$ . If  $c$  denote any point in  $(a, b)$ , then  $f(c)$  has a definite finite value; when  $x - c$  is small, so also is  $f(x) - f(c)$ ; and  $\lim_{x \rightarrow c} [f(x) - f(c)] = 0$ . To indicate all this, we say that  $f(x)$  is *continuous* at the point  $x = c$ . In general,

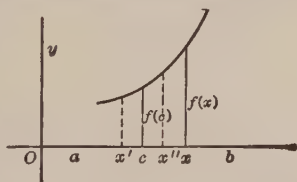


FIG. 3.

A function  $f(x)$  is said to be *continuous at the point  $x = c$*  when  $f(c)$  has a definite finite value and  $\lim_{x \rightarrow c} f(x) = f(c)$ .

A point where either of these conditions is not satisfied is called a *point of discontinuity*.

A function  $f(x)$  is said to be *continuous in the interval  $(a, b)$*  when it is continuous at every point of the interval, it being sufficient at the end points that  $\lim_{x \rightarrow a} f(x) = f(a)$ ,  $\lim_{x \rightarrow b} f(x) = f(b)$ .

By turning to §§ 7, 8, and 12, it will be seen that we have already proved that a polynomial in  $x$  is continuous for all finite values of  $x$ ; a rational fraction in  $x$ , reduced to its lowest terms, is discontinuous at points where its denominator vanishes, but continuous for all other finite values of  $x$ ; the function  $\sqrt[n]{x}$  is continuous for all finite positive values of  $x$  and for  $x = 0$ .

**17. Points of discontinuity.** The only points of discontinuity that immediately concern us are points  $x = c$  at which a function  $f(x)$  becomes infinite in the sense  $\lim_{x \rightarrow c} |f(x)| = \infty$ .

But, as the following examples show, the definition of function, § 14, makes other discontinuities possible.

The condition for continuity,  $\lim_{x \rightarrow c} f(x) = f(c)$ , implies that, any positive number  $\delta$  being given, however small, it is possible to find an interval  $(x', x'')$  containing  $c$  as an interior point (Fig. 3), and such that the difference between every two values of  $f(x)$  in  $(x', x'')$  is  $< \delta$ . A point  $c$  for which this is not the case is a point of discontinuity.

**EXAMPLE 1.** Both  $y = 1/x$  and  $y = 1/x^2$  are continuous for all values of  $x$  except 0. But  $\lim_{x \rightarrow 0} 1/|x| = \infty$ ,  $\lim_{x \rightarrow 0} 1/x^2 = \infty$ .

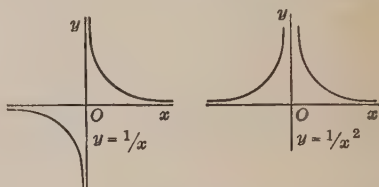


FIG. 4.

Hence both functions are discontinuous at  $x = 0$ . In any interval  $(x', x'')$  containing 0, both  $1/x$  and  $1/x^2$  take values whose difference will exceed any given number.

**EXAMPLE 2.** The function  $y = f(x)$ , where  $f(x) = x - 1$  when  $x \leq 1$ , but  $f(x) = x$  when  $x > 1$ , is discontinuous at  $x = 1$ ; for  $f(1) = 0$  but  $\lim_{x \rightarrow 1} f(x) = 1$ . In any interval

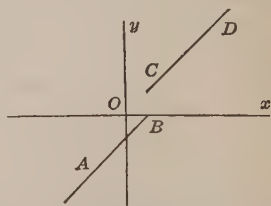


FIG. 5.

$(x', x'')$  containing 1 as an interior point,  $f(x)$  takes values whose difference exceeds 1. The graph consists of the half-lines  $AB$ ,  $CD$ , the point  $C$  excluded.



**EXAMPLE 3.** As  $x \rightarrow 0$ , the function  $\sin (1/x)$  continually oscillates between  $-1$  and  $1$  and therefore does not approach a limit. Hence  $\sin (1/x)$  is discontinuous at  $x = 0$ . Therefore the convention made in § 13, 2, gives no meaning to  $\sin (1/0)$ .

**18. Properties of continuous functions.** As will be proved later, it follows from the definition of a continuous function that

1. If  $f(x)$  is continuous in  $(a, b)$ , then among the different values of  $f(x)$  in  $(a, b)$  there is a greatest<sup>1</sup> value  $M$  and a least value  $m$ .

2. If  $f(x)$  is continuous in  $(a, b)$ , then between every two values  $x_1$  and  $x_2$  of  $x$  in  $(a, b)$ ,  $f(x)$  takes at least once every value between  $f(x_1)$  and  $f(x_2)$ .

These theorems are obvious geometrically if we assume that, in  $(a, b)$ ,  $y = f(x)$  has a graph of the kind described in § 15.

**EXAMPLE.** Suppose that the graph of  $y = f(x)$  is the curve arc  $CDEF$ . Here  $m = AC$  and  $M = GE$ . If  $x_1 = OX_1$ ,  $x_2 = OX_2$ , we have  $X_1P_1 = f(x_1)$ ,  $X_2P_2 = f(x_2)$ .

Let  $OY'$  represent any number between  $X_1P_1$  and  $X_2P_2$ . The parallel to  $Ox$  through  $Y'$  will meet the unbroken arc  $P_1P_2$  in some point  $P'$ , and if the abscissa  $OX'$  of  $P'$  be  $x'$ , we have  $f(x') = x'P' = OY'$ .

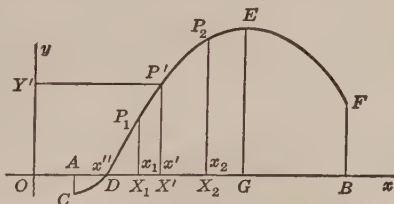


FIG. 6.

Observe that correspond-

ing to the fact that  $f(a) = AC$  is  $-$ , and  $f(b) = BF$  is  $+$ , there is a value of  $x$  in  $(a, b)$  for which  $f(x)$  is 0, namely the value  $x'' = OD$ .

**19. Polynomials in  $x$ .** This name is given to functions of the form  $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ , where  $n$  is a positive integer and  $a_0, a_1, \dots, a_n$  are constants.

<sup>1</sup> It is by no means true of every infinite set of numbers that it contains a greatest and a least number. Thus the set consisting of all numbers  $x$  such that  $2 < x < 3$  contains no greatest or least number.

A value  $b$  of  $x$  such that  $f(b)$  is 0 is called a *root* of  $f(x)$  or of the equation  $f(x) = 0$ .

1. The following discussion leads us to the most expeditious method of computing the value of  $f(x)$  for a given value of  $x$  and to the theorem:

*$f(x)$  is exactly divisible by  $x - b$  when and only when  $f(b) = 0$ .*

Suppose  $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$  to be divided by  $x - b$ . Let  $\phi(x) = a_0x^2 + C_1x + C_2$  denote the quotient, and  $R$  the remainder,  $C_1, C_2, R$  being constants whose values are to be determined. Then

$$\begin{aligned} a_0x^3 + a_1x^2 + a_2x + a_3 &\equiv (a_0x^2 + C_1x + C_2)(x - b) + R \\ &\equiv a_0x^3 + (C_1 - a_0b)x^2 + (C_2 - C_1b)x + (R - C_2b) \end{aligned}$$

Equating coefficients of like powers of  $x$ ,

$$C_1 - a_0b = a_1 \quad \therefore C_1 = a_0b + a_1$$

$$C_2 - C_1b = a_2 \quad \therefore C_2 = C_1b + a_2 = a_0b^2 + a_1b + a_2$$

$$R - C_2b = a_3 \quad \therefore R = C_2b + a_3 = a_0b^3 + a_1b^2 + a_2b + a_3 = f(b)$$

Hence (1)  $f(x) \equiv (x - b)\phi(x)$  when and only when  $f(b) = 0$ ; and

$$\begin{array}{r|l} a_0 & a_1 & a_2 & a_3 & | & b \\ a_0b & C_1b & C_2b & & & \\ \hline a_0 & C_1 & C_2 & R & = & f(b) \end{array}$$

(2) we may compute  $C_1, C_2, f(b)$  successively as in the accompanying scheme:  $C_1$  being got by adding  $a_0b$  to  $a_1$ , then  $C_2$  by adding  $C_1b$  to  $a_2$ , and so on.

This process is called *synthetic division* (or substitution).

EXAMPLE 1. Find the value of  $f(x) = 3x^4 - 8x^2 + 10x - 5$  when  $x = -2$ .

$$\begin{array}{r|l} 3 & +0 & -8 & +10 & -5 & | & -2 \\ -6 & +12 & -8 & -4 & & \\ \hline 3 & -6 & +4 & +2 & -9 & \end{array}$$

$$\text{Hence } f(-2) = -9$$

This is the remainder in the division of  $f(x)$  by  $x - (-2) = x + 2$ . The quotient is  $3x^3 - 6x^2 + 4x + 2$ .

EXAMPLE 2. Prove that 3 and  $-1/2$  are roots of  $2x^4 - 5x^3 + x^2 - 10x - 6 = 0$

2. For polynomials  $f(x)$  with integral coefficients we may deduce the following theorem from the fact that, if  $b$  is a root, then  $a_0b^n + \dots + a_{n-1}b + a_n = 0$ :

*Let the coefficients of  $f(x)$  be integers. If an integer  $b$  be a root of  $f(x)$ , then  $b$  is a factor of  $a_n$ . And if a rational fraction  $\beta/\alpha$  (in its lowest terms) be a root, then  $\beta$  is a factor of  $a_n$  and  $\alpha$  of  $a_0$ .*

In consequence of this theorem, the rational roots, if any, may be found by a limited number of trials. If one such root has been found, the search for the rest may be shortened by using the fact that if  $b$  be a root of  $f(x)$ , then, since  $f(x) \equiv (x-b)\phi(x)$ , the remaining roots of  $f(x)$  are the roots of the "depressed polynomial"  $\phi(x)$ .

The irrational roots, if any, of  $f(x)$  or the depressed polynomial  $\phi(x)$  may be found approximately by applications of the theorem § 18, 2. :

*If  $f(a)$  and  $f(b)$  have opposite signs, there is a root of  $f(x)$  between  $a$  and  $b$ .*

We seek first to locate the roots between consecutive integers, then between consecutive tenths, and so on. The method will be explained later.

EXAMPLE 3. Find the roots of  $3x^4 + 4x^3 - 19x^2 - 8x + 12 = 0$ .

EXAMPLE 4. Show that  $2x^3 - 3x^2 - 9x + 8 = 0$  has roots in the  $x$ -intervals  $(0, 1)$ ,  $(2, 3)$ ,  $(-2, -1)$ .

3. Let  $f(x) = a_0x^n + \dots$  denote a given polynomial of degree  $n$  with real coefficients. It is the fundamental theorem of Algebra that every polynomial  $f(x)$  has at least one root. By aid of this theorem and the theorem that if  $f(b) = 0$ , then  $f(x) \equiv (x-b)\phi(x)$ , where  $\phi(x)$  is a polynomial of degree  $n-1$ , it can be proved that  $n$  numbers  $\beta_1, \beta_2, \dots, \beta_n$  exist such that  $f(x) \equiv a_0(x-\beta_1)(x-\beta_2) \dots (x-\beta_n)$ .

Evidently  $f(\beta_1) = 0$ , and so on. It is therefore said that

*Every algebraic equation  $f(x) = 0$  of degree  $n$  has  $n$  roots.*

Certain of these roots may be equal. Those, if any, which are imaginary occur in pairs of the form  $a + bi$ ,  $a - bi$ , where  $a$  and  $b$  are real and  $i = \sqrt{-1}$ . But

$$[x - (a + bi)][x - (a - bi)] = x^2 - 2ax + (a^2 + b^2).$$

Hence

*Every polynomial  $f(x)$  with real coefficients is a product of factors of the first or second degree with real coefficients.*

**20. Rational functions.** Any function which can be reduced to a polynomial in  $x$  or to a fraction whose numerator and denominator are such polynomials is called a *rational function of  $x$ , integral or fractional*.

**21. Algebraic functions.** A function  $y = \phi(x)$  is called *algebraic* if it satisfies an *algebraic equation in  $x, y$* , that is, an equation  $f(x, y) = 0$  in which  $f(x, y)$  is a polynomial in  $x, y$ . Thus  $y = \sqrt{x}$  is algebraic: it satisfies  $y^2 - x = 0$ .

A rational function is algebraic. Thus  $y = 1/x$  satisfies  $xy - 1 = 0$ .

Functions, such as  $y = \sin x$ ,  $y = \log x$ , which are not algebraic, are called *non-algebraic* or *transcendental*.

### EXERCISE I

1. If  $f(x) = x^5 - x^3 + 3x^2 - x + 4$ , find  $f(0)$ ,  $f(1)$ ,  $f(-1)$ ,  $f(6)$ .
2. If  $f(x) = x^m$ , then  $f(x)f(y) = x^m y^m = (xy)^m = f(xy)$ .
3. If  $f(x) = a^x$ , show that  $f(x)f(y) = f(x + y)$ .
4. If  $f(x) = \log x$ , show that  $f(x) + f(y) = f(xy)$ ; also  $f(x^n) = nf(x)$ .
5. If  $f(x) = x + 1/x$ , show that  $[f(x)]^2 = f(x^2) + 2$ ; also  $[f(x)]^3 = f(x^3) + 3f(x)$ .
6. If  $y = f(x) = (1 - x)/(1 + x)$ , show that  $x = f(y)$ .
7. Prove by § 7 that, if two functions  $f(x)$  and  $\phi(x)$  are both continuous at  $x = c$ , so also are the functions  $f(x) + \phi(x)$  and  $f(x)\phi(x)$ ; also  $f(x)/\phi(x)$  unless  $\phi(c) = 0$ .
8. Show that, by the convention in § 13,  $f(x) = (x^2 - 1)/(x - 1)$  is continuous at  $x = 1$ .
9. Show that by the same convention  $f(x) = x \sin (1/x)$  is continuous at  $x = 0$ .
10. At what points, if any, are the following functions discontinuous?
 

|                                    |                                                |
|------------------------------------|------------------------------------------------|
| 1. $\frac{2x + 1}{x^3 - x}$        | 2. $\frac{(x - 1)(x + 1)^2}{(x - 1)^2(x + 1)}$ |
| 3. $\frac{x}{x^2 + 4}$             | 4. $\frac{x^2 - 5x + 4}{x^2 - 7x + 6}$         |
| 5. $\frac{2 - x}{(4 - x^2)^{1/2}}$ | 6. $\frac{1}{x^2 - 1} - \frac{1}{x(x^2 - 1)}$  |

11. Find the graph of  $y = |x|$ . Is this function discontinuous at  $x = 0$ ?

12. Find the graph of  $y = 1/(1 - |x|)$ . What discontinuities has this function?

13. The function  $y = \operatorname{sgn} x$ , read "sign  $x$ ," has the value  $-1$ ,  $0$ , or  $1$  according as  $x$  is negative,  $0$ , or positive. Draw its graph. What discontinuities does it have?

14. Given that  $f(x) = -x$  when  $x < 0$ ,  $f(x) = 2x + 1$  when  $x \geq 0$ . Show that  $f(x)$  is discontinuous at  $x = 0$ . Find the graph of  $y = f(x)$ .

15. Show that  $y$  is defined as a function of  $x$  by the statement: " $y$  is  $0$  or  $1$  according as  $x$  is rational or irrational." Show also that this function is discontinuous for all values of  $x$ . Can its graph be represented?

16. Illustrate the fact that  $f(x) = 1/x$  is discontinuous at  $x = 0$  by finding a point between  $0$  and  $.1$  where  $f(x) - f(.1) > 10^6$ .

17. Illustrate that  $f(x) = x^2 + 3x$  is continuous at  $x = 0$  by finding an interval  $(x', x'')$  in which  $|f(x)| < .01$ .

18. Find the limiting value of  $\left(x + \frac{1}{x}\right) / \left(2x - \frac{1}{x}\right)$  when  $x \rightarrow \infty$ ; also when  $x \rightarrow 0$ .

19. Show that  $y^2 - x = 0$  defines  $y$  as a "two-valued function" of  $x$  in the  $x$ -interval  $(0, \infty)$ . What two one-valued functions  $y = f(x)$  express this same relation of  $y$  to  $x$ ?

20. Given that (1)  $u$  increases and  $v$  decreases; (2)  $u$  remains  $< v$ ; (3)  $v - u \rightarrow 0$ ; prove by the theorems of § 5 that  $u$  and  $v$  approach one and the same limit.

21. Construct geometrically (using the straight line and circle only) the points in Fig. 1 which correspond to the numbers  $10/3$ ,  $-5/6$ ,  $\sqrt{2}$ ,  $\sqrt{3}$ . The points corresponding to  $\sqrt[3]{2}$  and  $\pi$  cannot be so constructed, but we assume that they exist. It will be proved later that to every point  $A$  on the line corresponds a real number  $a$  in the sense explained in § 1; but it is an assumption, a geometric postulate, that to every real number  $a$  corresponds a point  $A$ .

## II. THE DERIVATIVE

**22. The differential calculus.** The differential calculus is concerned first of all with the problem: A variable  $y$  being defined as a function of another variable  $x$  by a given equation of the form  $y = f(x)$ , it is required to find how  $y$  varies when  $x$  varies continuously. The problem is solved by aid of the function defined in the following section.

**23. The derivative.** 1. First consider the function  $y = x^2$ . Let  $x_1$  denote any given value of  $x$ , and  $y_1$  the corresponding value of  $y$ , so that

$$y_1 = x_1^2 \quad (1)$$

Again let  $\Delta x$  denote any change<sup>1</sup> in the value of  $x$ , and  $\Delta y$  the corresponding change in the value of  $y$ . We call  $\Delta x$ ,  $\Delta y$  corresponding *increments* of  $x$ ,  $y$ . They may be positive or negative.

Since  $y = x^2$ , if the increment  $\Delta x$  be added to  $x_1$ , an increment  $\Delta y$  will be added to  $y_1$  such that

$$y_1 + \Delta y = (x_1 + \Delta x)^2 \quad (2)$$

Subtract (1) from (2) and simplify the result; we find

$$\Delta y = 2 x_1 \Delta x + (\Delta x)^2 \quad (3)$$

Divide both members of (3) by  $\Delta x$ ; we get

$$\frac{\Delta y}{\Delta x} = 2 x_1 + \Delta x \quad (4)$$

The right member of (4) shows that if we treat  $\Delta x$  as a variable and cause it to approach 0 as limit, the ratio  $\Delta y/\Delta x$  will approach a definite limit, namely the number represented

<sup>1</sup> The letter  $\Delta$ , read "delta," has no numerical meaning in this connection. It is merely part of the symbol for an increment.



by  $2x_1$ . This number  $2x_1$  is called the *derivative of  $y = x^2$  at the point  $x = x_1$* . Thus the derivatives at the points  $x = -1, 0, 3$  are  $-2, 0, 6$ .

The numbers  $2x_1$  are values of the function  $2x$ . This function  $2x$  is called the *derivative of the function  $x^2$* . It may be obtained directly from  $y = x^2$  by using  $x$  instead of  $x_1$  to denote the initial value of  $x$ .

2. Let  $y = f(x)$  be any given function and let  $x$  denote any value of  $x$  for which  $f(x)$  is continuous, § 16. If the increment  $\Delta x$  be added to  $x$ , an increment  $\Delta y$  will be added to  $y$  such that

$$y = f(x) \quad \text{becomes} \quad y + \Delta y = f(x + \Delta x)$$

Subtract the first equation from the second; we get

$$\Delta y = f(x + \Delta x) - f(x) \quad (5)$$

By hypothesis,  $f(x)$  is continuous for the value of  $x$  under consideration; hence, § 16,  $\Delta y \rightarrow 0$  when  $\Delta x \rightarrow 0$ .

Divide both members of (5) by  $\Delta x$ ; we obtain

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (6)$$

Here  $[f(x + \Delta x) - f(x)]/\Delta x$  represents a known expression in  $x$  and  $\Delta x$  and, in the cases that we shall meet, when  $\Delta x \rightarrow 0$  this expression will approach a definite expression in  $x$  as limit. This last expression, or function, is called the *derivative of  $y = f(x)$*  and is denoted by the symbol  $f'(x)$ . And  $[f(x + \Delta x) - f(x)]/\Delta x \rightarrow f'(x)$  means that for every value  $c$  of  $x$  for which  $[f(c + \Delta x) - f(c)]/\Delta x$  approaches a definite number as limit when  $\Delta x \rightarrow 0$ , that number is  $f'(c)$ . Therefore, by definition,

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (7)$$

The derivative of a function  $f(x)$  is the limit approached by the ratio of the increment of  $f(x)$  to the increment of  $x$  when the increment of  $x$  approaches the limit 0.



EXAMPLE 1. If  $f(x) = x^3$ , find  $f'(x)$ .

$$\begin{aligned} f(x + \Delta x) - f(x) &= (x + \Delta x)^3 - x^3 \\ &= 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 \end{aligned}$$

Hence  $\frac{f(x + \Delta x) - f(x)}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2$

Therefore  $f'(x) = \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2.$

EXAMPLE 2. Find the derivative of  $x^4$ ; also the derivative of  $3x^2 + 2x$  at  $x = -1$ .

**24. Differentiation.** The process of finding the derivative of a function  $y = f(x)$  is called *differentiation*. It may be indicated by writing  $D_x$  before  $y$  or  $f(x)$ : thus,  $D_x y$ , read “ $D$  of  $y$ .” We may also use  $D_x y$  as a symbol for  $f'(x)$ .

EXAMPLE 1. Differentiate  $y = 1/x$ .

Since  $y = 1/x$ , we have  $y + \Delta y = 1/(x + \Delta x)$  and therefore

$$\Delta y = \frac{1}{x + \Delta x} - \frac{1}{x} = -\frac{\Delta x}{x(x + \Delta x)}$$

Hence  $\frac{\Delta y}{\Delta x} = -\frac{1}{x(x + \Delta x)}$

and therefore (§ 7)  $D_x y = -\lim_{\Delta x \rightarrow 0} \frac{1}{x(x + \Delta x)} = -\frac{1}{x^2}$

The reasoning holds good for all values of  $x$  except 0.

EXAMPLE 2. Differentiate  $y = \sqrt{x}$ .

Here  $\frac{\Delta y}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}$

When  $\Delta x \rightarrow 0$  this expression approaches the form  $0/0$ . But, if both numerator and denominator be multiplied by  $\sqrt{x + \Delta x} + \sqrt{x}$ , the numerator becomes  $\Delta x$ , and, canceling this with the  $\Delta x$  in the denominator, we have

$$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}$$

Hence (§ 8)  $D_x y = \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$

The reasoning holds good for all positive values of  $x$  and  $x + \Delta x$ . For  $x = 0$  we have  $\Delta y/\Delta x = 1/\sqrt{\Delta x}$ , and this  $\rightarrow \infty$  when  $\Delta x \rightarrow 0$ .

EXAMPLE 3. Differentiate the following functions:

$$\begin{array}{llll} 1. \sqrt{1-x} & 2. \frac{1}{\sqrt{x}} & 3. \frac{2x-1}{3x+1} & 4. \frac{1}{x^2} \quad 5. \frac{x^2+1}{x+1} \end{array}$$

**25. On the existence of a derivative.** 1. At a point  $x = c$  where the ratio  $[f(c + \Delta x) - f(c)]/\Delta x$  does not approach a finite limit when  $\Delta x \rightarrow 0$ , there is, properly speaking, no derivative. Nevertheless, in case  $\lim_{x \rightarrow c} |f'(x)| = \infty$ , it is customary to say that  $f(x)$  has an *infinite derivative* at  $x = c$ .

EXAMPLE 1. Both  $1/x$  and its derivative  $-1/x^2$  are infinite at  $x = 0$ .

2. A function  $f(x)$  cannot have a finite derivative at  $x = c$  unless it is continuous at  $x = c$ : i.e. unless  $f(c + \Delta x) - f(c) \rightarrow 0$  when  $\Delta x \rightarrow 0$ .

On the other hand,  $f(x)$  may be continuous at  $x = c$  and yet not have a finite derivative at  $x = c$ .

EXAMPLE 2. At  $x = 0$ ,  $\sqrt{x}$  is continuous but its derivative  $1/2\sqrt{x}$  is  $\infty$ .

EXAMPLE 3. The function  $f(x) = x \sin(1/x)$  is continuous at  $x = 0$  if we make  $f(0) = \lim_{x \rightarrow 0} f(x) = 0$ . But  $[f(0 + \Delta x) - f(0)]/\Delta x = \sin(1/\Delta x)$  does not approach a limit when  $\Delta x \rightarrow 0$ . Hence  $x \sin(1/x)$  has no derivative at  $x = 0$ .

EXAMPLE 4. The function  $y = |x|$ , whose graph consists of the half-lines  $OA$  and  $OB$ , is continuous, but does not have a derivative at  $O$ ; for

$$\frac{DP}{OD} = -1 \quad \therefore \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = -1$$

$$\frac{EQ}{OE} = 1 \quad \therefore \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = 1$$

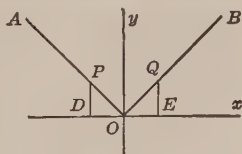


FIG. 7.

It may be said, however, that at  $O$  the function has a *left-hand derivative*,  $-1$ , and a *right-hand derivative*,  $1$ .

**26. Derivative of a constant.** A constant  $c$  does not change when  $x$  is given the increment  $\Delta x$ . Hence  $\Delta c = 0$   
 $\therefore \Delta c/\Delta x = 0 \quad \therefore \lim_{\Delta x \rightarrow 0} \Delta c/\Delta x = 0$ , that is,

$$D_x c = 0 \quad (1)$$

In like manner, if  $y$  be a variable which is independent of  $x$ , then  $D_x y = 0$ . For  $\Delta y$  denotes the increment which  $y$  receives as a consequence of giving  $x$  the increment  $\Delta x$ , and, if

$y$  be independent of  $x$ , its value is not affected by a change in the value of  $x$ . Hence  $\Delta y = 0$  and therefore  $\lim_{\Delta x \rightarrow 0} \Delta y / \Delta x = 0$ .

Thus  $\Delta x$  itself is a variable which is independent of  $x$ . Hence

$$D_x(\Delta x) = 0 \quad (2)$$

**27. Derivative of a sum.** Let  $y = u + v$ , where  $u, v$  denote functions of  $x$  which have finite derivatives for the values of  $x$  under consideration. When  $x$  is given the increment  $\Delta x$ , the functions  $u, v, y$  receive increments  $\Delta u, \Delta v, \Delta y$  such that  $\Delta y = \Delta u + \Delta v$  and therefore

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$

from which it follows, § 7, on making  $\Delta x \rightarrow 0$ , that

$$D_x y = D_x u + D_x v \quad (3)$$

The same reasoning applies to  $u - v, u \pm v \pm w$ , and so on. Hence

*The derivative of the algebraic sum of any finite number of functions is the algebraic sum of their derivatives.*

**28. Derivative of  $cu$ .** When  $x$  is given the increment  $\Delta x$ , and therefore  $u$  the increment  $\Delta u$ , then  $cu$  and  $u/c$  take the increments  $c\Delta u$  and  $\Delta u/c$ . Dividing  $c\Delta u, \Delta u/c$  by  $\Delta x$  and making  $\Delta x \rightarrow 0$ , we get

$$D_x cu = cD_x u \quad D_x(u/c) = D_x u/c \quad (4)$$

To find the derivative of  $cu$  or  $u/c$ , replace  $u$  by  $D_x u$ .

**29. Derivative of  $x^n$ .** Let  $y = x^n$ , where  $n$  is a positive integer. When  $x$  is given the increment  $\Delta x$ ,  $y$  takes an increment  $\Delta y$  such that

$$\Delta y = (x + \Delta x)^n - x^n$$

Set  $x + \Delta x = x'$  and therefore  $\Delta x = x' - x$ . Then

$$\frac{\Delta y}{\Delta x} = \frac{x'^n - x^n}{x' - x} = x'^{n-1} + x'^{n-2}x + \dots + x^{n-1}$$

There are  $n$  terms in this polynomial, and when  $\Delta x \rightarrow 0$  and therefore  $x' \rightarrow x$ , each of these terms  $\rightarrow x^{n-1}$ . Hence

$$D_x x^n = nx^{n-1} \quad (5)$$

Thus  $D_x x = 1$ ,  $D_x x^2 = 2x$ ,  $D_x x^3 = 3x^2$ ,  $D_x x^4 = 4x^3$ , and so on.

EXAMPLE 1. By the formulas (1), (3), (4), (5), we have

$$D_x 5x^7 = 5 \cdot 7x^6 = 35x^6, \quad D_x x^5/10 = 5x^4/10 = x^4/2,$$

$$D_x (2x^8 + 4x^4 - 7) = D_x 2x^8 + D_x 4x^4 - D_x 7 = 16x^7 + 16x^3.$$

EXAMPLE 2. If  $f(x) = (x - a)(x - 3a)$ , find  $f'(x)$ .

$$f(x) = x^2 - 4ax + 3a^2. \quad \text{Hence } f'(x) = 2x - 4a.$$

EXAMPLE 3. Differentiate each of the following with respect to  $x$ .

1.  $4 - 3x + 5x^2 - 7x^3$

2.  $2x^{12} - 10x^{11} + x^4 + 35$

3.  $x^3 - cx^2 + c^2x - c^3$

4.  $3(x - 2)(x + 4)x^2$

5.  $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-2}x^2 + a_{n-1}x + a_n$

6.  $(x - 1)(x - 2)(x - 3)$

7.  $x(1 - x)(1 + x) + x^2 + 5$

30. Slope property of the derivative. Let  $P$  and  $Q$  be the points  $(x_1, y_1)$  and  $(x_1 + \Delta x, y_1 + \Delta y)$  of the graph of  $y = f(x)$ .

Since  $PD = \Delta x$  and  $DQ = \Delta y$ , we have

$$\frac{\Delta y}{\Delta x} = \tan DPQ = \tan xSP$$

When  $\Delta x \rightarrow 0$ ,  $Q$  moves along the curve toward  $P$  as limit

and the line  $QPS$  turns about  $P$  toward a limiting position  $PT$  such that

$$\tan xTP = \lim_{\Delta x \rightarrow 0} \tan xSP = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(x_1) \quad (1)$$

By definition,  $PT$  is the tangent to the curve at  $P$ . Hence

*The derivative of  $y = f(x)$  at  $x = x_1$ , is the slope of the tangent to the graph at the point  $[x_1, f(x_1)]$ .*

The slope of the tangent to a curve at any point is also called the *slope of the curve* at that point. When  $f'(x)$  is con-

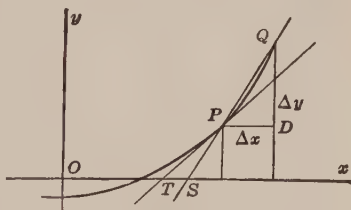


FIG. 8.

tinuous, the slope changes gradually from point to point and the curve is *smooth*.

The equation of the line through the point  $(x_1, y_1)$  and having the slope  $m$  is  $y - y_1 = m(x - x_1)$ ; hence the equation of the tangent to the curve  $y = f(x)$  at the point  $(x_1, y_1)$  is

$$y - y_1 = f'(x_1)(x - x_1) \quad (2)$$

EXAMPLE 1. Find the tangent and normal to the curve  $y = x^3$  at the point whose  $x$  is 2. The point is  $(2, 8)$ . Since  $f'(x) = 3x^2$ , we have  $f'(2) = 12$ . Hence the equation of the tangent is  $y - 8 = 12(x - 2)$ , or  $12x - y - 16 = 0$ . The normal, being perpendicular to the tangent, has the slope  $-1/12$ . Hence the equation of the normal is  $y - 8 = (-1/12)(x - 2)$ , or  $x + 12y - 98 = 0$ .

EXAMPLE 2. Find the points of the curve  $y = x^3$  where the slope is 3.

Since  $f'(x) = 3x^2 = 3$ , we have  $x = \pm 1$ . Hence the points are  $(1, 1)$  and  $(-1, -1)$ .

EXAMPLE 3. At what point of the parabola  $y = x^2 - 3x$  is the slope 0?

EXAMPLE 4. Find the tangent and normal to  $y = x^3 - 12x$  at the point whose  $x$  is  $-1$ ; also at the origin. What are the equations of the tangents parallel to  $Ox$ ?

EXAMPLE 5. For what values of  $x$  is the slope of  $y = 2x^3 - 9x^2 + 12x$  zero, positive, negative? At what points is the slope 12?

**31. Sign of the derivative.** Consider the values of  $y = f(x)$  in the order of increasing values of  $x$ . Let  $x_1$  be any value of  $x$  such that  $f'(x_1)$  is finite and not 0, and let  $y_1$  be the corresponding value of  $y$ ;

If  $f'(x_1)$  is positive, the value  $y_1$  of  $y$  is immediately preceded by lesser values only and followed by greater values only.

If  $f'(x_1)$  is negative, the value  $y_1$  of  $y$  is immediately preceded by greater values only and followed by lesser values only.

For when  $|\Delta x|$  is sufficiently small,  $\Delta y / \Delta x$  has the same sign<sup>1</sup> as its limit  $f'(x_1)$ , therefore  $\Delta y$  the same sign as  $f'(x_1)\Delta x$ .

<sup>1</sup> Suppose that  $u \rightarrow c$ , where  $c \neq 0$ . Then ultimately  $|u - c|$  remains  $< |c|$ . But  $u = c + (u - c)$ , and therefore when  $|u - c| < |c|$  the sign of  $u$  is that of  $c$ .

Hence

If  $f'(x_1)$  is  $+$ , then according as  $\Delta x \lesseqgtr 0$  we have  $\Delta y \lesseqgtr 0$  and therefore  $y_1 + \Delta y \lesseqgtr y_1$ .

If  $f'(x_1)$  is  $-$ , then according as  $\Delta x \lesseqgtr 0$  we have  $\Delta y \gtrless 0$  and therefore  $y_1 + \Delta y \gtrless y_1$ .

**32. Theorem.** *If  $f'(x)$  is positive throughout the  $x$ -interval  $(a, b)$ , then  $f(x)$  continually increases as  $x$  increases from  $a$  to  $b$ . If  $f'(x)$  is negative throughout  $(a, b)$ ,  $f(x)$  continually decreases as  $x$  increases from  $a$  to  $b$ .*

For let  $x_1$  and  $x_2$  be any two values of  $x$  such that  $a \leq x_1 < x_2 \leq b$ . By § 25, 2.,  $f(x)$  is continuous in  $(a, b)$ ; hence it has a greatest value  $M$  and a least value  $m$  in the interval  $(x_1, x_2)$ , § 18, 1.

If  $f'(x)$  is positive throughout  $(a, b)$ , then  $f(x_2)$  is  $M$ ; for every other value of  $f(x)$  in  $(x_1, x_2)$  is followed by greater values of  $f(x)$  in  $(x_1, x_2)$ , § 31. In like manner,  $f(x_1)$  is  $m$ . Hence  $f(x_1) < f(x_2)$ .

Similarly, if  $f'(x)$  is negative throughout  $(a, b)$ , then  $f(x_1) > f(x_2)$ .

**EXAMPLE 1.** When does  $y = x^3 - 3x$  increase and when decrease as  $x$  increases?

$$f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$$

In interval  $(-\infty, -1)$ ,  $f'(x)$  is  $+$

$\therefore y$  increases  $(-\infty \text{ to } 2)$

In interval  $(-1, 1)$ ,  $f'(x)$  is  $-$

$\therefore y$  decreases  $(2 \text{ to } -2)$

In interval  $(1, \infty)$ ,  $f'(x)$  is  $+$

$\therefore y$  increases  $(-2 \text{ to } \infty)$

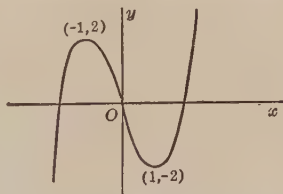


FIG. 9.

The graph rises from an infinite distance below  $Ox$  to the "maximum point"  $(-1, 2)$ , then falls to the "minimum point"  $(1, -2)$ , then rises to an infinite distance above  $Ox$  (§ 11, Ex. 1).

**EXAMPLE 2.** Deal similarly with the following, drawing their graphs.

1.  $y = x^3$    2.  $y = x^2 - 8x + 15$    3.  $y = x - x^2$    4.  $y = x^2(x - 1)^2$ .

**33. Approximate values of  $\Delta y$ .** Suppose  $f'(x_1) \neq 0$ . When  $|\Delta x|$  is sufficiently small,  $\Delta y/\Delta x$  differs but little numerically from its limit  $f'(x_1)$ , and therefore  $\Delta y$  but little from  $f'(x_1)\Delta x$ , that is, the difference is small as compared with  $f'(x_1)\Delta x$ . Hence, for small values of  $\Delta x$

*$f'(x_1)\Delta x$  is an approximate value of  $\Delta y$*

**EXAMPLE 1.** How much approximately does  $y = x^2$  change when  $x$  changes from 20 to 20.1?  $f'(x_1) = 2x_1 = 40$ ,  $\Delta x = .1 \therefore \Delta y = 40(.1) = 4$  approximately. The exact value of  $\Delta y$  is 4.01.

**EXAMPLE 2.** Find the approximate change in  $y = x^3 + 4x^2$  when  $x$  changes from 12 to 11.98.

**EXAMPLE 3.** If  $x$  start at 15, how much approximately may it change and change the value of  $y = 2x^2 + 5x$  by not more than .2?

**34. Differentials.** 1. The product  $f'(x)\Delta x$  is called the *differential* of  $y = f(x)$  and is denoted by  $dy$  or  $df(x)$ . Hence, by definition,

$$dy = f'(x)\Delta x \quad (1)$$

When  $y = x$ , then  $f'(x) = 1$ , and (1) becomes  $dx = \Delta x$ . Hence the definition (1) implies that the differential of the *independent* variable is the same as its increment. We may therefore also write (1) in the form

$$dy = f'(x)dx \quad (2)$$

**EXAMPLE.** Thus  $d(x^3) = 3x^2dx$ ,  $d(2x^2 + 3x + 5) = (4x + 3)dx$ .

2. The geometric meaning of  $dy$  is shown in Fig. 10; also its relation to  $\Delta y$ .  $P$  and  $Q$  are the points  $(x, y)$  and  $(x + \Delta x, y + \Delta y)$ ,  $PE$  is the tangent to the curve  $y = f(x)$  at  $P$ ,

$$PD = \Delta x = dx \quad DQ = \Delta y$$

$$DE = \tan DPE \cdot \Delta x = f'(x)dx = dy$$

3. Dividing both members of (2) by  $dx$ , we get

$$\frac{dy}{dx} = f'(x) \quad (3)$$

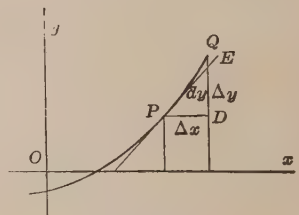


FIG. 10.



The new symbol  $dy/dx$  thus obtained for the derivative is the one most used. We shall ordinarily use it instead of  $D_x y$ . Thus the equation of the tangent at a curve point  $(x_1 y_1)$ , § 30(2), is written

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1) \quad (4)$$

4. Instead of  $D_x$  itself, we often use  $\frac{d}{dx}$ . Thus  $\frac{d}{dx}x^2 = 2x$ .

**35. Rate of change of a function.** 1. When  $y$  is so related to  $x$  that the ratio  $\Delta y/\Delta x$  of any corresponding changes in their values has a constant value,  $r$ , then  $y$  is said to vary uniformly with  $x$ , and  $r$  is called the *rate of change of  $y$  with respect to  $x$* . The numerical value of  $r$  is the number of units change in  $y$  per unit change in  $x$  and the sign of  $r$  indicates whether the rate is one of increase or decrease.

If  $x_1, y_1$  denote any particular pair of corresponding values of  $x$  and  $y$ , and  $x, y$  themselves any other pair, we may set  $\Delta x = x - x_1, \Delta y = y - y_1$  in the equation  $\Delta y/\Delta x = r$ , which gives

$$y - y_1 = r(x - x_1) \quad (1)$$

Therefore, if  $y$  varies uniformly with  $x$ , then  $y$  is connected with  $x$  by an equation of the first degree whose graph is a straight line having  $r$  for its slope. The converse is true; for, if  $y = mx + c$ , then  $\Delta y/\Delta x = m$ . Thus, if  $y = 6x + 5$ , then  $\Delta y/\Delta x = 6 = r$ .

2. When  $y$  is connected with  $x$  by an equation  $y = f(x)$  which is not of the first degree,  $y$  does not vary uniformly with  $x$ .

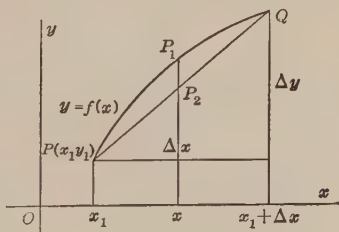


FIG. 11.

Let  $x_1, y_1$  be corresponding values of  $x, y$ , and  $\Delta x, \Delta y$  corresponding increments, and suppose  $f'(x_1)$  to exist and to be finite.

When  $x$  varies continuously from  $x_1$  to  $x_1 + \Delta x$ ,  $y$  varies continuously from  $y_1$  to  $y_1 + \Delta y$ , the corresponding point  $P_1(x, y)$  moving on the arc  $PQ$  of the graph of  $y = f(x)$ . But the result of this variation is the same as that effected by a uniform variation at the rate  $\Delta y/\Delta x$  in which the corresponding point  $P_2$  moves on the chord  $PQ$ ; and when  $\Delta x \rightarrow 0$ , the interval  $(x_1, x_1 + \Delta x)$  closes in on the point  $x_1$ , the arc  $PQ$  approaches coincidence<sup>1</sup> with the chord  $PQ$ , and the rate  $\Delta y/\Delta x$  approaches the limit  $f'(x_1)$ . This limit is therefore called the rate of change of  $y = f(x)$  with respect to  $x$  at the point  $x = x_1$ : that is, by definition,

*The rate of change of  $y = f(x)$  with respect to  $x$  at  $x = x_1$  is*

$$\frac{dy_1}{dx_1} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(x_1) \quad (2)$$

The ratio  $\Delta y/\Delta x$  is called the *mean rate* of change of  $y = f(x)$  with respect to  $x$  in the interval  $(x_1, x_1 + \Delta x)$ . Hence the rate at  $x = x_1$  may be defined as the limit approached by this mean rate as  $\Delta x \rightarrow 0$ .

Observe that the rate at  $x = x_1$  equals the slope of the graph at  $x = x_1$ .

EXAMPLE 1. If  $y = x^2$ , then  $dy/dx = 2x$ . Hence at  $x = 3$ , the rate of change of  $y$  with respect to  $x$  is 6, a rate for which  $y$  would increase 6 units when  $x$  increased 1 unit.

EXAMPLE 2. If  $x$  be the length in inches of an edge of a cube and  $V$  its volume (in cu. in.), then  $V = x^3 \therefore dV/dx = 3x^2$ . Hence, when  $x = 5$ , the rate of change of  $V$  with respect to  $x$  is 75 cu. in./in.

EXAMPLE 3. For what values of  $x$  is the rate of change of  $x^3 - 3x^2$  with respect to  $x$  equal to 9?

EXAMPLE 4. The current in an electric circuit varies inversely as the resistance. If  $c = 40$  when  $R = 6$ , find the rate of change of  $c$  with respect to  $R$  when  $R = 12$ .

EXAMPLE 5. Find a formula for the rate of change of the area  $A$  of a circle with respect to its radius  $r$ ; also for the rate of change of  $r$  with respect to  $A$ .

<sup>1</sup> By this is meant that  $P_2P_1/\Delta x \rightarrow 0$  when  $\Delta x \rightarrow 0$ .

EXAMPLE 6. Find the rate of change of the area of an equilateral triangle with respect to its altitude when the altitude is 8.

**36. Rectilinear motion.** Suppose that a point  $P$  is moving in a straight line. Let  $s$  denote its distance, in feet, from a fixed point  $O$  on the line  $t$  seconds from some given instant, the sign of  $s$  being  $+$  or  $-$  according as the direction from  $O$  to  $P$  is that taken as positive or negative on the line.

1. If  $P$  traverses equal spaces in equal intervals of time, its motion is said to be *uniform*. For in this case  $s$  varies uniformly with  $t$ , the ratio  $\Delta s/\Delta t$  of corresponding changes in  $s$  and  $t$  being constant. This constant ratio — the rate of change of  $s$  with respect to  $t$  — is called the *velocity* of the motion (in feet per second) and is denoted by  $v$ .

The equation connecting  $s$  and  $t$  has the form  $s = vt + s_0$ , where  $s_0$  is the distance of  $P$  from  $O$  when  $t = 0$ .

2. If the motion of  $P$  is not uniform but yet is in accordance with some known law expressed by a given equation  $s = f(t)$ , we define its velocity at any given instant  $t_1$  as the rate of change of  $s = f(t)$  with respect to  $t$  at  $t_1$ , that is, as the value at  $t_1$  of

$$v = \frac{ds}{dt} \quad (1)$$

The sign of  $v$  at  $t_1$  indicates the direction of the motion at  $t_1$ ; for  $s$  is increasing when  $ds/dt$  is  $+$ , decreasing when  $ds/dt$  is  $-$ , § 31.

The numerical value of  $v$  is called the *speed*.

3. The rate of change of  $v$  with respect to  $t$  at any instant  $t_1$  is called the *acceleration* at  $t_1$ . It is the value at  $t_1$  of the variable or constant  $\alpha$  defined by the formula:

$$\alpha = \frac{dv}{dt} \quad (2)$$

When  $\alpha$  is  $+$ , the velocity is increasing; when  $-$ , decreasing.

**EXAMPLE 1.** The distance  $s$  in feet that a body dropped from rest will fall in  $t$  seconds, the resistance of the air being disregarded, is approximately

$$s = 16 t^2$$

Since  $v = ds/dt = 32 t$ , the velocity at the end of the 1st, 2nd ... second is 32, 64 ... ft./sec. Since  $\alpha = dv/dt = 32$ , the acceleration is constant: 32 ft./sec<sup>2</sup>.

**EXAMPLE 2.** A point  $P$  moves in a straight line. At the end of  $t$  seconds its distance from  $O$  is  $s = t^3 - 7 t^2 + 8 t$  feet. Discuss its motion.

Here  $v = 3 t^2 - 14 t + 8 = (3 t - 2)(t - 4)$ , which is  $+$  in the  $t$ -interval  $(0, 2/3)$ ,  $-$  in the interval  $(2/3, 4)$ , and  $+$  when  $t > 4$ . Hence  $P$  first moves forward for  $2/3$  of a second, then backward for  $3\frac{1}{3}$  seconds, then forward again.

Also  $\alpha = 6 t - 14$ , which is  $-$  for  $t < 7/3$  and  $+$  for  $t > 7/3$ .

Hence the velocity first decreases for  $7/3$  seconds, then increases.

**EXAMPLE 3.** Let  $P$  be moving along a straight line, and let  $s$  denote its distance from  $O$  at the time  $t$  (before or after the instant of reference according as the sign of  $t$  is  $-$  or  $+$ ). In each of the following cases discuss the motion of  $P$  as in Ex. 2; also find the velocity and acceleration at the time indicated.

1.  $s = 2 t^3 - 21 t^2 + 60 t - 200$ ,  $t = 3$
2.  $s = t^4 - 2 t^2$ ,  $t = -5$ .

**EXAMPLE 4.** If a projectile  $P$  is shot vertically upward from a platform  $s_0$  feet from the ground, with an initial speed  $v_0$  ft./sec., at the end of  $t$  seconds it will be  $s = s_0 + v_0 t - 16 t^2$  feet above the ground. If  $s_0 = 20$ ,  $v_0 = 1024$ , find the two instants at which  $P$  is 14100 ft. above the ground and its velocities at these instants.

## EXERCISE II

1. Find the derivative and the differential of each of the following:

$$(1) 3x^4 - x^3/6 + (x^2 - 8x)/2 + 4 \qquad (2) x(x-1)(x^2 + x + 1)$$

2. When does  $y = x^2$  decrease and when increase? Draw the graph carefully, first plotting the points where  $x = 0, \pm 1/2, \pm 1, \pm 2, \pm 3$ . Find the tangent and normal where  $x = 1$ . Find the angles which it makes<sup>1</sup> with the line  $y - x - 2 = 0$ .

3. When does  $y = x^3 - 3x^2 - 9x$  increase and when decrease? Find its graph. Find the tangent and normal where  $x = 1$ . Find the points where the slope is  $-9$ .

<sup>1</sup> If two curves  $C_1$  and  $C_2$  intersect at the point  $P$  and their slopes at  $P$  are  $m_1$  and  $m_2$ , the angle  $\theta$  between  $C_1$  and  $C_2$  at  $P$  is given by the formula  $\tan \theta = (m_1 - m_2)/(1 + m_1 m_2)$ .

4. The sides of a rectangle are  $x + 2$  and  $2x + 3$ . At what rate with respect to  $x$  is its area increasing when  $x$  increasing passes through the value 5?

5. What is the change approximately in this area (Ex. 4) when  $x$  increases from 5 to 5.25?

6. Approximately how much does  $y = x^3 - 2x^2 + 5$  change when  $x$  changes from 10 to 9.96?

7. By how much approximately may  $x$ , starting at 7, increase and  $y = x^4 + x^2$  change not more than .3?

8. The edge of a cubical box is supposed to be a foot long. How much approximately may this be in error and the actual volume differ from 1 cu. ft. by not more than 1 cu. in.?

9. Find approximate formulas for the area of a narrow circular ring and the volume of a thin spherical shell.

10. If  $y = x^3 + 3x^2$ , find  $\Delta y - dy$  in terms of  $x$  and  $\Delta x$ ; also its value when  $x = 10$  and  $\Delta x = .1$ .

11. A soap bubble is being blown; at what rates with respect to its radius  $r$  are its surface and volume increasing when  $r = 3$ ?

12. The equation of motion of a ball rolling down a certain inclined plane is  $s = 5t^2$  (in feet and seconds). Find  $v$  when  $t = 4$ ;  $t$  and  $s$  when  $v = 10$ ; and  $\alpha$ .

13. A ball which starts rolling up a certain incline is at the end of  $t$  seconds at the distance (in feet)  $s = 3t - t^2$  from the starting point. What is its initial velocity? When does it begin rolling downward?

14. The equations of motion of two points  $P_1$  and  $P_2$  on the same line are  $s_1 = 2t^2 + 2$ ,  $s_2 = t^2 + t + 1$ . Show that  $P_1$  and  $P_2$  are nearest each other when  $t = 1/2$ .

15. Show that if  $s = 16t^2$ , then  $v = 8\sqrt{s}$ . What is the rate of change of  $v$  with respect to  $s$  when  $s = 8$ ? when  $s = 20$ ? See §24, Ex. 2.

16. Let  $y = f(x)$  be a function whose rate of change  $f'(x)$  is continuous and increases (or decreases) throughout the interval  $(x_1, x_1 + \Delta x)$ , Fig. 11. Show by §18, 2 that there is a point  $x'$  in  $(x_1, x_1 + \Delta x)$  such that  $f'(x') = \Delta y / \Delta x$  and therefore  $\Delta y = f'(x')\Delta x$ .

### III. DERIVATIVES OF ALGEBRAIC FUNCTIONS

**37. Derivative of a product.** 1. Let  $y = uv$ , where  $u$  and  $v$  denote functions of  $x$  which have finite derivatives for the values of  $x$  under consideration. Give  $x$  the increment  $\Delta x$ . Then  $u, v, y$  receive increments  $\Delta u, \Delta v, \Delta y$  such that

$$y = uv \quad \text{becomes} \quad y + \Delta y = (u + \Delta u)(v + \Delta v)$$

Subtract the first equation from the second and divide by  $\Delta x$ .

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x}(v + \Delta v) + u \frac{\Delta v}{\Delta x}$$

Let  $\Delta x \rightarrow 0$ . Then  $v + \Delta v \rightarrow v$ , § 25, 2., and, by § 7, we have

$$\frac{d}{dx}(uv) = v \frac{du}{dx} + u \frac{dv}{dx} \quad (1)$$

2. Let  $y = uvw$ . By (1), we have

$$\begin{aligned} \frac{d}{dx}(uvw) &= \frac{du}{dx} \cdot vw + \frac{d}{dx}(vw) \cdot u \\ &= \frac{du}{dx} vw + \frac{dv}{dx} wu + \frac{dw}{dx} uv \end{aligned} \quad (2)$$

A repetition of this process leads to the rule:

*To differentiate the product of two or more functions, multiply the derivative of each function by the product of the other functions and add the products thus obtained.*

**38. Derivative of a quotient.** Let  $y = u/v$ , where  $v \neq 0$ . We have

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}$$

$$\therefore \frac{\Delta y}{\Delta x} = \left[ v \frac{\Delta u}{\Delta x} - u \frac{\Delta v}{\Delta x} \right] / v(v + \Delta v)$$

Let  $\Delta x \rightarrow 0$ . By § 7, we have

$$\frac{d}{dx} \frac{u}{v} = \left[ v \frac{du}{dx} - u \frac{dv}{dx} \right] / v^2 \quad (3)$$

*The derivative of a fraction is denominator times derivative of numerator, minus numerator times derivative of denominator, all divided by square of denominator.*

EXAMPLE 1. 
$$\begin{aligned} \frac{d}{dx} (x-a)(x-b)(x-c) &= \frac{d}{dx} (x-a) \cdot (x-b)(x-c) \\ &+ \frac{d}{dx} (x-b) \cdot (x-c)(x-a) + \frac{d}{dx} (x-c) \cdot (x-a)(x-b) \\ &= (x-b)(x-c) + (x-c)(x-a) + (x-a)(x-b). \end{aligned}$$

EXAMPLE 2. 
$$\frac{d}{dx} \frac{x^2}{3x+5} = \frac{(3x+5)2x - x^2 \cdot 3}{(3x+5)^2} = \frac{3x^2 + 10x}{(3x+5)^2}$$

EXAMPLE 3. Differentiate each of the following with respect to  $x$ .

- |                               |                                   |                        |
|-------------------------------|-----------------------------------|------------------------|
| 1. $(ax+b)(cx+d)$             | 2. $(x^2+a^2)(x^2+b^2)$           | 3. $(x^2+c^2)/x^2$     |
| 4. $\frac{2x^2-8x+5}{x^3-10}$ | 5. $\frac{(x-1)(2-3x)}{x^2(3-x)}$ | 6. $\frac{ax+b}{cx+d}$ |
| 7. $(x^2-3x+4)(x^2+2x-1)$     | 8. $(2x+3)(x-4)(5-2x)$            |                        |

39. **Derivative of  $x^n$ .** It is true for *all rational values of  $n$*  that

$$\frac{d}{dx} x^n = nx^{n-1} \quad (1)$$

1. Let  $y = x^{p/q}$ , where  $p, q$  are positive integers. We have

$$\Delta y = (x + \Delta x)^{p/q} - x^{p/q}$$

Set  $x^{1/q} = z$  and  $(x + \Delta x)^{1/q} = z'$

Then  $x = z^q$  and  $x + \Delta x = z'^q \quad \therefore \Delta x = z'^q - z^q$

Also  $x^{p/q} = z^p$  and  $(x + \Delta x)^{p/q} = z'^p \quad \therefore \Delta y = z'^p - z^p$

Hence 
$$\frac{\Delta y}{\Delta x} = \frac{z'^p - z^p}{z'^q - z^q} = \frac{z'^{p-1} + z'^{p-2}z + \dots + z^{p-1}}{z'^{q-1} + z'^{q-2}z + \dots + z^{q-1}}$$

When  $\Delta x \rightarrow 0$ , each of the  $p$  terms in the numerator  $\rightarrow z^{p-1}$  and each of the  $q$  terms in the denominator  $\rightarrow z^{q-1}$ , § 8. Hence

$$\frac{d}{dx} x^{p/q} = \frac{pz^{p-1}}{qz^{q-1}} = \frac{p}{q} z^{p-q} = \frac{p}{q} (x^{1/q})^{p-q} = \frac{p}{q} x^{p/q-1} \quad (a)$$



2. Let  $s$  denote any positive rational number. By (a) and § 38 (3)

$$\frac{d}{dx} x^{-s} = \frac{d}{dx} \frac{1}{x^s} = -\frac{s x^{s-1}}{x^{2s}} = (-s) x^{-s-1} \quad (b)$$

Both (a) and (b) are of the form (1), which proves the theorem.

EXAMPLE 1.  $\frac{d}{dx} x^{1/3} = \frac{1}{3} x^{-2/3}, \quad \frac{d}{dx} \sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}},$   
 $\frac{d}{dx} \frac{1}{x} = \frac{d}{dx} x^{-1} = -x^{-2}, \quad \frac{d}{dx} \frac{x^2 + 1}{x^{1/2}} = \frac{d}{dx} (x^{3/2} + x^{-1/2}) = \frac{3}{2} x^{1/2} - \frac{1}{2} x^{-3/2}.$

EXAMPLE 2. Differentiate the following:

$$\sqrt[3]{x^5}, \quad x^{-7}, \quad 5/x^4, \quad 2/5 x, \quad (2x^2 - 3x^{1/2} + 1)/2x.$$

**40. Derivative of a function of a function.** 1. Let  $y$  be a function of a variable  $u$  which is itself a function of  $x$ : that is, let

$$y = f(u) \quad \text{where} \quad u = \phi(x)$$

and suppose that, for all values of  $x$  under consideration and the corresponding values of  $u$ , the derivatives  $D_u y = f'(u)$ ,  $D_x u = \phi'(x)$  exist and are finite. Then

$$D_x y = D_u y D_x u = f'(u) \phi'(x) \quad (1)$$

*The derivative of  $y$  with respect to  $x$  is the derivative of  $y$  with respect to  $u$  times the derivative of  $u$  with respect to  $x$ .*

For give  $x$  the increment  $\Delta x$  and let  $\Delta u$  and  $\Delta y$  be the corresponding increments of  $u = \phi(x)$  and  $y = f(u)$ . We have

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \frac{\Delta u}{\Delta x}$$

Let  $\Delta x \rightarrow 0$ . Then, by hypothesis,  $\Delta u/\Delta x \rightarrow$  the finite limit  $\phi'(x)$ ; therefore also  $\Delta u \rightarrow 0$  and  $\Delta y/\Delta u \rightarrow f'(u)$ . Hence<sup>1</sup>  $\Delta y/\Delta x \rightarrow f'(u) \phi'(x)$ .

<sup>1</sup> It is not permissible to express  $\Delta y/\Delta x$  in the form  $[\Delta y/\Delta u][\Delta u/\Delta x]$  when  $\Delta u$  is 0. Hence the proof given above does not suffice for a function  $u = \phi(x)$  and a value of  $x$  such that  $\Delta u$  is 0 for certain values of  $\Delta x$  as small as we please but  $\neq 0$ . But it can be shown that in such a case both  $\phi'(x)$  and  $D_x f[\phi(x)]$  are 0 at the point  $x$  in question, so that the formula (1) is true at that point. An example is  $u = x^2 \sin(1/x)$  at  $x = 0$ .

2. If we multiply both members of (1) by  $dx$  we get

$$D_x y \, dx = f'(u) \phi'(x) dx$$

But, by § 34 (2),  $D_x y \, dx = dy$  and  $\phi'(x) dx = du$ . Hence

$$dy = f'(u) du \quad (2)$$

By (2), we can replace  $f'(u)$  in (1) by  $dy/du$  just as, by § 34 (3), we can replace  $D_x y$  by  $dy/dx$  and  $\phi'(x)$  by  $du/dx$ . If this be done, (1) becomes the easily remembered identity

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad (3)$$

3. Observe that (2) shows that if  $y = f(u)$ , then  $dy$  may be expressed in terms of  $du$  in the same manner, § 34 (2), as if  $u$  were the independent variable.

EXAMPLE. If  $y = u^3$  and  $u = x^2 + x$ , find  $dy/dx$ .

$$\frac{dy}{dx} = \frac{d}{du} u^3 \cdot \frac{d}{dx} (x^2 + x) = 3 u^2 (2x + 1) = 3(x^2 + x)^2 (2x + 1)$$

The most frequent application of (1) or (3) is in differentiating a function given in the form  $f(u)$  where  $u$  denotes some expression in  $x$ . Thus, § 39 (1),

$$\frac{d}{dx} u^n = \frac{du^n}{du} \frac{du}{dx} = n u^{n-1} \frac{du}{dx} \quad (4)$$

EXAMPLES.  $\frac{d}{dx} (3x - 5)^4 = \frac{d(3x - 5)^4}{d(3x - 5)} \frac{d(3x - 5)}{dx} = 4(3x - 5)^3 \cdot 3.$

$$\frac{d}{dx} \left( \frac{1-x}{1+x} \right)^3 = 3 \left( \frac{1-x}{1+x} \right)^2 \cdot \frac{d}{dx} \left( \frac{1-x}{1+x} \right) = 3 \left( \frac{1-x}{1+x} \right)^2 \cdot \frac{-2}{(1+x)^2}.$$

$$\frac{d}{dx} \frac{a}{(bx+c)^2} = a \frac{d}{dx} (bx+c)^{-2} = -2a(bx+c)^{-3} b = -\frac{2ab}{(bx+c)^3}$$

$$\frac{d}{dx} \sqrt{1-x^2} = \frac{d}{dx} (1-x^2)^{1/2} = \frac{1}{2} (1-x^2)^{-1/2} (-2x) = -\frac{x}{\sqrt{1-x^2}}$$

### EXERCISE III

Differentiate each of the following with respect to the variable of which it is a function. If the expression can be more easily differentiated in another form, first reduce it to that form.

1.  $x^3(5 - 3x)^4$       2.  $(1 + x)^3(1 - 2x)^4$       3.  $(1 + x^2)^3(1 - 2x^2)^6$   
 4.  $\frac{1 + 2t}{3 - t}$       5.  $\left(\frac{1 + 2t}{3 - t}\right)^6$       6.  $\frac{(2x^2 - x)^2}{x^4 + 2}$   
 7.  $x(1 - x)^2(1 + x)^3$       8.  $\sqrt[3]{1 + 2x}$       9.  $\sqrt{bx + c}$   
 10.  $\sqrt[3]{x(1 + x)}$       11.  $\sqrt{1 + 2t} \cdot \sqrt{1 - 3t}$       12.  $\sqrt{a^2 - x^2}$   
 13.  $\frac{z}{\sqrt{z^2 + a^2}}$       14.  $\frac{1}{\sqrt{x + \sqrt{x + 2}}}$       15.  $\left(\frac{1 - x^2}{1 + x^2}\right)^{1/2}$   
 16.  $\frac{\sqrt{ax^2 + b}}{\sqrt{cx^2 + d}}$       17.  $\left(\frac{y^{3/2}}{3} + \frac{y^{5/2}}{5}\right)^{1/2}$       18.  $\frac{x^{2/3} + \sqrt[3]{x} - 2x^{-1}}{x}$   
 19.  $\frac{(1 + 2x)^2}{(1 + 3x)^4}$ ; also reduced to the form  $(1 + 2x)^2(1 + 3x)^{-4}$ .

20.  $x\sqrt{x + 5} + (x + 5)^{3/2}$       21.  $(x + 2)\sqrt{x^2 + 3x + 2}$   
 22. Prove: If  $y = u^{1/2}$  and  $u = a^2 - x^2$ , then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = -\frac{x}{\sqrt{a^2 - x^2}}$$

23. Prove: If  $y = u^{1/2}$ ,  $u = \frac{1 - v}{1 + v}$ ,  $v = x^2$ , then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dv} \frac{dv}{dx} = \frac{-2x}{(1 + x^2)\sqrt{1 - x^4}}$$

24. If  $x$  is increasing at the rate of 10 ft./sec., at what rate with respect to the time  $t$  is the diagonal of the rectangle whose sides are  $x$  and  $x + 4$  increasing when  $x = 12$  feet?

**41. Derivatives of higher orders.** The derivative of the derivative of  $y = f(x)$  is called its *second derivative* and is denoted by  $D^2_x y$  or  $f''(x)$ . Similarly the derivative of the second derivative is called the *third derivative*:  $D^3_x y = f'''(x)$ ; and so on.

Thus, if  
 then

$$\begin{aligned} y &= 2x^3 - 3x^2 + 5x - 7 = f(x) \\ D_x y &= 6x^2 - 6x + 5 = f'(x) \\ D^2_x y &= 12x - 6 = f''(x) \\ D^3_x y &= 12 = f'''(x) \end{aligned}$$

and  $D^4_x y$  and all subsequent derivatives are 0.

The differential of  $dy = f'(x)dx$  is called the *second differential* of  $y$  and is denoted by  $d^2y$ ; that is, § 34,

$$d^2y = D_x(dy)dx = D_x[f'(x)dx]dx$$

But since  $dx = \Delta x$  is independent of  $x$  and therefore  $D_x(dx)$  is 0, § 26, (2), we have  $D_x[f'(x)dx] = f''(x)dx$ . Hence

$$d^2y = [f''(x)dx]dx = f''(x)(dx)^2 \quad \text{and} \quad \frac{d^2y}{dx^2} = f''(x) \quad (1)$$

where  $dx^2$  stands for  $(dx)^2$ . Similarly, if  $d^3y = D_x(d^2y)dx$ , and so on, we get

$$\frac{d^3y}{dx^3} = D^3_x y, \quad \frac{d^4y}{dx^4} = D^4_x y, \quad \dots, \quad \frac{d^ny}{dx^n} = D^n_x y \quad (2)$$

These differential symbols for the derivatives of higher order are the ones most frequently used.

EXAMPLE 1. Find the 1st, 2nd,  $\dots$   $n$ th derivatives of

$$y = 1/(x + 2) = (x + 2)^{-1}.$$

$$\frac{dy}{dx} = (-1)(x + 2)^{-2}, \quad \frac{d^2y}{dx^2} = (-1)(-2)(x + 2)^{-3}, \dots,$$

$$\frac{d^ny}{dx^n} = (-1)^n n! (x + 2)^{-(n+1)} = \frac{(-1)^n n!}{(x + 2)^{n+1}}$$

EXAMPLE 2. Find the successive derivatives of  $x^4 - 2x^3 + 5x^2 - x$ .

EXAMPLE 3. Find the second derivative of  $\sqrt{1 + x^2}$ .

EXAMPLE 4. If  $y = 1/(1 - x)$ , find  $d^3y/dx^3$ ,  $d^5y/dx^5$ ,  $d^ny/dx^n$ .

EXAMPLE 5. Find the  $n$ th derivative of

$$(a) \ 2/(3x + 4), \quad (b) \ 1/(x + 3)^3, \quad (c) \ (1 + x)^{1/2}.$$

EXAMPLE 6. Show that, if the successive derivatives of  $u, v$  are  $u_1, u_2, \dots; v_1, v_2, \dots$ ,

$$1. \ D^2_x uv = u_2v + 2u_1v_1 + uv_2$$

$$2. \ D^3_x uv = u_3v + 3u_2v_1 + 3u_1v_2 + uv_3$$

and so on, the coefficients being those in the expansion of  $(a + b)^n$  by the Binomial Theorem.<sup>1</sup>

EXAMPLE 7. Find 2nd and 3rd derivatives of  $x(1 - x)^{1/2}$  by setting  $u = (1 - x)^{1/2}$ ,  $v = x$  in Ex. 6, 1., 2.

**42. Differentiation of implicit algebraic functions.** 1. Let  $f(x, y)$  denote a polynomial in  $x, y$ , that is, a sum of terms of

<sup>1</sup> This formula for the  $n$ th derivative of the product of two functions is called Leibnitz's theorem.

the type  $cx^my^n$ , where  $m, n$  are positive integers or 0; then  $f(x, y) = 0$  is called an *algebraic equation* in  $x, y$ .

Suppose that  $f(x, y) = 0$  can be solved for  $y$  in terms of  $x$ , and let  $y = \phi(x)$  denote one of the solutions; the function  $y = \phi(x)$  is called an *algebraic function*, and it is said to be *defined implicitly* by  $f(x, y) = 0$ . It can be proved that such functions have derivatives.

The graph of  $f(x, y) = 0$  consists of the graphs of all the functions  $y = \phi(x)$ .

EXAMPLE 1. The equation  $y^2 - x^3 = 0$  has the solutions  $y = x^{3/2}$  and  $y = -x^{3/2}$ . Hence both  $y = x^{3/2}$  and  $y = -x^{3/2}$  are algebraic functions defined implicitly by  $y^2 - x^3 = 0$ . Both have derivatives. The graphs of  $y = x^{3/2}$  and  $y = -x^{3/2}$  are  $OA$  and  $OB$ . Together they form the graph  $AOB$  of  $y^2 - x^3 = 0$ .

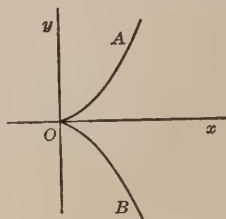


FIG. 12.

2. Let  $D_x f(x, y)$  denote the result of differentiating  $f(x, y)$  term by term, regarding  $y$  as a function of  $x$ .

EXAMPLE 2. Thus, if

$$f(x, y) = y^2 + 2xy + 3x^2 + 4y - 6x + 11$$

then

$$\begin{aligned} D_x f(x, y) &= 2y \frac{dy}{dx} + 2\left(y + x \frac{dy}{dx}\right) + 6x + 4 \frac{dy}{dx} - 6 \\ &= (2y + 6x - 6) + (2y + 2x + 4) \frac{dy}{dx} \end{aligned}$$

If  $f(x, y) = 0$  is satisfied by  $y = \phi(x)$ , then  $D_x f(x, y) = 0$  is satisfied by  $y = \phi(x)$ ,  $dy/dx = \phi'(x)$ . For  $f[x, \phi(x)] \equiv 0$  and therefore  $D_x f[x, \phi(x)] \equiv 0$ . Hence

If  $y$  denote a function of  $x$  defined by  $f(x, y) = 0$ , then corresponding values of  $x, y, dy/dx$  satisfy the equation  $D_x f(x, y) = 0$ .

3. As Ex. 2 shows,  $D_x f(x, y) = 0$  can be reduced to the form  $M + N(dy/dx) = 0$ , where  $M, N$  involve  $x, y$  only. It therefore gives the slope  $dy/dx$  at any point  $P$  of the graph  $C$  of  $f(x, y) = 0$  where  $N \neq 0$ . For this  $dy/dx$  is the slope

at  $P$  of the graph  $C_1$  of one of the functions  $y = \phi(x)$ , and  $C_1$  is a part of  $C$ .

EXAMPLE 3. Thus, differentiating  $y^2 - x^3 = 0$  gives  $2y \frac{dy}{dx} - 3x^2 = 0$ .

Hence at the point  $(1, 1)$  we have  $2 \frac{dy}{dx} - 3 = 0 \quad \therefore \frac{dy}{dx} = \frac{3}{2}$ .

4. As Ex. 2 also shows,  $M$  may be got by differentiating  $f(x, y)$  with respect to  $x$ , treating  $y$  as a constant.<sup>1</sup> It is therefore called the *partial derivative of  $f(x, y)$  with respect to  $x$*  and is denoted by  $\partial f / \partial x$ . Similarly  $N = \partial f / \partial y$ . Hence the equation  $D_x f(x, y) = 0$  may be written

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0 \quad (1)$$

and gives the slope  $dy/dx$  of  $C$  at any point  $P$  where  $\partial f / \partial y \neq 0$ .

It will be shown later that at a point  $P$  where  $\partial f / \partial y = 0$  and  $\partial f / \partial x \neq 0$ , the slope of  $C$  is infinite.

A point  $P$  where  $\partial f / \partial x = 0$  and  $\partial f / \partial y = 0$  is called a *singular point* of  $C$ . We shall find that at such a point  $C$  may have one tangent, more than one, or none.<sup>2</sup> The point  $O$  in Fig. 12 is a singular point of the curve  $y^2 - x^3 = 0$ .

5. If  $u$  be given as a function of  $x$  by two equations of the form  $u = F(x, y)$ ,  $f(x, y) = 0$ , and it be known that it has

<sup>1</sup> That this is true for any  $f(x, y) = \sum c_{mn} x^m y^n$  follows from the fact that

$$\frac{d}{dx} (x^m y^n) = m x^{m-1} y^n + x^m n y^{n-1} \frac{dy}{dx} = \frac{\partial}{\partial x} (x^m y^n) + \frac{\partial}{\partial y} (x^m y^n) \frac{dy}{dx}$$

<sup>2</sup> In deriving (1) we have supposed that  $P$  belongs to but one of the solutions  $y = \phi(x)$  of  $f(x, y) = 0$  and that  $dy/dx = \phi'(x)$  is finite at  $P$ . This is true when  $\partial f / \partial y \neq 0$  at  $P$ . But when  $\partial f / \partial y = 0$ ,  $P$  belongs to more than one  $y = \phi(x)$ ; the slopes of these  $y = \phi(x)$ 's at  $P$  are  $\infty$  when  $\partial f / \partial x \neq 0$ , but they may be finite and distinct, equal, or imaginary when  $\partial f / \partial x = 0$ . These several possibilities are illustrated at the point  $O$  on the graphs of  $y^2 - x = 0$ ,  $y^2 - x^2 = 0$ ,  $y^2 - x^3 = 0$ ,  $y^2 + x^2 = 0$ .

It may be added that, generally speaking, when the degree of  $f(x, y)$  in  $y$  exceeds 4 the functions  $y = \phi(x)$  cannot be found by solving  $f(x, y) = 0$  algebraically — that is, by means of radicals — but in a sense and by methods which will be explained later.

an  $x$ -derivative when  $y$  has one, the method of implicit differentiation enables us to find  $du/dx$  in terms of  $x, y$ .

EXAMPLE 4. Given  $u = (x^2 + y^2)^{1/2}$ , and  $xy - 2 = 0$ , find  $du/dx$ .

$$\text{Diff'g} \quad u^2 = x^2 + y^2 \quad \text{gives} \quad u \frac{du}{dx} = x + y \frac{dy}{dx}$$

$$\text{Diff'g} \quad xy - 2 = 0 \quad \text{gives} \quad y + x \frac{dy}{dx} = 0$$

$$\text{Eliminating } dy/dx \text{ and solving for } du/dx, \quad \frac{du}{dx} = \frac{x^2 - y^2}{x(x^2 + y^2)^{1/2}}$$

EXAMPLE 5. If  $u = y/x$  and  $y^2 - 3xy + 2x^2 - 4y = 0$ , find  $du/dx$  in terms of  $x, y$ .

**43. Tangents.** In the equation  $y - y_1 = \frac{dy_1}{dx_1} (x - x_1)$

replace  $\frac{dy_1}{dx_1}$  by  $-\frac{\partial f}{\partial x_1} / \frac{\partial f}{\partial y_1}$ , § 42 (1), and simplify. We obtain

$$(x - x_1) \frac{\partial f}{\partial x_1} + (y - y_1) \frac{\partial f}{\partial y_1} = 0 \quad (1)$$

EXAMPLE 1. After showing that the point  $(1, 2)$  is on the curve  $y^2 - 2xy + 2x^2 - 2x = 0$ , find the equation of the tangent at this point.

$$\text{At } (1, 2) \quad \frac{\partial f}{\partial x} = -2y + 4x - 2 = -2 \quad \frac{\partial f}{\partial y} = 2y - 2x = 2$$

Hence the tangent is  $-2(x - 1) + 2(y - 2) = 0$ , or  $x - y + 1 = 0$ .

EXAMPLE 2. At what points of the circle  $x^2 + y^2 = 5$  is the slope  $-2$ ?

$$\text{Diff'g} \quad x^2 + y^2 = 5 \text{ and simplifying gives} \quad x + y \frac{dy}{dx} = 0$$

At the required points  $dy/dx = -2$ . Hence these points satisfy  $x - 2y = 0$  and  $x^2 + y^2 = 5$  and therefore are  $(2, 1)$  and  $(-2, -1)$ .

### EXERCISE IV

1. Find the tangents and normals to the following curves:

1.  $x^3 + x^2y + y^3 = 9$  at  $(-1, 2)$
2.  $y^2 + 3xy + 2x - 3y = 0$  at  $(0, 0)$
3.  $2x^2y - 3y^2 - 2x + 7 = 0$  at  $(2, 3)$
4.  $x^2 + xy + y^2 - 1 = 0$  at  $(0, 1)$



2. Find the tangents to  $3x^2 - 2xy + y^2 - 3y = 0$  at the points whose abscissa is 2.

3. Find the points of  $2x^2 + y^2 - 4x = 0$  where the tangents are parallel to  $x + y = 0$ .

4. Find the points of  $5x^2 - 2xy + 2y^2 = 45$  where the slope is 0; also the points where the slope is  $\infty$ . What are the equations of the tangents at these points?

5. Show that the hyperbolas  $y^2 - x^2 = c$  and  $xy = c'$  cut at right angles.

6. Show that the tangent at the point  $(x_1, y_1)$  of the hyperbola  $2xy = a$  is  $xy_1 + yx_1 = a$ , and that the area of the triangle bounded by  $Ox$ ,  $Oy$  and the tangent is  $a$ .

7. A point  $P$  is moving on  $y^2 - 2xy - 3x^2 = 5$ . As  $P$  passes through  $(-1, 2)$ , what are the rates of change with respect to  $x$  of  $y$ , of  $OP$ , and of the slope at  $P$ ?

**44. Time rates.** If two variables  $x, y$  are connected by an equation of the form  $y = F(x)$ , then, § 40, their time rates of change  $dx/dt, dy/dt$  are connected by the equation  $dy/dt = F'(x)dx/dt$ . Also

*If  $x, y$  are connected by an algebraic equation  $f(x, y) = 0$ , then  $x, y, dx/dt, dy/dt$  are connected by the equation got by differentiating  $f(x, y) = 0$  with respect to  $t$ , regarding  $x, y$  as functions of  $t$ .*

For since  $y$  is a function of  $x$ , and  $x$  of  $t$ , we have, §§ 40, 42,

$$\begin{aligned}\frac{d}{dt}f(x, y) &= \frac{d}{dx}f(x, y)\frac{dx}{dt} \\ &= \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right)\frac{dx}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.\end{aligned}$$

But  $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$ . Hence  $\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$ .

In applying this theorem to problems, care must be taken not to substitute given values of  $x$  or  $y$  in  $f(x, y) = 0$  before differentiating.

**EXAMPLE 1.** Water is running into a conical receiver 8 ft. deep and 8 ft. across at the top at the rate 4 cu. ft./min. How fast is the surface rising when the depth is 5 ft.?

The volume of the water is  $V = \frac{1}{3} \pi r^2 h$

But  $r = h/2$ . Hence in terms of  $h$   $V = \frac{1}{12} \pi h^3$

Differentiating with respect to  $t$   $\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$ .

Subst'g  $\frac{dV}{dt} = 4$ ,  $h = 5$ ,

we find  $\frac{dh}{dt} = .204$  ft./min..

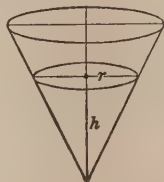


FIG. 13.

**EXAMPLE 2.** The ends of  $PQ$ , 13 in. long, move  $P$  on  $Ox$ ,  $Q$  on  $Oy$ . When  $OP = 5$  in.,  $OP$  is increasing at the rate 3 in./sec.; at what rates are  $OQ$  and  $OPQ$  then changing?

Let  $x = OP$ ,  $y = OQ$ ,  $u = OPQ$ . Then

$$(a) \quad x^2 + y^2 = 169 \quad (1)$$

$$\therefore x \frac{dx}{dt} + y \frac{dy}{dt} = 0 \quad (2)$$

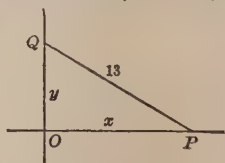


FIG. 14.

When  $x = 5$ , (1) gives  $y = 12$ . Also  $\frac{dx}{dt} = 3$   $\therefore$ , by (2),  $\frac{dy}{dt} = -\frac{5}{4}$

$$(b) \quad u = \frac{1}{2} xy \quad \therefore \frac{du}{dt} = \frac{1}{2} \left[ y \frac{dx}{dt} + x \frac{dy}{dt} \right] = 14 \frac{1}{4}.$$

### EXERCISE V

1. A man is approaching a flag pole 100 ft. high at the rate 3 ft./sec. At what rate is his distance from the top of the pole changing when he is 50 ft. from its foot?

2.  $P$  and  $Q$  move on  $Ox$  and  $Oy$  in such a manner that  $OPQ$  is always 10 sq. in. Find  $OP$ ,  $OQ$  at the time when  $OP$  increases twice as fast as  $OQ$  decreases.

3. Sand poured on the ground is forming a conical pile whose altitude is always  $2/3$  the radius of its base. How fast is the sand falling if when the height of the pile is 3 ft. it is increasing at the rate 2 in./min.?

4. A baseball diamond is a square 90 ft. to a side. A batter is running to first base at the rate 25 ft./sec. When he has run halfway, at what rate is his distance from second base changing?

5. Two ships  $A$  and  $B$  are steaming eastward, at the rates 14 and 16 m./h. respectively.  $A$  starts at 1 P.M. and  $B$  at 3 P.M.,  $B$ 's starting point being 5 m. due south of  $A$ 's. Are they approaching or separating at 6 P.M., and how fast?

6. A light  $A$  is on the ground 40 ft. from the wall  $BC$  of a building. A man 6 ft. tall is walking toward  $BC$  at the rate 4 ft./sec. At what rate is his shadow moving down  $BC$  when he is 35 feet from  $A$ ?

7. A light  $A$  is at the top of a pole 60 ft. high. From a point on the ground 25 ft. from the base of the pole a ball  $B$  is tossed upward with an initial velocity of 30 ft./sec. At what rate is  $B$ 's shadow moving along the ground one second later?

8. A point  $P$  is moving on the line  $3x - 4y - 12 = 0$  with the constant speed 5 in./sec. How fast is  $OP$  changing when the  $x$  of  $P$  is  $-2$ ?

9. An aeroplane which is flying due west at an altitude of 1 m. and with the velocity 100 m./h. passes vertically over an automobile which is running due north 40 m./h. At what rate are the two separating 1 hour later?

10. Gas is escaping from a spherical balloon at the rate 1 cu. ft./min. How fast is the radius  $r$  of the balloon decreasing, and how fast is its surface shrinking, when  $r = 10$  ft.?

11. A ship is moving 600 ft./min. A man in its stern, and 20 ft. above the water, is pulling in, at the rate 200 ft./min., a rope attached to a row boat. When there is 52 ft. of rope out, at what rate is the boat running through the water?

12. A solution is filtering through a conical filter, 18 in. deep and 12 in. across at the top, into a cylindrical vessel whose diameter is 10 in. When the depth of the solution in the filter is 12 in., its level is falling at the rate 1 in./min. At what rate is its level in the cylinder then rising?

13. A light  $A$  is 20 ft. from a wall and 10 ft. above the center of a path which is at right angles to the wall. A man 6 ft. tall is walking on the path toward the wall at the rate 2 ft./sec. When he is 4 ft. from the wall, how fast is his shadow moving up it?

14. Two ships,  $A$  and  $B$ , leave the same port on the same day,  $A$  at noon and  $B$  at 2 P.M. If  $A$  runs 10 mi./h. on a course due north, and  $B$  runs 12 mi./h. on a course due east, how fast are they moving apart at 10 P.M.?

15. A point  $P$  is moving on the upper half of the circle  $x^2 + y^2 - 4x = 0$ . When the abscissa is 3 in., it is increasing at the rate 5 in./sec. At what rates are its ordinate and its distance from the origin then changing?

## IV. MAXIMA AND MINIMA

**45. Maximum and minimum values.** Let  $y = f(x)$  be a function which has a derivative, finite or infinite, for all values of  $x$  under consideration. If when  $x$ , increasing, passes through the value  $x_1$ ,  $y$  ceases to increase and begins to decrease, then  $f(x_1)$  is called a *maximum value* of  $y$  (as at  $A$  and  $C$ ); if  $y$

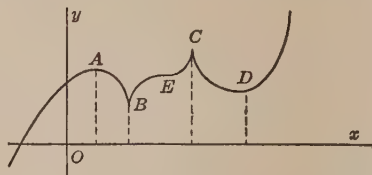


FIG. 15.

ceases to decrease and begins to increase, then  $f(x_1)$  is called a *minimum value* of  $y$  (as at  $B$  and  $D$ ).<sup>1</sup>

In both cases  $f'(x_1)$  is 0 or  $\infty$ ; for if  $f'(x_1)$  were  $+$ , or  $-$ ,  $y$  would continue to increase, or decrease, as  $x$  passed through the value  $x_1$ .

If as  $x$  passes through  $x_1$ ,  $f'(x)$  changes sign from  $+$  to  $-$ , then  $f(x_1)$  is a maximum; from  $-$  to  $+$ , a minimum. Hence the following rule:

*To find the maximum and minimum values of  $y = f(x)$ , determine all the real values  $x_1$  of  $x$  for which  $f'(x)$  is 0 or  $\infty$ . Then for each such  $x_1$  find whether or not  $f'(x)$  changes sign as  $x$  increases through  $x_1$ .*

*If  $\text{sgn } f'(x)$  changes  $+$  to  $-$ ,  $f(x_1)$  is a maximum value.*

*If  $\text{sgn } f'(x)$  changes  $-$  to  $+$ ,  $f(x_1)$  is a minimum value.*

*If  $\text{sgn } f'(x)$  does not change,  $f(x_1)$  is neither a maximum nor minimum.*

<sup>1</sup> Hence a maximum value  $f(x_1)$  may not be the greatest of all values of  $f(x)$ , but it is the greatest value in some interval extending to both sides of  $x_1$ ; similarly for a minimum value.

The points of the graph of  $y = f(x)$  at which  $y$  is a maximum or minimum and  $f'(x)$  is 0 are called the *turning points* of the curve. Thus  $A$  and  $D$  in Fig. 15.

**EXAMPLE 1.** Find the maximum and minimum values of the function  $y = x^3 - 6x^2 + 9x - 1$ . Also its graph.

Solving  $f'(x) = 3x^2 - 12x + 9 = 0$   
we get  $x = 1$  or  $3$ .

Hence  $f'(x) = 3(x - 1)(x - 3)$

As  $x$  increases through 1,  $\text{sgn } f'(x)$  changes from  $(-)(-) = +$  to  $(+)(-) = -$ ; hence  $f(1) = 3$  is a maximum value.

At  $x = 3$ ,  $\text{sgn } f'(x)$  changes  $-$  to  $+$ ; hence  $f(3) = -1$  is a minimum value. In the  $x$  intervals  $(-\infty, 1)$ ,  $(1, 3)$ ,  $(3, \infty)$ ,  $\text{sgn } f'(x)$  is  $+$ ,  $-$ ,  $+$ ; hence the graph in Fig. 16.

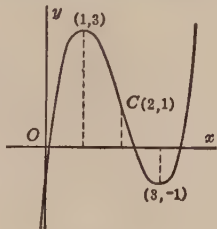


FIG. 16.

**EXAMPLE 2.** Investigate  $y = (x - 2)^{2/3} + 1$  for maximum and minimum values.

Here  $f'(x) = 2/3(x - 2)^{-1/3}$ , which at  $x = 2$  becomes  $\infty$  and changes sign  $-$  to  $+$ ; hence  $f(2) = 1$  is a minimum value. In the  $x$  interval  $(-\infty, 2)$ ,  $y$  decreases from  $\infty$  to 1; in  $(2, \infty)$ ,  $y$  increases from 1 to  $\infty$ .

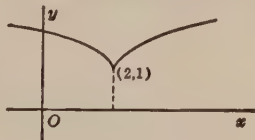


FIG. 17.

**EXAMPLE 3.** Find the maxima and minima of the following; also their graphs.

1.  $y = x^2 - 4x + 3$

2.  $y = 1 + 2x - x^2$

3.  $y = x^5$

4.  $y = x^3 + 2x^2 - 4x - 5$

5.  $y = 2 + 3x + 4x^2 - x^3$

6.  $y = \sqrt{2x - x^2}$

**46. Concavity. Points of inflection.** 1. Suppose that  $f'(x)$  has a derivative  $f''(x)$ , finite or infinite, at every point  $P$  of the curve  $y = f(x)$ ; and suppose the  $x$  of  $P$  to be increasing. By § 32,

When  $f''(x)$  is  $+$ , the slope  $f'(x)$  is increasing, hence the tangent at  $P$  is turning counter-clockwise and the part of the curve near  $P$  is *concave upward*. This is the case at any point but  $C$  on  $ABC$  in Fig. 18.

When  $f''(x)$  is  $-$ ,  $f'(x)$  is decreasing, hence the tangent at  $P$  is turning clockwise and the part of the curve near  $P$  is *concave downward*. Thus at any point but  $C$  on  $CDE$ .

At a point where  $f''(x)$  becomes 0 or  $\infty$  and *changes sign*, the sense in which the tangent is turning is reversed. Thus at  $C$ . To one side of  $C$  the curve is concave

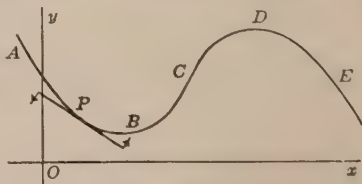


FIG. 18.

upward, to the other concave downward, so that the curve crosses the tangent. Such a point  $C$  is called a *point of inflection*.

2. Evidently a point where  $f'(x)$  is 0 and the curve is concave upward is a minimum point; concave downward, a maximum point. Hence

*If  $f'(x_1)$  is 0 and  $f''(x_1)$  is  $+$ , then  $f(x_1)$  is a minimum.*

*If  $f'(x_1)$  is 0 and  $f''(x_1)$  is  $-$ , then  $f(x_1)$  is a maximum.*

This test is less general but sometimes easier to apply than that of § 45.

EXAMPLE 1. In Ex. 1, § 45, we have  $f''(x) = 6(x - 2)$  which is 0,  $-$  to  $+$ , at  $x = 2$ . Hence the point of inflection  $C(2, 1)$ . At the maximum point  $B(1, 3)$  we have  $f''(1) = -6$  i.e. negative; at the minimum point  $D(3, 1)$ ,  $f''(3) = 6$  i.e. positive.

EXAMPLE 2. Find the points of inflection of  $y = x^3(x - 2)$ .  
 $f'(x) = 2x^2(2x - 3)$        $f''(x) = 12x(x - 1)$

1.  $f'(x)$  is 0 at  $x = 0$  without changing sign, and at  $x = 3/2$ ,  $-$  to  $+$ .

Hence the only turning point is the minimum point  $C(\frac{3}{2}, -\frac{27}{8})$ .

2.  $f''(x)$  is 0 at  $x = 0$ ,  $+$  to  $-$ , and at  $x = 1$ ,  $-$  to  $+$ . Hence the points of inflection  $O(0, 0)$  and  $B(1, -1)$ , where the slopes, found from  $f'(x)$ , are 0 and  $-2$ . Sgn  $f''(x)$  shows that the curve is concave upward from  $x = -\infty$  to  $O$ , then downward to  $B$ , then upward.

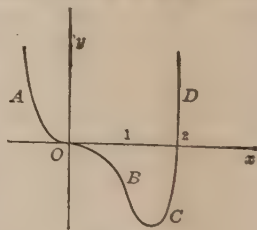


FIG. 19.



**EXAMPLE 3.** Find the turning points and points of inflection of the following; also their graphs.

1.  $y = 3x - x^3$

2.  $y = x^4$

3.  $y = x^{1/3}$

4.  $y = (x^2 - 1)^2$

5.  $y = 6x^2 - 4 - x^4$

6.  $y = x^3 + 6x$

7.  $2y = 2x^3 - 3x^2 - 12x + 6$

8.  $3y = x^3 - 3x^2 - 9x + 15$

9.  $y = x^4 - 2x^3 + 2x - 1$

**47. Greatest and least values.** 1. If  $f(x)$  is continuous in the interval  $(ab)$ , it has a greatest value  $M$  in  $(ab)$ , § 18. It may take the value  $M$  either at  $x = a$  or  $b$  or at some point  $x = x_1$  between  $a$  and  $b$ . In the latter case,  $M = f(x_1)$  is the greatest of the maximum values of  $f(x)$  in  $(ab)$ . Similarly for the least value  $m$  of  $f(x)$  in  $(ab)$ .

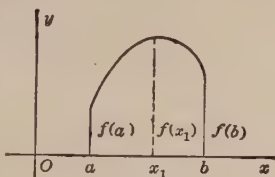


FIG. 20.

2. If  $f'(x)$  exists throughout  $(ab)$  and is 0 or  $\infty$  at one and but one point  $x = x_1$  between  $a$  and  $b$ , then  
 $f(x_1) = M$ , if  $f(x_1) > f(a), f(b)$ ;  $f(x_1) = m$ , if  $f(x_1) < f(a), f(b)$

**48. Problems.** Each of the following problems is concerned with some magnitude which varies continuously subject to certain restrictions, and its greatest or least value is sought. The given conditions enable one to express the numerical measure of the magnitude as a function of a single variable in some interval  $(ab)$ . We seek the greatest or least value of the function in this interval.

**EXAMPLE 1.** Find the largest box open at the top that can be made of a piece of tin  $a$  in. square by cutting out equal square pieces at the corners and bending upward the projecting portions which remain.

Let  $x$  be the height of the box; it is restricted to values between 0 and  $a/2$ .

Then if  $u$  denote the volume of the box,  $u = x(a - 2x)^2$

Hence, diff'g and simplifying,

$$\frac{du}{dx} = (a - 6x)(a - 2x),$$



At  $x = a/6$ ,  $du/dx$  vanishes,  $+$  to  $-$ ; and  $a/6$  is between 0 and  $a/2$ . Hence  $a/6$  is the height of the largest box. Its volume is  $2a^3/27$ . This also follows from § 47, 2.

At  $x = a/2$ ,  $du/dx$  also vanishes,  $-$  to  $+$ , but as this value of  $x$  is not between 0 and  $a/2$  it is to be discarded. It gives, of course, a minimum value of the unrestricted function  $x(a - 2x)^2$ , namely the value 0. The box is nonexistent for this value.

**EXAMPLE 2.** What points of the parabola  $y^2 = 4ax$  are nearest the point  $C(c, 0)$ ?

$$CP^2 = (x - c)^2 + y^2 = (x - c)^2 + 4ax \quad (x > 0)$$

Since  $CP$  is least when  $CP^2$  is least, we seek the least value of

$$u = (x - c)^2 + 4ax \quad (x > 0)$$

$\frac{du}{dx} = 2(x - c) + 4a$  vanishes,  $-$  to  $+$ ,  
at  $x = c - 2a$

Hence  $x = c - 2a$  is the abscissa of the two curve points nearest  $C$  when  $x = c - 2a > 0$ , i.e. when  $c > 2a$ . But when  $c \leq 2a$ , then  $du/dx = 2[x + (2a - c)]$  is always  $+$  for  $x > 0$   $\therefore CP$  takes its least value when  $x = 0$ , and  $O$  is the curve point nearest  $C$ .

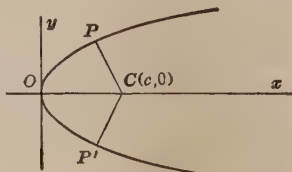


FIG. 21.

**EXAMPLE 3.** Find 1. the cylinder of greatest volume inscribed in a given right circular cone; 2. the cylinder of greatest curved surface; 3. the cylinder of greatest total surface.

1. Let  $r = BC$ ,  $a = BA$ ,  $x = BF$ ,  $y = BD$

$\therefore V = \pi x^2 y$ . We must express  $V$  in terms of  $x$  (or  $y$ ) only. But

$$\frac{FG}{FC} = \frac{BA}{BC} \quad \therefore \frac{y}{r - x} = \frac{a}{r} \quad \therefore y = \frac{a}{r}(r - x).$$

Hence  $V = \frac{\pi a}{r} x^2(r - x)$  ( $0 < x < r$ ), which will

be found to take its greatest value when  $x = (2/3)r$ .

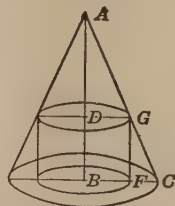


FIG. 22.

2. The area of the curved surface  $S_1 = 2\pi xy = (2\pi a/r)(rx - x^2)$  is greatest when  $x = r/2$ .

3. The total area  $S_2 = 2\pi[(a/r)(rx - x^2) + x^2]$  is greatest when  $x = ar/2(a - r)$ . But we must have  $ar/2(a - r) < r$   $\therefore r < a/2$ . If  $r \geq a/2$ , then as  $x$  increases from 0 to  $r$ ,  $S_2$  increases from 0 to  $2\pi r^2$ .

## EXERCISE VI

1. From a point on the hypotenuse of a right-angled triangle perpendiculars are drawn to the other two sides. When is the rectangle thus formed a maximum?

2. Prove that a square is the greatest rectangle of given perimeter; also that it is the greatest rectangle that can be inscribed in a circle.

3. The total area of a page is 96 sq. in., the combined widths of the margins at the top and bottom 3 in., at the sides 2 in. For what dimensions of the page is the printed area greatest?

4. What is the least number of sq. ft. of lumber that will form an open tank with a square base and a capacity of 108 cu. ft.?

5. For what height will the cost of a box with square base and 324 cu. ft. capacity be least, if the bottom costs 4 cts., the top 5 cts., and the sides 3 cts. per sq. in.?

6. The axes  $Ox$ ,  $Oy$  are met at  $P$ ,  $Q$  by a variable line through the point  $(2, 3)$ . Find the triangle  $OPQ$  of least area.

7. The sides  $AB$ ,  $AC$  of a triangle  $ABC$  are met by a parallel to  $BC$  in the points  $D$ ,  $E$ , and these points are joined to a fixed point  $F$  in  $BC$ . For what position of the line  $DE$  will the triangle  $DEF$  be a maximum?

8. Find the greatest rectangle that can be inscribed in the ellipse  $x^2 + 4y^2 = 4$ .

9. Find greatest rectangle that can be inscribed in the space bounded by  $y^2 = 8x$  and  $x = 4$ .

10. A ship  $A$  is sailing from E. to W., 8 m./h., and a ship  $B$  from N. to S., 7 m./h. At noon  $A$  is 10 m. east of  $B$ 's course and  $B$  is 14 m. north of  $A$ 's course. When will their distance apart be least?

11. A manufacturer can now ship a cargo of 100 tons at a profit of \$5 per ton. He estimates, however, that by waiting he can add 20 tons per week to the shipment, but that the profit on all that he ships will be reduced 25 cts. per ton per week. How long will it be to his advantage to wait?

12. A man is in a rowboat 3 m. from the nearest point of the shore and wishes to reach a point of the shore 6 m. from this nearest point as quickly as possible. He thinks that he can row 4 m./h. and move on shore 5 m./h. To what point of the shore should he row?

13. Find (1) the cylinder of greatest volume that can be inscribed in a sphere of radius  $a$ ; (2) the cylinder of greatest surface; (3) the cone of greatest volume.

14. The cone  $K$  has the altitude  $a$ , and  $r$  is the radius of its base  $B$ . Find the greatest cone  $K'$  which has the center of  $B$  for vertex and the circle in which  $K$  is cut by some plane parallel to  $B$  for its base.

15. The points  $A$  and  $B$  are to the same side of a given line. For what point  $P$  on the line is  $AP^2 + BP^2$  least? For what point is  $AP + BP$  least?

16. A line drawn from the corner  $A$  of the rectangle  $ABCD$  meets the side  $BC$  at  $E$  and  $DC$  produced at  $F$ . When is the sum of the triangles  $ABE$ ,  $FCE$  least?

17. A point  $A$  is at the distance  $h$  above the center  $C$  of a sphere of radius  $r$ . A cone is constructed having  $A$  for vertex, and for base the circle got by cutting the sphere by a plane perpendicular to  $CA$ . When is this cone largest?<sup>1</sup>

18. A sheet of paper  $ABCD$  has the width  $AB = a$ . A triangle  $BEF$  is formed by folding the sheet in such a manner that the corner  $B$  falls on the edge  $AD$ . When is the area of  $BEF$  least? When is the length of  $EF$  least?

19. What points of the ellipse  $x^2 + 4y^2 = 8$  are nearest the point  $(1, 0)$ ?

20. How high above the center  $C$  of a circular plot of grass 30 ft. across should an electric light  $A$  be placed to produce the greatest brightness at points  $P$  on the circumference, it being assumed that the brightness at  $P$  varies inversely as  $AP^2$  and directly as  $\sin CPA$ ?

21. If the cost of propelling a boat varies as the cube of the speed generated, what is the most economical speed against a 3 m./h. current?

22. Let  $A, B, C$  be three fixed points on a given straight line; find the point  $P$  the sum of the squares of whose distances from  $A, B, C$  is least.

23. If the perimeter of an isosceles triangle be given, show that the area is greatest when the triangle is equilateral.

24. Let  $A(a, 0)$  and  $B(-a, 0)$  be two fixed points on the  $x$ -axis, and  $P, Q$  two movable points,  $P$  on the lower half of the  $y$ -axis, and  $Q$  vertically below  $A$ . If  $BP + PA + AQ$  is constant, for what position of  $P$  is the midpoint of  $PQ$  as far as possible from the  $x$ -axis?

25. Let  $A$  be a fixed point on the  $x$ -axis, and  $P, Q$  variable points,  $P$  on the  $x$ - and  $Q$  on the  $y$ -axis, and such that  $AP = PQ$ . Prove that the area of the triangle  $OPQ$  is greatest when  $OP$  is one third of  $OA$ .

<sup>1</sup> The plane may be above  $C$  or below  $C$ .

26. If the cost of fuel for running a steamboat be proportional to the square of the speed generated and is \$8 an hour when a speed of 8 m./h. is generated, what is the most economical speed against a 5 m./h. current, the fixed charges being \$18 per hour?

49. **Implicit functions.** The maximum and minimum values of a function which is given by two equations of the form  $u = \phi(x, y)$ ,  $f(x, y) = 0$  may be found as follows:

EXAMPLE 1. Find the greatest rectangle that can be inscribed in a circle of radius  $a$ .

We have  $u = 4xy$  (1) where  $x^2 + y^2 = a^2$  (2)

Eq's (1), (2) define  $u$  as a function of  $x$ :  $u = 4x\sqrt{a^2 - x^2}$  (3)

But we may solve the problem without using (3); for differentiating (1) and (2), regarding  $y$  and  $u$  as functions of

$x$ , and setting  $\frac{du}{dx} = 0$ ,

$$y + x \frac{dy}{dx} = 0 \quad x + y \frac{dy}{dx} = 0$$

$$\text{Hence, eliminating } \frac{dy}{dx}, \quad y^2 - x^2 = 0$$

$$\therefore y = x \quad (4)$$

Evidently the rectangle thus obtained is a square with half side  $x = a/\sqrt{2}$ , and area

$u = 2a^2$ . That it is the greatest rectangle follows from § 47, 2. For  $x = a/\sqrt{2}$  is between the extreme values, 0 and  $a$ , of  $x$ , and is the only value of  $x$  between 0 and  $a$  for which  $du/dx$  is 0. And the value of  $u$  for  $x = a/\sqrt{2}$ , namely  $2a^2$ , is greater than those for  $x = 0$  and  $x = a$ .

EXAMPLE 2. Find when the surface of a cylindrical tank of given content will be least (1) if the tank be completely enclosed; (2) if it be open at the top.

EXAMPLE 3. For what dimensions will the least material be required to construct a conical tent of given content?

EXAMPLE 4. Through the point  $(a, b)$  a line is drawn which meets  $Ox$  at  $P$  and  $Oy$  at  $Q$ . What is the least length that  $PQ$  can have?

EXAMPLE 5. Find the dimensions of the cone of greatest curved surface that can be inscribed in a given sphere.

EXAMPLE 6. Show that the maximum and minimum values of  $y$ , when defined as a function of  $x$  by an equation of the form  $f(x, y) = 0$ , may be found by solving the equations  $f(x, y) = 0$ ,  $\partial f / \partial x = 0$ .

EXAMPLE 7. If  $x^4 - x^2y + y^3 = 0$ , find the maximum value of  $y$ .

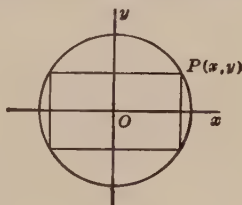


FIG. 23.

**50. Roots of algebraic equations.** 1. The real roots  $c$  of an algebraic equation  $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n = 0$  are pictured by the points where the graph of  $y = f(x)$  meets  $Ox$ . When we can solve the equation  $f'(x) = 0$ , we can readily make a rough determination of this graph as in §§ 45, 46. The curve thus got will at once indicate the whereabouts of the roots  $c$ ; it will also suggest what values of  $f(x)$  to compute in order to locate each root more precisely — by aid of the theorem (§ 18, 2):

*If  $f(a)$  and  $f(b)$  have opposite signs, then  $f(x) = 0$  has at least one root between  $a$  and  $b$ .*

**EXAMPLE 1.** Locate the roots of  $f(x) = x^3 - 3x + 1 = 0$ , using the graph of  $y = f(x)$ .  $f'(x) = 3x^2 - 3 = 0$  gives the turning points  $C(-1, 3)$ ,  $F(1, -1)$ . Since  $C$  and  $F$  are on opposite sides of  $Ox$ , there is a root  $E$  between  $-1$  and  $1$ ; and since  $f(0) = 1$  is  $+$ ,  $E$  is between  $0$  and  $1$ . Since  $f(-1)$  is  $+$  and  $f(-\infty)$  is  $-$ , there is a root  $B$  to the left of  $-1$ ; and since  $f(-2) = -1$  is  $-$ ,  $B$  is between  $-2$  and  $-1$ . Similarly, since  $f(1)$  is  $-$  and  $f(2) = 3$  is  $+$ , the third root  $G$  is between  $1$  and  $2$ .

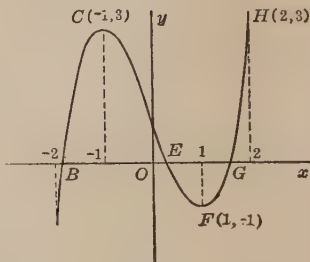


FIG. 24.

Moreover the line  $FH$  will cut the line segment  $1, 2$  in the ratio  $F1/2H = 1/3$ ; hence  $G$  is between  $1\frac{1}{4}$  and  $2$ .

2. If  $f(c)$  is 0, then  $x - c$  is a factor of  $f(x)$ ; and conversely. If  $f(x)$  contains  $(x - c)^r$ ,  $r \geq 1$ , and no higher power of  $x - c$ , as a factor, then  $c$  is called a *root of order  $r$* . We also call  $c$  a *simple root* when  $r = 1$ , a *multiple root* when  $r > 1$ . It is easy to show<sup>1</sup> that

*If  $c$  is a simple root of  $f(x) = 0$ , it is not a root of  $f'(x) = 0$ ; but if  $c$  is a root of order  $r > 1$  of  $f(x) = 0$ , it is a root of order  $r - 1$  of  $f'(x) = 0$ .*

<sup>1</sup> For if  $f(x) = (x - c)^r \phi(x)$ , then  $f'(x) = r(x - c)^{r-1} \phi(x) + (x - c)^r \phi'(x) = (x - c)^{r-1} [r\phi(x) + (x - c)\phi'(x)]$ .

Hence the graph of  $y = f(x)$  merely cuts  $Ox$  at  $x = c$  when  $r = 1$ , but touches  $Ox$  when  $r > 1$ , also crossing  $Ox$  when  $r$  is odd since in that case  $(x - c)^r$  changes sign at  $x = c$ .

EXAMPLE 2.  $f(x) = (x + 1)(x - 1)^2(x - 2)^3 = 0$  has the roots  $-1$ ,  $1$ ,  $2$ ; their orders are  $1$ ,  $2$ ,  $3$ .  $f'(x) = (x - 1)(x - 2)^2(6x^2 - 5x - 5)$  does not contain  $x + 1$  but does contain  $x - 1$  and  $(x - 2)^2$ . Draw a rough graph of  $y = f(x)$  showing that it crosses  $Ox$  at  $x = -1$ , touches at  $x = 1$ , touches and crosses at  $x = 2$ .

EXAMPLE 3. Show that if the degree  $n$  of  $f(x) = a_0x^n + \dots + a_n$  is odd, then  $f(-\infty)$  and  $f(\infty)$  have opposite signs and therefore that the sum of the orders of the real roots  $c$  of  $f(x) = 0$  is odd, being at least one. How is it when  $n$  is even?

EXAMPLE 4. Draw a typical graph of  $y = f(x)$  for the case  $n = 5$  and  $a_0$  positive, showing that the number of times it crosses  $Ox$  may be  $5$ ,  $3$ , or  $1$ .

EXAMPLE 5. Show that between two points where the curve  $y = f(x)$  meets  $Ox$  there is at least one turning point, and between two turning points at least one point of inflection.

EXAMPLE 6. Show that  $x^3 - 3px + q = 0$  has three real roots when  $q^2 < 4p^3$ .

EXAMPLE 7. Draw graphs of the following, locating the points where each curve meets  $Ox$ .

1.  $y = x^2(x - 2)^3$

2.  $2y = 2x^3 - 3x^2 - 12x + 6$

3.  $y = x^4 + 4x^3 - 8x^2 + 2$

**51. Graphs of rational fractional functions of  $x$ .** Every such function can be reduced to the form :

$$F(x) = \frac{f(x)}{\phi(x)} = \frac{a_0x^n + a_1x^{n-1} + \dots + a_n}{b_0x^p + b_1x^{p-1} + \dots + b_p} \quad (1)$$

where  $f(x)$  and  $\phi(x)$  have no common factor.

Evidently  $F(x)$  is  $0$  when  $f(x)$  is  $0$ ;  $F(x)$  is  $\infty$  when  $\phi(x)$  is  $0$ ; and for all other finite values of  $x$ ,  $f(x)$  is continuous and not  $0$ . Let  $|x| \rightarrow \infty$ . Then, § 11,

$$F(x) \rightarrow 0 \text{ when } n < p \qquad F(x) \rightarrow \frac{a_0}{b_0} \text{ when } n = p$$

$$|F(x)| \rightarrow \infty \text{ when } n > p$$



As the cases  $y = 1/x$ ,  $y = 1/x^2$ , p. 10, Ex. 1, illustrate, the graphs of fractional functions  $y = F(x)$  may have "asymptotes."

If as a point  $P$  moves out on an infinite curve branch  $C$  it tends to move along a definite line  $l$ , so that the distance  $DP \rightarrow 0$  and the slope at  $P \rightarrow$  the slope of  $l$ , then  $l$  is called an *asymptote* of  $C$ .

1. If  $x - c$  is a factor of  $\phi(x)$  in (1),  $c$  being real, then the line  $x - c = 0$  is an asymptote (like  $l'$ ) of the curve  $y = F(x)$ .

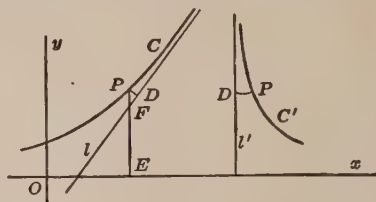


FIG. 25.

2. If  $n - p \geq 1$  in (1), we can reduce  $y = F(x)$  to the form  $y = mx + b + \psi(x)$ , where  $m$ ,  $b$ , one or both, may be 0, and  $\psi(x)$  denotes a *proper*<sup>1</sup> fraction, so that when  $|x| \rightarrow \infty$  then  $\psi(x) \rightarrow 0$  and also, as it is easy to show,  $\psi'(x) \rightarrow 0$ . Hence, by the following theorem,  $y = mx + b$  is an asymptote (like  $l$ ) of  $y = F(x)$ .

Thus  $y = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$ ; hence  $y = x + 1$  is an asymptote. So also is  $x - 1 = 0$ , by 1.

If  $F(x) = mx + b + \psi(x)$ , and  $\psi(x)$  and  $\psi'(x) \rightarrow 0$  when  $|x| \rightarrow \infty$ , then the line  $y = mx + b$  is an asymptote of the curve  $y = F(x)$ .

For let  $l$  in Fig. 25 be the line  $y = mx + b$  and  $C$  part of the curve  $y = F(x)$ . Then for  $x = OE$ , we have  $mx + b + \psi(x) = EP$ , and  $mx + b = EF$ .  $\therefore FP = \psi(x)$ .

Hence when  $\psi(x) \rightarrow 0$ , then  $FP \rightarrow 0$  and therefore also  $DP = FP \sin DFP \rightarrow 0$ .

Also  $F'(x) = m + \psi'(x) \rightarrow m$ , since  $\psi'(x) \rightarrow 0$ .

<sup>1</sup> A rational fraction in  $x$  is called a *proper fraction* when its numerator is of lower degree than its denominator.



Observe that for sufficiently great values of  $|x|$ ,  $C$  is above or below  $l$  according as  $\psi(x) = FP$  is positive or negative.<sup>1</sup>

EXAMPLE 1. Find the graph of  $y = (x^2 - x + 1)/(x - 1)$ .

1. Since  $y$  becomes infinite at  $x = 1$ ,  $-$  to  $+$ , the line  $x - 1 = 0$  ( $l_1$ ) is an asymptote to a branch  $DE$  below (at left) and to a branch  $FG$  above (at right).

2. By division,  $y = x + \frac{1}{x-1}$  and  $1/(x-1) \rightarrow 0$  when  $|x| \rightarrow \infty$ .

Hence the line  $y = x$  ( $l_2$ ) is an asymptote at  $x = -\infty$  to a branch  $AB$  (below  $l_2$  since  $1/(x-1)$  is  $-$  as  $x \rightarrow -\infty$ ) and at  $x = \infty$  to a branch  $KL$  above  $l_2$ .

3. We find  $F'(x) = x(x-2)/(x-1)^2$  which vanishes at  $x = 0$ ,  $+$  to  $-$ , and at  $x = 2$ ,  $-$  to  $+$ , and gives the turning points  $C(0, -1)$  and  $H(2, 3)$ . The asymptotes and the sign of  $F'(x)$  show that as  $x$  increases from  $-\infty$  to  $\infty$  the curve follows the course  $ABCDEFHKL$ .

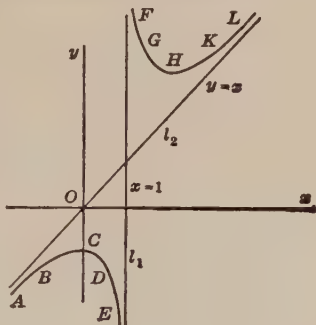


FIG. 26.

EXAMPLE 2. Find the graph of  $y = x/(x^2 - 5x + 4)$ .

$$F(x) = \frac{x}{(x-1)(x-4)} \quad \therefore F'(x) = -\frac{(x+2)(x-2)}{(x^2 - 5x + 4)^2}$$

The curve cuts  $Ox$  at  $x = 0$ . It has the vertical asymptotes  $x - 1 = 0$  and  $x - 4 = 0$ . Since

$F(x)$  is a proper fraction its asymptote  $l$  of Fig.

25 is  $y = 0$ .  $\text{Sgn } F(x)$  is  $-$

in the  $x$  interval  $(-\infty, 0)$ ,

$+$  in  $(0, 1)$ ,  $-$  in  $(1, 4)$ ,  $+$

in  $(4, \infty)$ . These facts

suffice to show that the

curve must be of the type

indicated in Fig. 27. To

determine it more precisely we find the turning points given by  $F'(x) = 0$ .

They are  $B(-2, -1/9)$  and  $E(2, -1)$ .

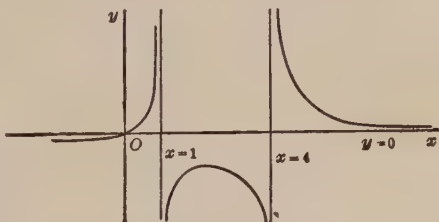


FIG. 27.

<sup>1</sup> Since  $FP = y - mx - b$ , we have the theorem: When the point  $P(x, y)$  is not on the line  $y - mx - b = 0$ , then  $y - mx - b$  represents the distance  $FP$  from the line to  $P$  measured parallel to  $Oy$ . It is  $+$  or  $-$  according as  $P$  is above or below the line.

## EXERCISE VII

Find the graphs of the following (1-12):

1.  $y = \frac{x-2}{x+1}$

2.  $y = \frac{2x^2-3x}{x-2}$

3.  $y = \frac{x^2-1}{x^2-4}$

4.  $y = \frac{x^2+x-2}{x^2-x-2}$

5.  $y = \frac{2x^2}{x^2-1}$

6.  $y = \frac{2x}{x^2+1}$

7.  $y = \frac{x^2+2x-3}{x^2-8x+12}$

8.  $y = \frac{x^3}{(x-1)^2}$

9.  $y = \frac{x^3}{x^2+1}$

10.  $y = \frac{x^3}{x^2-1}$

11.  $y = \frac{2x^2-x}{x^2-1}$

12.  $y = \frac{x^3}{x^2-x-6}$

13. Prove that when  $(x-\beta)^r$ ,  $r > 1$ , is a factor of  $f(x)$ , the curve  $y = f(x)/\phi(x)$  touches the  $x$ -axis at  $x = \beta$ , crossing it or not according as  $r$  is odd or even.

14. Prove that when  $(x-\beta)^r$ ,  $r \geq 1$ , is a factor of  $\phi(x)$ , the infinite value taken by  $y = f(x)/\phi(x)$  at  $x = \beta$  is a maximum or minimum if  $r$  is even, but not if  $r$  is odd; also that the sense in which the tangent to the curve turns as  $x$  increases changes at  $x = \beta$  if  $r$  is odd but not if  $r$  is even.

15. Show that two of the roots of  $x^3 - 3px + q = 0$  are equal if  $q^2 = 4p^3$ .

## V. INVERSE FUNCTIONS

**52. Theorem.** *If between  $x = a$  and  $x = b$ ,  $y = f(x)$  is a one valued, continuous, and increasing function of  $x$ , then also between  $x = a$  and  $x = b$ ,  $x$  is a one valued, continuous, and increasing function of  $y$ .*

1. For since  $f(x)$  is continuous in the  $x$ -interval  $(a, b)$ , and  $f(a)$  and  $f(b)$  are its least and greatest values in  $(a, b)$ , to each value  $y_1$  of  $y$  between  $f(a)$  and  $f(b)$  corresponds a value  $x_1$  of  $x$  such that  $f(x_1) = y_1$ , § 18, 2.; and there can be but one such value since  $f(x)$  is an increasing function and therefore can take no value  $y_1$  more than once. Hence, by the definition of function, § 14, between  $x = a$  and  $x = b$ ,  $x$  is a one valued function of  $y$ , namely a function  $x = \phi(y)$  such that

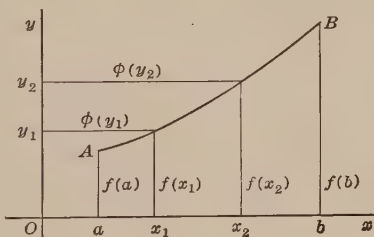


FIG. 28.

$$\text{If } y_1 = f(x_1) \text{ then } x_1 = \phi(y_1) \quad (1)$$

2. The function  $x = \phi(y)$  increases with  $y$ . If  $y_1 < y_2$ , then  $x_1 < x_2$ ; for were  $x_1 \geq x_2$ , then  $y_1 \geq y_2$ .

3. The function  $x = \phi(y)$  is continuous. For let  $x_1, y_1$  be corresponding values of  $x, y$  as in (1); it is to be proved that  $\lim_{y \rightarrow y_1} \phi(y) = \phi(y_1)$ . Compare § 8.

First let  $y$  increase toward  $y_1$  as limit. Then  $x = \phi(y)$  increases but remains  $< \phi(y_1)$  and therefore (§ 5) approaches some limit  $l (< \phi(y_1))$ . But, since  $f(x)$  is continuous, when

$x \rightarrow l$ , then  $f(x) \rightarrow f(l)$ . Hence  $f(l) = y_1$  and therefore  $l = \phi(y_1)$ .

Similarly, when  $y \rightarrow y_1$ , then  $\phi(y) \rightarrow \phi(y_1)$ . Therefore

$$\lim_{y \rightarrow y_1} \phi(y) = \phi(y_1).$$

Similarly, if between  $x = a$  and  $x = b$ ,  $y = f(x)$  is a one valued, continuous, and decreasing function of  $x$ , then also between  $x = a$  and  $x = b$ ,  $x$  is a one valued, continuous, and decreasing function of  $y$ .

**53. The inverse of a function.** The function  $x = \phi(y)$  is called the *inverse* of the function  $y = f(x)$  in the  $x$ -interval  $(a, b)$ .

If instead of increasing or decreasing throughout  $(a, b)$ ,  $y = f(x)$  first increases to a maximum value at  $x = c$ , say, and then decreases,  $y = f(x)$  has one inverse  $x = \phi_1(y)$  in the interval  $(a, c)$ , and another,  $x = \phi_2(y)$ , in  $(c, b)$ ; and so in general. If the equation  $y = f(x)$  can be solved for  $x$  in terms of  $y$ , these inverses are the several solutions.

**EXAMPLE 1.** The inverses of the function  $y = (x^2 - 1)^2$ , got by solving for  $x$ , are

1.  $x = -(1 + y^{1/2})^{1/2}$ ,  $x$  in  $(-\infty, -1)$
2.  $x = -(1 - y^{1/2})^{1/2}$ ,  $x$  in  $(-1, 0)$
3.  $x = (1 - y^{1/2})^{1/2}$ ,  $x$  in  $(0, 1)$
4.  $x = (1 + y^{1/2})^{1/2}$ ,  $x$  in  $(1, \infty)$

The graphs of these four inverses are the parts of the curve  $y = (x^2 - 1)^2$  marked 1, 2, 3, 4.

They are separated at the points  $B, C, D$  where  $f'(x)$  vanishes and changes sign.

**EXAMPLE 2.** Find the inverses of the following and their graphs:

1.  $y = 3x - 2$
2.  $y^* = x^2$
3.  $y = x^2 - 2x$
4.  $y = x/(x + 2)$

**EXAMPLE 3.** Find the graphs of the inverses of  $y = x^3 - 4x^2 + 4x$ .

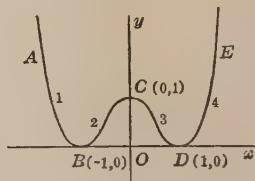


FIG. 29.

**54. The derivatives  $dy/dx$  and  $dx/dy$ .** If  $y = f(x)$  has a derivative  $f'(x)$  which is continuous and not 0 at  $x = x_1$ , then  $y = f(x)$  is an increasing or a decreasing function in the

neighborhood of the point  $(x_1, y_1)$ , § 31, and therefore has an inverse  $x = \phi(y)$  in that neighborhood, §§ 52, 53.

The function  $x = \phi(y)$  has a derivative  $\phi'(y)$  at  $(x_1, y_1)$ ; and

$$\phi'(y_1) = 1/f'(x_1) \quad (1)$$

For  $\Delta x/\Delta y = 1 \div \Delta y/\Delta x$ , and  $\Delta x \rightarrow 0$  when  $\Delta y \rightarrow 0$ , § 52, 3.; hence

$$\lim_{\Delta y \rightarrow 0} \frac{\Delta x}{\Delta y} = 1 \bigg/ \lim_{\Delta y \rightarrow 0} \frac{\Delta y}{\Delta x} = 1 \bigg/ \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}, \text{ or } \phi'(y_1) = 1/f'(x_1)$$

We have  $f'(x_1) = dy_1/dx_1$ ,  $\phi'(y_1) = dx_1/dy_1$ ; hence, omitting subscripts,

$$\frac{dx}{dy} = 1 \bigg/ \frac{dy}{dx}, \text{ when } \frac{dy}{dx} \text{ is continuous and not } 0 \quad (2)$$

EXAMPLE. Thus  $y = x^2$  gives  $dy/dx = 2x$ ; the inverse  $x = y^{1/2}$  gives  $dx/dy = 1/2 y^{1/2} = 1/2 x$ .

**55. The derivative  $dy/dx$  when  $x = \phi(t)$ ,  $y = \psi(t)$ .** Suppose that  $x$  and  $y$  are given as functions of another variable  $t$  by equations of the form  $x = \phi(t)$ ,  $y = \psi(t)$ , where  $\phi(t)$  and  $\psi(t)$  are one valued and have continuous derivatives for all values of  $t$  under consideration. In the neighborhood of any value of  $t$  for which  $\phi'(t) \neq 0$ , the function  $x = \phi(t)$  has an inverse  $t = f(x)$ , § 54; and the equations  $y = \psi(t)$ ,  $t = f(x)$  combined define  $y$  as a function of  $x$ . Hence, §§ 40, 54,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt} = \frac{\psi'(t)}{\phi'(t)}, \text{ when } \phi'(t) \neq 0 \quad (3)$$

**56. Parametric equations of curves.** To any assigned value of  $t$  correspond definite values of  $x = \phi(t)$  and  $y = \psi(t)$  and therefore a definite point  $P$  of which these values are the coordinates. When  $t$  varies continuously,  $P$  will trace a curve. The equations  $x = \phi(t)$ ,  $y = \psi(t)$  are called the *parametric equations* of this curve in terms of the *parameter*  $t$ . Its slope at the point  $(x_1, y_1)$  corresponding to  $t = t_1$  can be found by the formula § 55 (3).

**EXAMPLE 1.** The equations  $x = t^2$ ,  $y = 2t$  represent the parabola  $y^2 = 4x$ . For if we eliminate  $t$  between  $x = t^2$ ,  $y = 2t$ , we get  $y^2 = 4x$ . As  $t$  increases,  $-\infty$  to  $\infty$ , the entire curve is traced in the sense  $AOB$ . At the point  $t = 1$ , we have  $x = 1$ ,  $y = 2$ ,  $dx/dt = 2t = 2$ ,  $dy/dt = 2$ . Hence  $dy/dx = 2/2 = 1$ , and the tangent at the point is  $y - 2 = (x - 1)$ .

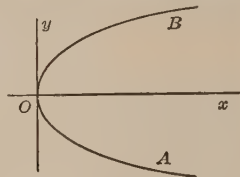


FIG. 30.

**EXAMPLE 2.** Find the tangent to  $x = t^2$ ,  $y = t^3$  at  $t = -2$ . Find the  $x, y$  equation of the curve. Trace it.

**EXAMPLE 3.** Show that  $x = at + b$ ,  $y = ct + d$  represent a straight line through the point  $(b, d)$  and having the slope  $c/a$ .

## EXERCISE VIII

1. If  $y = x^3 - 2x$ , find  $dx/dy$  when  $x = 2$ . At what points is  $dx/dy = 7$ ?

2. Find the inverses of  $y = x^2 + 2x - 3$  and their graphs.

3. Find the tangents to the following curves at the points indicated.

1.  $x = 3t^2 + 2t$ ,  $y = 1/t$ , at  $t = -2$

2.  $x = 25 - t^2$ ,  $y = t^3$ , at  $t = 4$

4. Find the  $x, y$  equation of the curve  $x = 1/t$ ,  $y = 2 - t$ . Trace the curve as  $t$  increases from  $-\infty$  to  $\infty$ .

5. Show that  $x = \frac{t^2 - 1}{t^2 + 1}$ ,  $y = \frac{2t}{t^2 + 1}$  represent the circle  $x^2 + y^2 = 1$ .

6. Verify the following:  $\frac{d^2x}{dy^2} = \frac{d}{dx} \left[ 1 / \frac{dy}{dx} \right] \frac{dx}{dy} = - \frac{d^2y}{dx^2} / \left( \frac{dy}{dx} \right)^3$ .

7. Show, if  $x = \phi(t)$ ,  $y = \psi(t)$ , that

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left[ \frac{dy}{dx} \right] \frac{dt}{dx} = \left[ \frac{dx}{dt} \frac{d^2y}{dt^2} - \frac{dy}{dt} \frac{d^2x}{dt^2} \right] / \left( \frac{dx}{dt} \right)^3$$

8. By aid of 6. and 7. find the points of inflection of  $x = y^3 - 3y^2$ , and  $x = t^2 - t$ ,  $y = t^3 + t$ .

9. A steamship is moving through the water 30 ft./sec. A ball is dropped from a point of a mast 64 ft. vertically above a point A on the deck. Neglecting the resistance of the air, show that the path of the ball is an arc of a parabola. At what angle will it strike the deck?

## VI. THE TRIGONOMETRIC FUNCTIONS

**57. Length of a curve arc.** A curve arc is called *convex* when the line joining any two of its points lies wholly within the space bounded by the arc and its chord.

In the convex arc  $AB$  inscribe a variable polygon  $AP_1P_2 \cdots P_nB$ , the lengths of whose sides all approach the limit 0 as  $n$  increases. It will be proved later that as  $n \rightarrow \infty$  the length of the polygon  $AP_1P_2 \cdots P_nB$  approaches a definite limit. This limit is called the *length of the arc*  $AB$ .

Join  $AB$ , and draw the tangents  $AT, BT$ . By elementary geometry, and § 5,

$AB < AP_1P_2 \cdots P_nB < ATB$ ; hence  $AB < \text{arc } AB < ATB$

Suppose that  $B$  is made to move along the curve toward  $A$  as limit. The ratio  $ATB/AB$  will approach 1 as limit.<sup>1</sup> Hence the ratio  $\text{arc } AB/AB$ , which lies between  $ATB/AB$  and 1, will also approach 1 as limit. Therefore

*The limit of the ratio of a convex arc to its chord, when the arc approaches 0, is 1.*

**58. The circle.** Let the arc  $AB$  be the circumference of a circle of radius  $r$ . The perimeter of an inscribed regular

<sup>1</sup> For  $AT + TB = AD \sec \theta + DB \sec \phi = AB \sec \theta + DB(\sec \phi - \sec \theta)$

$$\therefore \frac{ATB}{AB} = \sec \theta + \frac{DB}{AB} (\sec \phi - \sec \theta)$$

Let  $B \rightarrow A$ . Then  $\theta, \phi \rightarrow 0 \therefore \sec \theta, \sec \phi \rightarrow 1$

$$\therefore \frac{ATB}{AB} \rightarrow 1.$$

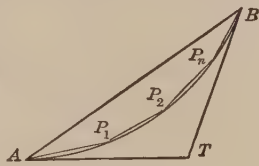


FIG. 31.

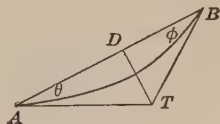


FIG. 32.



polygon of 4, 8, 16, ...,  $2^n$  sides can be computed in terms of  $r$ . The limit which it approaches when  $n \rightarrow \infty$  is  $2\pi r$ , where  $\pi$  denotes an irrational number whose value to the fifth decimal figure is 3.14159. Hence the length of the circumference of a circle of radius  $r$  is

$$c = 2\pi r$$

$$\pi = 3.14159 \dots$$

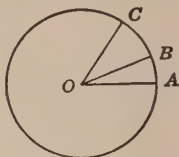


FIG. 33.

### 59. Circular measure of angle. Let $AOB$

be any angle. With  $O$  as center and any radius  $r$  describe a circle cutting  $OA$  and  $OB$  at  $A$  and  $B$ . Let  $l$  denote the length of the arc  $AB$ . It follows from the definition of length of circular arc that the ratio  $l/r$  is the same for all values of  $r$ . This ratio is called the *circular measure* of the angle  $AOB$ . Let  $\theta$  denote its value. Then

$$\theta = l/r \quad \text{and} \quad l = r\theta \quad (1)$$

When  $l = r$ , we have  $\theta = 1$ . Hence the *unit of circular measure* is the angle  $AOC$  subtended by an arc  $AC$  equal to  $r$ . This angle is called the *radian*. And since  $AOB:AOC = \text{arc } AB:\text{arc } AC = l:r$ , the circular measure  $\theta$  of  $AOB$  is the ratio of  $AOB$  to the radian  $AOC$ .

When  $AOB = 360^\circ$ , we have  $l = 2\pi r \therefore \theta = 2\pi$ . Hence the circular measures of  $360^\circ$ ,  $180^\circ$ ,  $90^\circ$ ,  $1^\circ$  are  $2\pi$ ,  $\pi$ ,  $\pi/2$ ,  $\pi/180$ . Conversely a radian  $= (180/\pi)^\circ$ .

In the calculus the measure of an angle always means its circular measure.

**60. Theorem.** When  $\theta \rightarrow 0$ , the ratio  $\sin \theta / \theta \rightarrow 1$ .

$$\text{For} \quad \frac{\sin \theta}{\theta} = \frac{2r \sin \theta}{2r\theta} = \frac{\text{chd } B'B}{\text{arc } B'B}$$

$$\text{But } \lim_{\theta \rightarrow 0} \frac{\text{chd } B'B}{\text{arc } B'B} = 1, \text{ § 57.}$$

$$\text{Hence } \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

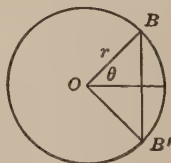


FIG. 34.

**61. Continuity of the trigonometric functions.** 1. The functions  $\sin \theta$  and  $\cos \theta$  are continuous for all finite values of  $\theta$ . For in the identity

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \quad (1)$$

set  $A = \theta + \Delta\theta$ ,  $B = \theta$ . We obtain

$$\sin(\theta + \Delta\theta) - \sin \theta = 2 \cos \left( \theta + \frac{\Delta\theta}{2} \right) \sin \frac{\Delta\theta}{2} \quad (2)$$

Here  $\cos \left( \theta + \frac{\Delta\theta}{2} \right)$  is between  $-1$  and  $1$ , and  $\sin \frac{\Delta\theta}{2} \rightarrow 0$  when  $\Delta\theta \rightarrow 0$ , § 60. Hence

$$\lim_{\Delta\theta \rightarrow 0} [\sin(\theta + \Delta\theta) - \sin \theta] = 0 \quad (3)$$

The continuity of  $\cos \theta$  follows from that of  $\sin \theta$ ; for

$$\cos \theta = \sin \left( \theta + \frac{\pi}{2} \right).$$

2. It follows from 1. that  $\tan \theta = \sin \theta / \cos \theta$  and  $\sec \theta = 1 / \cos \theta$  are continuous except when  $\cos \theta = 0$ , that is, when  $\theta$  is some odd multiple of  $\pi/2$ ; also that  $\cot \theta = \cos \theta / \sin \theta$  and  $\operatorname{cosec} \theta = 1 / \sin \theta$  are continuous except when  $\sin \theta = 0$ , that is, when  $\theta$  is 0 or some even multiple of  $\pi/2$ .

**62. Derivative of  $\sin x$ .** 1. Let  $x$  denote the circular measure of any angle. If  $y = \sin x$ , then

$$\begin{aligned} \Delta y &= \sin(x + \Delta x) - \sin x \\ &= 2 \cos \left( x + \frac{\Delta x}{2} \right) \sin \frac{\Delta x}{2} \end{aligned} \quad [\S 61 (2)]$$

$$\text{and} \quad \frac{\Delta y}{\Delta x} = \cos \left( x + \frac{\Delta x}{2} \right) \frac{\sin(\Delta x/2)}{\Delta x/2}$$

When  $\Delta x \rightarrow 0$ , the first factor on the right approaches the limit  $\cos x$ , § 61, and the second factor approaches the limit 1, (§ 60). Hence

$$\frac{d}{dx} \sin x = \cos x$$

Therefore also, by § 40, if  $u$  has a finite  $x$  derivative,

$$\frac{d}{dx} \sin u = \cos u \frac{du}{dx} \quad (1)$$

2. Replacing  $u$  by  $\pi/2 - u$  in (1), we have

$$\begin{aligned} \frac{d}{dx} \cos u &= \frac{d}{dx} \sin \left( \frac{\pi}{2} - u \right) = \cos \left( \frac{\pi}{2} - u \right) \frac{d}{dx} \left( \frac{\pi}{2} - u \right) \\ &= - \sin u \frac{du}{dx} \end{aligned} \quad (2)$$

3. The case of  $\tan u$  will illustrate the method of finding the derivatives of the remaining functions. The results are tabulated in § 63.

$$\frac{d}{dx} \tan u = \frac{d}{dx} \frac{\sin u}{\cos u} = \frac{\cos^2 u + \sin^2 u}{\cos^2 u} \frac{du}{dx} = \sec^2 u \frac{du}{dx}$$

### 63. Table of the derivatives of the trigonometric functions.

- |                                                                                          |                                                                     |
|------------------------------------------------------------------------------------------|---------------------------------------------------------------------|
| 1. $\frac{d}{dx} \sin u = \cos u \frac{du}{dx}$                                          | 2. $\frac{d}{dx} \cos u = - \sin u \frac{du}{dx}$                   |
| 3. $\frac{d}{dx} \tan u = \sec^2 u \frac{du}{dx}$                                        | 4. $\frac{d}{dx} \cot u = - \operatorname{cosec}^2 u \frac{du}{dx}$ |
| 5. $\frac{d}{dx} \sec u = \sec u \tan u \frac{du}{dx}$                                   |                                                                     |
| 6. $\frac{d}{dx} \operatorname{cosec} u = - \operatorname{cosec} u \cot u \frac{du}{dx}$ |                                                                     |

EXAMPLE. Find the derivatives of  $\sin(1 - 2x)$ ,  $\cos^2 x$ ,  $\sec x^3$ ,  $\tan^4 x^2$ .

$$\frac{d}{dx} \sin(1 - 2x) = \cos(1 - 2x) \frac{d}{dx} (1 - 2x) = -2 \cos(1 - 2x).$$

$$\frac{d}{dx} \cos^2 x = 2 \cos x \frac{d}{dx} \cos x = -2 \cos x \sin x = -\sin 2x.$$

$$\frac{d}{dx} \sec x^3 = \sec x^3 \tan x^3 \frac{d}{dx} x^3 = 3x^2 \sec x^3 \tan x^3.$$

$$\begin{aligned} \frac{d}{dx} \tan^4 x^2 &= 4 \tan^3 x^2 \cdot \frac{d}{dx} \tan x^2 = 4 \tan^3 x^2 \sec^2 x^2 \cdot \frac{d}{dx} x^2 \\ &= 8x \tan^3 x^2 \sec^2 x^2. \end{aligned}$$

## EXERCISE IX

Differentiate the following functions 1-12.

1.  $\sin 5x$ ,  $\cos (x/3)$ ,  $\tan (2x - 4)$ ,  $\sec (2x/3)$ ,  $\cot 6x$ ,  $\operatorname{cosec} 9x$ .

2.  $\sin 2x^3$ ,  $\cos \sqrt{x}$ ,  $\tan x^{2/3}$ ,  $\cot x^5$ ,  $\sec (1/x)$ ,  $\operatorname{cosec} (1/x^2)$ .

3.  $\sin^3 x$ ,  $\sqrt{\cos x}$ ,  $\tan^4 3x$ ,  $\cot^3 (1 - x)$ ,  $\sin^6 x^4$ ,  $\sqrt{\cos x^2}$ ,  $\tan^3 (1/x)$ .

4.  $x \sin x + \cos x$       5.  $\tan x - x$       6.  $\sec^4 x - \tan^4 x$

7.  $\cos^2 3x - \sin^2 3x$       8.  $\sqrt{1 - \cos x}$       9.  $\sin^2 x \cos^2 x$

10.  $\sin \frac{2x}{1-x^2}$       11.  $\tan \sqrt{\frac{1+x}{1-x}}$       12.  $\sec \frac{1+x}{1-x}$

13. Show that  $\frac{d}{dx} \sin x = \sin \left( x + \frac{\pi}{2} \right)$ ,  $\frac{d^n}{dx^n} \sin x = \sin \left( x + n \frac{\pi}{2} \right)$ ;

similarly,  $\frac{d^n}{dx^n} \cos x = \cos \left( x + n \frac{\pi}{2} \right)$ .

14. Find the maximum and minimum values of the following:

(1)  $\sin x + \cos x$       (2)  $2 \sin x + \cos x$       (3)  $3 \sec x - \tan^2 x$

(4)  $\sin x + \cos 2x$       (5)  $2 \sin x + \cos 2x$       (6)  $2 \sin x + \sin 2x$

15. Show that  $\sin \theta / \theta$  continually decreases as  $\theta$  increases from 0 to  $\pi/2$ .

16. A right circular cone whose vertical angle is  $2\theta$  is inscribed in a sphere of radius  $a$ . For what value of  $\theta$  is its volume greatest? its curved surface?

17. The four sides of a quadrilateral are given. Show that the area of the quadrilateral is greatest when its opposite angles are supplementary.

18. Find when the area of a circular sector of given perimeter is greatest.

19. Show, by substitution, that the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ has the parametric equations}$$

$$x = a \cos \phi \quad y = b \sin \phi$$

Show that the equation of the tangent at  $x_1 = a \cos \phi_1$ ,  $y_1 = b \sin \phi_1$  is

$$\frac{x}{a} \cos \phi_1 + \frac{y}{b} \sin \phi_1 = 1$$

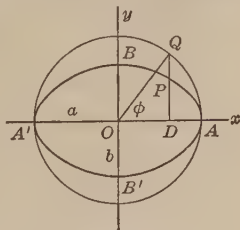


FIG. 35.

If  $P$  be any point of the ellipse and  $Q$  the point where  $DP$  produced meets the circle on the major axis  $A'A$  as diameter, show that  $\phi = DOQ$ .

20. As  $\phi$  varies,  $\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1$  remains always tangent to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . If  $OX$  and  $OY$  denote its  $x$ - and  $y$ -intercepts, find when  $OXY$  is least; also when  $XY$  is least.

21. Show that for every rectilinear motion defined by an equation of the form  $s = A \sin kt + B \cos kt$ , the acceleration is proportional to  $|s|$  and is directed toward  $O$ .

64. Inverses of  $y = \sin x, \cos x, \tan x$ . 1. If  $x, y$  be interpreted as the rectangular coordinates of a point, the graph of  $y = \sin x$  between  $x = 0$  and  $x = \pi$  is the curve arc  $OCA$ , having the maximum point  $C(\pi/2, 1)$ , and symmetric with respect to the ordinate  $BC$  through this point. The graph between

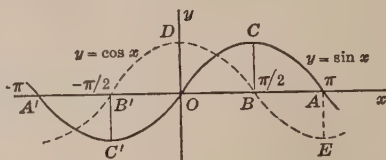


FIG. 36.

$x = -\pi$  and  $x = 0$  is the arc  $A'C'O$  symmetric to  $OCA$  with respect to  $O$  and having the minimum point  $C'(-\pi/2, -1)$ . Since  $\sin(x + 2k\pi) = \sin x$ ,  $k$  denoting any integer, the complete graph is the wave curve got by repeating the double arc  $A'C'OCA$  indefinitely to the left and right.

Between every two consecutive turning points of the graph,  $y = \sin x$  is an increasing or decreasing function and therefore has a one valued inverse, § 52. The inverse between the points  $C'$  and  $C$ , that is, in the  $x$ -interval  $(-\pi/2, \pi/2)$ , is called the *principal inverse*, and is denoted by  $x = \arcsin y$  or  $x = \sin^{-1} y$ . It is a continuous and increasing function of  $y$ , § 52.

2. Since  $\cos x = \sin(x + \pi/2)$ , the graph of  $y = \cos x$  is the same as that of  $y = \sin x$ , shifted the distance  $\pi/2$  to the left. It has a maximum point at  $D(0, 1)$  and a minimum point at  $E(\pi, -1)$ . The inverse of  $y = \cos x$  between  $D$  and  $E$ , that is, in the  $x$ -interval  $(0, \pi)$ , is called its *principal inverse* and is denoted by  $x = \arccos y$  or  $x = \cos^{-1} y$ . It is a continuous and decreasing function of  $y$ .

3. The graph of  $y = \tan x$  between  $x = -\pi/2$  and  $x = \pi/2$  is the infinite curve branch  $AOC$  having the lines  $x = -\pi/2$  and  $x = \pi/2$  for asymptotes, and symmetric with respect to  $O$ . Since  $\tan(x + k\pi) = \tan x$ ,  $k$  denoting any integer, the rest of the graph consists of repetitions of  $AOC$  to the left and right. Along each of these branches,  $y = \tan x$  has a one valued inverse. The inverse along  $AOC$ , that is, in the  $x$ -interval  $(-\pi/2, \pi/2)$ , is called the *principal inverse* and is denoted by  $x = \arctan y$  or  $x = \tan^{-1} y$ . It is a continuous and increasing function of  $y$ .

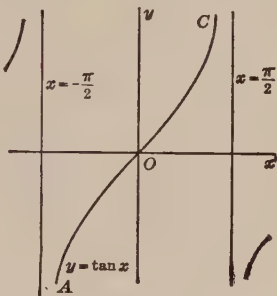


FIG. 37.

65. The functions  $\sin^{-1} x$ ,  $\cos^{-1} x$ ,  $\tan^{-1} x$ . We may interchange the rôles played by  $x$  and  $y$  in the preceding discussion. Thus  $y = \sin^{-1} x$  is the inverse of  $x = \sin y$  between  $y = -\pi/2$  and  $y = \pi/2$ .

If in any equation  $f(x, y) = 0$  (1) we interchange  $x$  and  $y$ , the equation becomes  $f(y, x) = 0$  (2). If  $P(a, b)$  be any point of the graph of (1), then  $P'(b, a)$  is a point of the graph of (2). But it is readily seen from Fig. 38 that  $P'$  is symmetric to  $P$  with respect to the line  $y = x$ . Hence the graph of (2) is symmetric to the graph of (1) with respect to this line  $y = x$ . It therefore follows from the figures in § 64 that the graphs of the functions  $y = \sin^{-1} x$ ,  $y = \cos^{-1} x$ ,  $y = \tan^{-1} x$  are as indicated below.

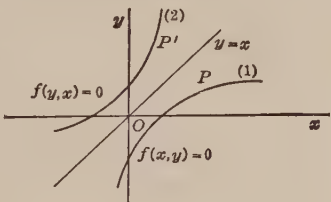


FIG. 38.

1. The functions  $y = \sin^{-1} x$ ,  $y = \cos^{-1} x$ . These functions exist in the  $x$ -interval  $(-1, 1)$  only. They are con-

tinuous in this interval. The graph of  $y = \sin^{-1} x$  is the curve arc  $AOB$ : it shows the manner in which, when  $x$  increases from  $-1$  to  $1$ ,  $y$  increases from  $-\pi/2$  to  $\pi/2$ . The graph of  $y = \cos^{-1} x$  is the curve arc  $CDE$ : it shows how, when  $x$  increases from  $-1$  to  $1$ ,  $y$  decreases from  $\pi$  to  $0$ . The arcs  $AOB$  and  $CDE$  are symmetric with respect to the line  $y = \pi/4$ . Hence  $\sin^{-1} x + \cos^{-1} x = \pi/2$ .

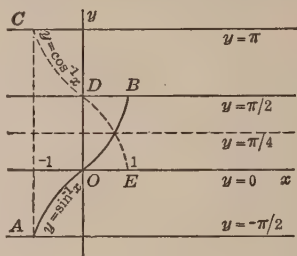


FIG. 39.

2. The function  $y = \tan^{-1} x$ . This function exists in the  $x$ -interval  $(-\infty, \infty)$ , and is continuous except at  $x = \pm \infty$ . Its graph is the infinite curve branch  $KOL$ : it shows how, when  $x$  increases from  $-\infty$  to  $\infty$ , then  $y$  increases from  $-\pi/2$  to  $\pi/2$ .

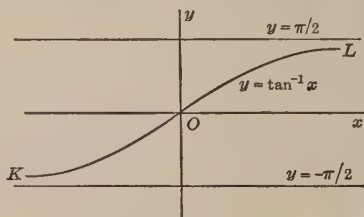


FIG. 40.

**66. Derivatives of  $\sin^{-1} x$ ,  $\cos^{-1} x$ ,  $\tan^{-1} x$ .** 1. Since  $y = \sin^{-1} x$  is the inverse of  $x = \sin y$ ,  $d(\sin^{-1} x)/dx$  is the reciprocal of  $d(\sin y)/dy$ , § 54. Hence

$$\frac{d}{dx} \sin^{-1} x = 1 / \frac{d}{dy} \sin y = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}} \quad (1)$$

For  $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$ , the radical having the  $+$  sign because  $\cos y$  is  $+$  in the  $y$ -interval  $(-\pi/2, \pi/2)$  to which  $\sin^{-1} x$  belongs.

In like manner, we have

$$\frac{d}{dx} \cos^{-1} x = 1 / \frac{d}{dy} \cos y = - \frac{1}{\sin y} = - \frac{1}{\sqrt{1-x^2}} \quad (2)$$

$$\frac{d}{dx} \tan^{-1} x = 1 / \frac{d}{dy} \tan y = \frac{1}{\sec^2 y} = \frac{1}{x^2 + 1} \quad (3)$$



By § 40, we obtain from (1), (2), (3) the more general formulas:

$$\begin{aligned} 1. \quad \frac{d}{dx} \sin^{-1} u &= \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \\ 2. \quad \frac{d}{dx} \cos^{-1} u &= -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \\ 3. \quad \frac{d}{dx} \tan^{-1} u &= \frac{1}{u^2+1} \frac{du}{dx} \end{aligned}$$

In 1. and 2.,  $x$  is restricted to values for which  $|u| < 1$ .

We may also derive (1), (2), (3) as follows: The function  $y = \sin^{-1} x$  is a solution of the equation  $\sin y = x$ , and, § 54, its derivative  $dy/dx$  is known to exist. Hence (by the reasoning in § 42), differentiating  $\sin y = x$  with respect to  $x$ , we have  $\cos y \cdot dy/dx = 1 \therefore dy/dx = 1/\cos y = 1/\sqrt{1-x^2}$ .

### EXERCISE X

Differentiate the following (1-20):

- |                                  |                                                                |                                  |
|----------------------------------|----------------------------------------------------------------|----------------------------------|
| 1. $\sin^{-1} x/a$               | 2. $\cos^{-1} 3x$                                              | 3. $\tan^{-1} x/a$               |
| 4. $\sin^{-1} (2-3x)$            | 5. $\sin^{-1} x^2$                                             | 6. $\tan^{-1} 1/x$               |
| 7. $\cos^{-1} 1/x$               | 8. $\sin^{-1} 1/x$                                             | 9. $\sin^{-1} (\cos x)$          |
| 10. $\tan^{-1} (\tan x)$         | 11. $\sin^{-1} (\sin 2x)$                                      | 12. $\cos^{-1} (1-x^2)^{1/2}$    |
| 13. $\cos^{-1} x^{1/2}$          | 14. $\sin^{-1} (1+2x)^{1/2}$                                   | 15. $\sin^{-1} [2(x-x^2)]^{1/2}$ |
| 16. $\sin^{-1} \sqrt{\sin x}$    | 17. $\tan^{-1} \frac{2x}{1-x^2}$                               | 18. $\tan^{-1} \frac{1+x}{1-x}$  |
| 19. $\tan^{-1} \frac{x+a}{1-ax}$ | 20. $\tan^{-1} \left( \frac{1-\cos x}{1+\cos x} \right)^{1/2}$ |                                  |

21. Express the complete solution of the equation  $\sin y = x$  in terms of  $\sin^{-1} x$ .

22. The points  $A$  and  $B$  are on  $Oy$ , 3 and 4 inches above  $O$ . At what points  $P$  on  $Ox$  is the angle  $APB$  the greatest?

23. We may define  $\cot^{-1} x$ ,  $\sec^{-1} x$ , and  $\operatorname{cosec}^{-1} x$  by the formulas:

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x \quad \sec^{-1} x = \cos^{-1} \frac{1}{x} \quad \operatorname{cosec}^{-1} x = \sin^{-1} \frac{1}{x}$$

Find the graphs of these functions and the formulas for their derivatives.

## VII. THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

**67. Exponential and logarithmic functions.** 1. Let  $a$  denote any number  $> 1$ . The symbol  $a^x$  is defined in elementary algebra for all rational values of  $x$ . Let  $c$  be an irrational number. It can be proved<sup>1</sup> that for all modes

<sup>1</sup>The proof is contained in the theorems that follow.

**Theorem 1.** *The function  $a^x$  increases as  $x$  increases through rational values.*

(1) If  $p, q$  are any positive integers, then  $a^{p/q} > 1$ . For  $a > 1 \therefore a^p > 1$   
 $\therefore \sqrt[q]{a^p} > 1$ , that is,  $a^{p/q} > 1$ .

(2) Let  $r, s$  be any two rationals,  $s > r$ . Then  $s - r > 0 \therefore a^{s-r} > 1$   
 $\therefore a^{s-r} a^r > a^r \therefore a^s > a^r$ .

**Theorem 2.** *If  $n \rightarrow \infty$  through integral values, then  $a^n \rightarrow \infty$  and  $a^{-n} \rightarrow 0$ .*  
 For since  $a > 1$ , we may set  $a = 1 + d$ , where  $d$  is positive. Then  
 $a^n = (1 + d)^n > 1 + nd$ . But  $\lim_{n \rightarrow \infty} (1 + nd) = \infty$ . Hence  $\lim_{n \rightarrow \infty} a^n = \infty$ .  
 Also  $a^{-n} = 1/a^n$ ; hence  $\lim_{n \rightarrow \infty} a^{-n} = 0$ .

**Theorem 3.** *If  $n \rightarrow \infty$  through integral values, then  $a^{1/n} \rightarrow 1$ ; therefore also  $a^{-1/n} = 1/a^{1/n} \rightarrow 1$ .*

For since  $a^{1/n} > 1$ , we may set  $a^{1/n} = 1 + d_n$ , where  $d_n$  is a positive number which varies with  $n$ . We then have  $a = (1 + d_n)^n > 1 + nd_n$

$\therefore d_n < (a - 1)/n \therefore \lim_{n \rightarrow \infty} d_n = 0 \therefore \lim_{n \rightarrow \infty} a^{1/n} = 1$ .

**Theorem 4.** *If  $c$  and  $x$  be rational, then  $\lim_{x \rightarrow c} a^x = a^c$ .*

(1) Let  $c = 0$ . We can choose an integer  $n$  such that  $-1/n < x < 1/n$ , which makes  $a^{-1/n} < a^x < a^{1/n}$ . When  $x \rightarrow 0$ , then  $n \rightarrow \infty$  and therefore  $a^{1/n}, a^{-1/n} \rightarrow 1 \therefore \lim_{x \rightarrow 0} a^x = 1 = a^0$ .

(2) Let  $c \neq 0$ . We have  $a^x = a^c \cdot a^{x-c} \therefore \lim_{x \rightarrow c} a^x = a^c \lim_{x \rightarrow c} a^{x-c} = a^c$ .

**Theorem 5.** *If  $c$  be irrational and  $x$  rational, then  $a^x$  approaches a limit when  $x \rightarrow c$ .*

(1) Let  $x \rightarrow c$  through increasing values and let  $g$  denote any rational  $> c$ . As  $x$  increases,  $a^x$  increases but remains less than  $a^g$ ; it therefore approaches a limit  $l (< a^g)$ , § 5.

of approach of  $x$  to  $c$  as limit, through rational values,  $a^x$  will approach one and the same limit; we represent this limit by the symbol  $a^c$ . Thus  $2^{\sqrt{3}} = \lim_{x \rightarrow \sqrt{3}} 2^x$ .

Having thus defined  $a^x$  for all real values of  $x$ , we can show that  $y = a^x$  is a one valued, continuous and increasing function of  $x$  which increases from 0 to 1 as  $x$  increases from  $-\infty$  to 0, and from 1 to  $\infty$  as  $x$  increases from 0 to  $\infty$ . Its graph is a curve of the type (1) in Fig. 41, as may be illustrated by taking the case  $y = 2^x$ , plotting the points corresponding to  $x = -3, -2, -1, 0, 1, 2, 3$ , and drawing a smooth curve through these points.

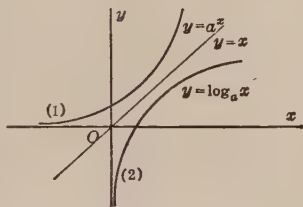


FIG. 41.

The function  $a^x$  is called an *exponential function* of  $x$ .

(2) Let  $x' \rightarrow c$  through any rational values. Since  $a^{x'} = a^x \cdot a^{x'-x}$ ,  
 $\lim_{x' \rightarrow c} a^{x'} = \lim_{x \rightarrow c} a^x \lim_{x, x' \rightarrow c} a^{x'-x} = l$ .

**Irrational exponents.** The limit  $l$  which  $a^x$  approaches when  $x \rightarrow c$  is denoted by  $a^c$ . Hence for  $c$  irrational and  $x$  rational we have  $a^c = \lim_{x \rightarrow c} a^x$ .

**Theorem 6.** The function  $a^x$  increases as  $x$  increases continuously.

For let  $b$  and  $c$  be any two real numbers,  $b < c$ , and  $x$  any rational such that  $b < x < c$ . We have  $a^b < a^x$  and  $a^x < a^c$ .  $\therefore a^b < a^c$ .

**Theorem 7.** The function  $a^x$  is continuous for every finite value  $c$  of  $x$ .

For let  $x \rightarrow c$  through any real values, and let  $x', x''$  be rationals such that  $x' < x < x''$ . Then  $a^{x'} < a^x < a^{x''}$ . When  $x', x'' \rightarrow c$ , then  $a^{x'}, a^{x''} \rightarrow a^c$  and therefore  $a^x \rightarrow a^c$ .

When  $0 < a < 1$ ,  $a^x$  decreases as  $x$  increases through rational values. By modifying the preceding discussion to suit this circumstance it is easy to show that in this case also  $a^x$  is continuous for all finite values of  $x$ .

**Theorem 8.** For any positive values of  $a, b$  and any real values of  $m, n$ , we have

$$1. a^m a^n = a^{m+n}$$

$$2. (a^m)^n = a^{mn}$$

$$3. (ab)^m = a^m b^m$$

For let  $x, y$  be rationals which  $\rightarrow m, n$ . We have  $a^x a^y = a^{x+y}$ ,  $(a^x)^y = a^{xy}$ ,  $(ab)^x = a^x b^x$ . But when  $x, y \rightarrow m, n$ , then  $a^x a^y = a^{x+y}$  gives  $a^m a^n = a^{m+n}$  and  $(ab)^x = a^x b^x$  gives  $(ab)^m = a^m b^m$ . Also when  $x \rightarrow m$ ,  $(a^x)^y = a^{xy}$  gives  $(a^m)^y = a^{my}$ , and this in turn, when  $y \rightarrow n$ , gives  $(a^m)^n = a^{mn}$ .

2. Since  $y = a^x$  is a continuous increasing function of  $x$  in the  $x$ -interval  $(-\infty, \infty)$ , it has a continuous increasing inverse in that interval, § 52. This inverse is written  $x = \log_a y$  and called the *logarithm of  $y$  to the base  $a$* . It is a continuous increasing function of  $y$  in the  $y$ -interval  $(0, \infty)$ .

In like manner,  $y = \log_a x$ , which is the inverse of  $x = a^y$ , is a continuous increasing function of  $x$  in the  $x$ -interval  $(0, \infty)$ . Its graph is the curve (2) symmetric to (1) with respect to the line  $y = x$ . (See § 65.)

**68. Properties of logarithms.** 1. As is shown in the footnote, Th. 8, for  $a$  and  $b$  positive, and  $\mu$  and  $\nu$  real, we have  $a^\mu a^\nu = a^{\mu+\nu}$ ,  $(a^\mu)^\nu = a^{\mu\nu}$ ,  $(ab)^\mu = a^\mu b^\mu$ .

2. By § 67, 2., to every positive number  $m$  corresponds a real number  $\mu$  such that  $a^\mu = m$ ; we call  $\mu$  the logarithm of  $m$  to the base  $a$  and write  $\mu = \log_a m$ : so that  $m = a^{\log_a m}$ .

Since  $a^0 = 1$  and  $a^1 = a$ , we have  $\log_a 1 = 0$  and  $\log_a a = 1$ . Since  $a > 1$ ,  $\log_a m$  is  $+$  or  $-$ , according as  $m \gtrless 1$ . Since  $\lim_{x \rightarrow \infty} a^x = \infty$  and  $\lim_{x \rightarrow \infty} a^{-x} = 0$ , we have  $\log_a \infty = \infty$ , and  $\log_a 0 = -\infty$ .

3. Suppose that  $a^\mu = m$ ,  $a^\nu = n$ , so that  $\mu = \log_a m$ ,  $\nu = \log_a n$ . Then

$$mn = a^\mu \cdot a^\nu = a^{\mu+\nu} \quad \text{that is,} \quad \log_a mn = \log_a m + \log_a n$$

$$m/n = a^\mu / a^\nu = a^{\mu-\nu} \quad \text{that is,} \quad \log_a m/n = \log_a m - \log_a n$$

$$m^r = (a^\mu)^r = a^{\mu r} \quad \text{that is,} \quad \log_a m^r = r \log_a m$$

$$\sqrt[s]{m} = \sqrt[s]{a^\mu} = a^{\mu/s} \quad \text{that is,} \quad \log_a \sqrt[s]{m} = [\log_a m]/s.$$

4. Suppose that  $m = a^\mu = b^\nu$ , so that  $\mu = \log_a m$  and  $\nu = \log_b m$ .

Then  $a^{\mu/\nu} = b$ . Hence  $\log_a b = \mu/\nu = \log_a m / \log_b m$ , and therefore

$$\log_b m = \frac{\log_a m}{\log_a b}; \text{ also setting } m = a, \log_b a = \frac{1}{\log_a b}$$

EXAMPLE. Express  $\log_{10} \sqrt[3]{2000} \sqrt[4]{25} / 256$  in terms of  $\log_{10} 2$ .

**69. The number  $e$ .** Consider the behavior of  $(1 + 1/n)^n$  as  $n$  increases. Its values for  $n = 1, 2, 3, 4$  are 2, 2.25, 2.37, 2.44. We shall prove that it continually increases with  $n$ , and approaches a number between 2 and 3 as limit when  $n \rightarrow \infty$ .

1. Let  $n$  take positive integral values only. The expansion of  $(1 + 1/n)^n$  by the binomial theorem is

$$1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \frac{1}{n^3} + \cdots + \frac{1}{n^n}$$

The sum of the first two terms is 2. The rest may be written

$$\begin{aligned} & \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \cdots \\ & + \frac{1}{n!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right) \end{aligned}$$

As  $n$  increases, the values of the first, second,  $\cdots$  of these terms increase (for  $1/n, 2/n, \cdots$  decrease); the number of the terms also increases; hence the sum increases. But it remains less than

$$\frac{1}{2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{2 \cdot 3 \cdots n} < \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 1$$

Hence  $(1 + 1/n)^n$  increases with  $n$  but remains less than  $2 + 1$  or 3. It therefore approaches some number between 2 and 3 as limit, § 5. This number is denoted by  $e$ . It will be shown later that its value to the third decimal figure is 2.718. Hence, by definition,

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = 2.718 \cdots$$

2. Let  $\mu$  denote a variable which  $\rightarrow \infty$  through positive real values of any kind, and  $n$  a variable integer such that

$$n \leq \mu < n + 1$$

Then,  $\left(1 + \frac{1}{n+1}\right)^{n+1} < \left(1 + \frac{1}{\mu}\right)^{\mu} < \left(1 + \frac{1}{n}\right)^{n+1}$

The first and last of these expressions may be written

$$\left(1 + \frac{1}{n+1}\right)^{n+1} \div \left(1 + \frac{1}{n+1}\right) \text{ and } \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)$$

and therefore both of them  $\rightarrow e$  when  $n \rightarrow \infty$ . Hence

$$\lim_{\mu \rightarrow \infty} \left(1 + \frac{1}{\mu}\right)^{\mu} = e$$

3. Finally, we have

$$\begin{aligned} \left(1 - \frac{1}{\mu}\right)^{-\mu} &= \left(\frac{\mu}{\mu-1}\right)^{\mu} = \left(1 + \frac{1}{\mu-1}\right)^{\mu} \\ &= \left(1 + \frac{1}{\mu-1}\right)^{\mu-1} \cdot \left(1 + \frac{1}{\mu-1}\right) \end{aligned}$$

Hence (by 2.), 
$$\lim_{\mu \rightarrow \infty} \left(1 - \frac{1}{\mu}\right)^{-\mu} = e$$

We have therefore proved that  $(1 + 1/n)^n \rightarrow e$  when  $n \rightarrow \infty$  or  $-\infty$  through real numbers of whatever kind.

**70. Natural logarithms.** This name is given to logarithms to the base  $e$ . They are practically the only logarithms used in the Calculus. In what follows,  $\log x$  will mean  $\log_e x$ .

**71. Derivatives of  $\log x$  and  $e^x$ .** 1. If  $y = \log x$ , then

$$\begin{aligned} \Delta y &= \log(x + \Delta x) - \log x = \log \frac{x + \Delta x}{x} \\ &= \log \left(1 + \frac{\Delta x}{x}\right) = \frac{\Delta x}{x} \log \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} \end{aligned}$$

Hence 
$$\frac{\Delta y}{\Delta x} = \frac{1}{x} \log \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

Let  $x/\Delta x = \mu$ , and therefore  $\Delta x/x = 1/\mu$ . Then

$$\lim_{\Delta x \rightarrow 0} \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} = \lim_{\mu \rightarrow \infty} \left(1 + \frac{1}{\mu}\right)^{\mu} = e$$

and since  $\log x$  is continuous, when  $(1 + 1/\mu)^\mu \rightarrow e$  then  $\log(1 + 1/\mu)^\mu \rightarrow \log e = 1$ .

Hence 
$$\frac{d}{dx} \log x = \frac{1}{x} \quad (1)$$

2. Since  $y = e^x$  is the inverse of  $x = \log y$ , we have, § 54, (1),

$$\frac{d}{dx} e^x = 1 / \frac{d}{dy} \log y = 1 / \frac{1}{y} = y = e^x$$

Hence 
$$\frac{d}{dx} e^x = e^x \quad (2)$$

3. By § 40, we derive from (1) and (2) the more general formulas

$$\frac{d}{dx} \log u = \frac{1}{u} \frac{du}{dx} \quad \frac{d}{dx} e^u = e^u \frac{du}{dx} \quad (3)$$

It is usually best to express the logarithm of a product in terms of the logarithms of the factors before differentiating it, and the like is true of the logarithm of a quotient, power, or root.

EXAMPLE 1.  $\frac{d}{dx} e^{-3x} = e^{-3x} \frac{d}{dx} (-3x) = -3 e^{-3x}$

$$\frac{d}{dx} \log \sin x = \frac{1}{\sin x} \frac{d}{dx} \sin x = \frac{\cos x}{\sin x} = \cot x$$

$$\begin{aligned} \frac{d}{dx} \log \sqrt{\frac{(x^2 + 2)^3}{x^2 + 1}} &= \frac{d}{dx} \frac{1}{2} [3 \log (x^2 + 2) - \log (x^2 + 1)] \\ &= \frac{1}{2} \left[ \frac{3 \cdot 2x}{x^2 + 2} - \frac{2x}{x^2 + 1} \right] = \frac{2x^3 + x}{(x^2 + 2)(x^2 + 1)} \end{aligned}$$

EXAMPLE 2. By § 68, 4.,  $\log_a x = \frac{\log x}{\log a} = \log x \cdot \log_a e$ .

Hence 
$$\frac{d}{dx} \log_a x = \frac{1}{x} \cdot \log_a e.$$

**72. Logarithmic differentiation.** It follows from § 67, 2. that a function  $u$ , when positive, may be expressed in the form  $u = e^v$ , where  $v = \log u$ . This gives

$$\frac{du}{dx} = \frac{d}{dx} e^v = e^v \frac{dv}{dx} = u \frac{dv}{dx} \quad (1)$$



Hence  $u$  has a derivative if its logarithm,  $v$  has one, and  $du/dx$  may be got by multiplying  $u$  by  $dv/dx$ . This process is called *logarithmic differentiation*.

EXAMPLE 1. Find the derivative of  $x^x$ .

Let  $y = x^x$ . Then  $\log y = x \log x$ . Hence, diff'g with respect to  $x$ ,  $\frac{1}{y} \frac{dy}{dx} = \log x + 1$ . But  $y = x^x$ . Hence  $\frac{dy}{dx} = x^x(\log x + 1)$ .

EXAMPLE 2. If  $y = a^u$ , then  $\log y = u \log a \quad \therefore \frac{1}{y} \frac{dy}{dx} = \frac{du}{dx} \log a$ .

Hence  $\frac{d}{dx} a^u = a^u \log a \frac{du}{dx}$ .

EXAMPLE 3. If  $y = u^n$ , then  $\log y = n \log u \quad \therefore \frac{1}{y} \frac{dy}{dx} = n \frac{1}{u} \frac{du}{dx}$ .

Hence  $\frac{d}{dx} u^n = n u^{n-1} \frac{du}{dx}$ .

EXAMPLE 4. Find the derivative of  $[x(1-x)^2/(x+1)^4]^{1/3}$  by logarithmic differentiation. Equating the expression to  $y$ , taking the logs of both members, diff'g with respect to  $x$ , we obtain finally

$$\frac{dy}{dx} = \frac{1}{3} \left[ \frac{1}{x} - \frac{2}{1-x} - \frac{4}{x+1} \right] \left[ \frac{x(1-x)^2}{(x+1)^4} \right]^{1/3}$$

### EXERCISE XI

Differentiate the following functions 1-20 with respect to  $x$ :

1.  $e^{4x}$ ,  $e^{kx}$ ,  $\sqrt{e^x}$ ,  $\sqrt[n]{e^x}$ ,  $\sqrt{1/e^x}$ ,  $e^{3x^2}$ ,  $e^{\sin x}$ ,  $e^{\tan 2x}$ ,  $e^{\sin^{-1} x}$ .
2.  $\log 5x$ ,  $\log (bx+c)$ ,  $\log e^{3x}$ ,  $\log (e^{x/2} + x)$ ,  $\log^3 x$ ,  $\log (\log x)$ .
3.  $\log (1-5x-3x^4)^2$ ,  $\log (\cos x)$ ,  $\log (\tan 3x)$ ,  $\log (\tan^{-1} x)$ ,  $\log \sqrt{1-3x}$ ,  $\log (1/x)$ .
4.  $\log \frac{x+a}{x-a}$     5.  $\log \sqrt{\frac{x+a}{x-a}}$     6.  $\log \sqrt{\frac{1+x^2}{1-x^2}}$     7.  $\log \sqrt{\frac{1+\cos x}{1-\cos x}}$
8.  $\log (\sec x + \tan x)$     9.  $\log (x + \sqrt{x^2 + a^2})$     10.  $\log \sqrt{e^{2x} + x}$
11.  $\frac{a}{2} (e^{x/a} - e^{-x/a})$     12.  $\log \frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2} - x}$     13.  $\frac{1-e^x}{1+e^x}$
14.  $x^{x^n}$     15.  $(\sin x)^{\tan 2x}$     16.  $(1-x)^x$
17.  $x^{1/3}(1-x)^{2/3}(1-2x^3)^{4/3}$

$$18. \sqrt[3]{\frac{x^2+1}{x^6(1-x)^4}} \quad 19. \frac{(2x-3)^2(1-5x)^4}{(x^3+1)(x^4+2)^2} \quad 20. x\sqrt[3]{2} + x\sqrt[3]{3}$$

$$21. \text{ Prove that (1) } \frac{d^n}{dx^n} e^{kx} = k^n e^{kx}$$

$$(2) \frac{d^n}{dx^n} a^x = a^x (\log a)^n \quad (3) \frac{d^n}{dx^n} \log x = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

22. Find the turning points and graphs of (1)  $y = xe^x$ , (2)  $y = x^2e^x$ .

23. What are the characteristics of the motion of a point  $P$  on  $Ox$  if the equation be  $x = e^{-t} \sin t$ ?

**73. Hyperbolic functions.** 1. In applied mathematics use is made of the functions  $\sinh x$  and  $\cosh x$  called *hyperbolic sine* and *hyperbolic cosine* and defined by the formulas

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad (1)$$

We also meet  $\tanh x = \sinh x / \cosh x$ , and so on.

2. Since, as shown<sup>1</sup> in Fig. 42,  $y = \sinh x$  is a continuous increasing function of  $x$  between  $x = -\infty$  and  $x = \infty$ , it has but one inverse, § 53. This inverse is denoted by  $x = \sinh^{-1} y$ . The equation

$$y = (e^x - e^{-x})/2$$

gives  $e^{2x} - 2ye^x - 1 = 0$

Hence<sup>2</sup>  $e^x = y + \sqrt{y^2 + 1}$

and  $x = \sinh^{-1} y = \log [y + \sqrt{y^2 + 1}] \quad (2)$

The function  $y = \cosh x$  decreases in the  $x$ -interval  $(-\infty, 0)$  and increases in the interval  $(0, \infty)$ . It therefore

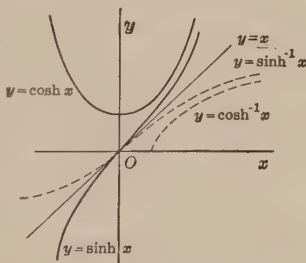


FIG. 42.

<sup>1</sup> The graph of  $y = \cosh x$  is a *catenary*, the curve in which a uniform chain hangs under the action of gravity. If the length of its  $y$ -intercept in terms of some new unit be  $a$ , and the coordinates of any point in terms of this unit be  $x', y'$ , the equation becomes  $y' = a \cosh (x'/a)$ .

<sup>2</sup> The solution  $e^x = y - \sqrt{y^2 + 1}$  makes  $e^x$  negative and therefore  $x$  imaginary.

has two inverses. We select that belonging to  $(0, \infty)$  as the principal inverse and denote it by  $x = \cosh^{-1} y$ .

$$\begin{array}{ll} \text{Solving} & y = (e^x + e^{-x})/2 \quad \text{for } x \\ \text{gives}^1 & x = \cosh^{-1} y = \log [y + \sqrt{y^2 - 1}] \end{array} \quad (3)$$

The inverses of  $x = \sinh y$  and  $x = \cosh y$  are  $y = \sinh^{-1} x$  and  $y = \cosh^{-1} x$ . The functions  $\sinh^{-1} x$  and  $\cosh^{-1} x$  are called the *inverse hyperbolic sine and cosine of  $x$* . Interchanging  $x$  and  $y$  in (2) and (3), we get

$$\begin{aligned} \sinh^{-1} x &= \log [x + \sqrt{x^2 + 1}] \\ \cosh^{-1} x &= \log [x + \sqrt{x^2 - 1}] \end{aligned} \quad (4)$$

**EXAMPLE 1.** Prove the following properties of the hyperbolic functions:

- |                                                                       |                                                                       |
|-----------------------------------------------------------------------|-----------------------------------------------------------------------|
| 1. $\cosh^2 x - \sinh^2 x = 1$                                        | 2. $\operatorname{sech}^2 x + \tanh^2 x = 1$                          |
| 3. $\frac{d}{dx} \sinh x = \cosh x$                                   | 4. $\frac{d}{dx} \cosh x = \sinh x$                                   |
| 5. $\frac{d}{dx} \sinh^{-1} \frac{x}{a} = \frac{1}{\sqrt{x^2 + a^2}}$ | 6. $\frac{d}{dx} \cosh^{-1} \frac{x}{a} = \frac{1}{\sqrt{x^2 - a^2}}$ |

**EXAMPLE 2.** Find the graph of  $y = \tanh x$ , showing that as  $x$  increases, from  $-\infty$  to  $\infty$ , the curve rises from the asymptote  $y = -1$  through  $O$  to the asymptote  $y = 1$ .

**EXAMPLE 3.** Define  $\tanh^{-1} x$  and show that  $\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}$ .

<sup>1</sup> The solution  $x = \log [y - \sqrt{y^2 - 1}] = -\log [y + \sqrt{y^2 - 1}]$  is the inverse belonging to the  $x$ -interval  $(-\infty, 0)$ .

## VIII. CURVES IN POLAR COORDINATES

**74. Polar coordinates.** Take, as the elements of reference, the origin  $O$  and the direction  $Ox$ . Then,  $P$  being any point in the plane, join  $OP$  and represent the measures of  $xOP$  and  $OP$  by  $\theta$  and  $r$ . We call  $\theta, r$  the *polar coordinates* of  $P$ , referred to  $O, Ox$ .

Conversely, to find  $P$  when  $\theta$  and  $r$  are given, we first draw from  $O$  the half-line which makes the angle  $\theta$  with  $Ox$ , and then on this half-line when  $r$  is  $+$ , or on the half-line produced through  $O$  when  $r$  is  $-$ , lay off  $OP$  of length  $|r|$ . Thus  $\theta = \pi/6, r = 3$  and  $\theta = -5\pi/6, r = -3$  represent the same point  $P$ .

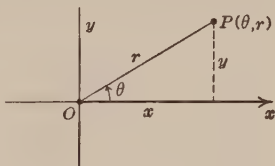


FIG. 43.

The relations between the polar coordinates  $\theta, r$  of  $P$  and its  $x, y$  coordinates referred to the rectangular axes  $Ox, Oy$  in Fig. 43 are

$$\begin{aligned} x &= r \cos \theta & r &= (x^2 + y^2)^{1/2} \\ y &= r \sin \theta & \tan \theta &= y/x \end{aligned}$$

**75. Angle between tangent and radius vector.** An equation of the form  $r = f(\theta)$  represents some curve  $C$ ; it is required to find the angle  $\psi$  made by the tangent at a point  $P(\theta, r)$  of  $C$  with the radius vector  $OP$ . Let  $Q$  be the point  $(\theta + \Delta\theta, r + \Delta r)$ . With  $O$  as center draw the circular arc  $PR$  of radius  $r$ ; also draw the chords  $PR$  and  $PQ$ .

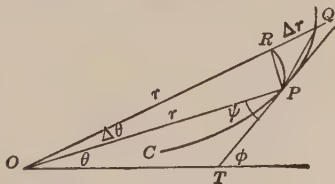


FIG. 44.

In the triangle  $PQR$ ,

$$\frac{\sin Q}{\sin P} = \frac{PR}{RQ}.$$

Since the angle  $P$  is the supplement of  $Q + R$ , we have  $\sin P = \sin (Q + R)$ .

Let  $\Delta\theta \rightarrow 0$ . Then  $Q \rightarrow \psi$  and  $R \rightarrow \pi/2$  and therefore

$$\frac{\sin Q}{\sin P} = \frac{\sin Q}{\sin (Q + R)} \rightarrow \frac{\sin \psi}{\sin \left( \psi + \frac{\pi}{2} \right)} = \tan \psi$$

Also  $RQ = \Delta r$ , and arc  $PR = r \Delta\theta$ . Hence, when  $\Delta\theta \rightarrow 0$ ,

$$\frac{PR}{RQ} = \frac{PR}{\text{arc } PR} \frac{r \Delta\theta}{\Delta r} \rightarrow r \frac{dr}{d\theta}$$

Therefore  $\tan \psi = r \frac{dr}{d\theta}$  (1)

The figure and proof correspond to the case in which  $r$  increases with  $\theta$ . When  $r$  decreases,  $dr/d\theta$  is  $-$  and (1) gives the obtuse angle made by the tangent with  $OP$ . In both cases  $\psi$  is the angle between  $OP$  and the half-tangent  $PT$  which, when  $\theta$  increases, follows  $P$  in its motion along  $C$ . The proof fails when  $dr/d\theta$  is 0; but (1) continues to hold good since  $\psi$  is then  $\pi/2$ .

Observe in Fig. 44 that  $\theta$ ,  $\psi$ , and the slope angle  $\phi$  satisfy

$$\phi = \theta + \psi \quad (2)$$

EXAMPLE 1. For the cardioid  $r = a(1 - \cos \theta)$ , show that  $\psi = \theta/2$ .

$$\tan \psi = r \frac{dr}{d\theta} = \frac{1 - \cos \theta}{\sin \theta} = \tan \frac{\theta}{2}$$

$$\therefore \psi = \frac{\theta}{2}.$$

To  $\theta = 0, \frac{\pi}{2}, \pi$  correspond the points  $O, A, B$ .  
 $A \left( \frac{\pi}{2}, a \right), B(\pi, 2a)$ .

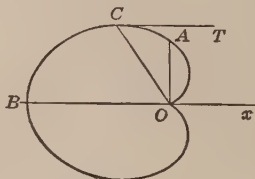


FIG. 45.

At the highest point  $C$ , we have

$$xOC + OCT = \pi. \text{ But } xOC = 2 OCT \quad \therefore xOC = \frac{2}{3}\pi \quad \therefore OC = \frac{3}{2}a.$$

EXAMPLE 2. Find  $\psi$  at the point of  $r = a \sin^3 (\theta/3)$  where  $\theta = \pi$ .

**76. Graphs of equations  $r = f(\theta)$ .** To find the graph  $C$  of an equation of the form  $r = f(\theta)$ , we compute  $r$  for significant values of  $\theta$  and pass a smooth curve through the points thus determined. If  $f(-\theta) = f(\theta)$ , the curve is symmetric with respect to  $Ox$  (§ 75, Ex. 1). To find whether  $C$  passes through  $O$ , we may set  $f(\theta) = 0$ ; its real roots, if any, are the angles at which  $C$  cuts  $Ox$  at  $O$ . The sign of  $dr/d\theta$  indicates when  $r$  increases with  $\theta$  and when  $r$  decreases. We can find the points where the slope is  $\infty$  and 0 by solving  $dx/d\theta = 0$  and  $dy/d\theta = 0$  for  $\theta$ , or by aid of the relation  $\phi = \theta + \psi$ .

**EXAMPLE 1.** Find the graph of  $r = a \cos 3\theta$ .

|           |     |         |          |          |          |          |        |
|-----------|-----|---------|----------|----------|----------|----------|--------|
| $\theta$  | 0   | $\pi/6$ | $2\pi/6$ | $3\pi/6$ | $4\pi/6$ | $5\pi/6$ | $\pi$  |
| $3\theta$ | 0   | $\pi/2$ | $\pi$    | $3\pi/2$ | $2\pi$   | $5\pi/2$ | $3\pi$ |
| $r$       | $a$ | 0       | $-a$     | 0        | $a$      | 0        | $-a$   |

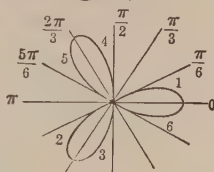


FIG. 46.

As  $\theta$  increases from 0 to  $\pi$  we trace successively the arcs 1, 2, 3, 4, 5, 6.

**EXAMPLE 2.** The curve  $r^2 = a^2 \cos 2\theta$  is called a lemniscate. Show that it has the shape indicated in Fig. 47. Show that for it

$$\psi = 2\theta + \pi/2.$$

Find its  $x, y$  equation.

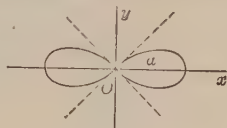


FIG. 47.

**EXAMPLE 3.** The curve  $r = e^{k\theta}$ ,  $k > 0$ , is called a logarithmic spiral. Show that it has the shape indicated in Fig. 48, and that when  $\theta \rightarrow -\infty$  the tracing point approaches  $O$  asymptotically. Show also that  $\tan \psi$  has the constant value  $1/k$ .

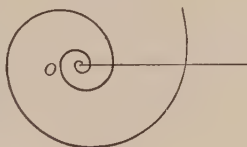


FIG. 48.

## EXERCISE XII

1. Show that  $r = a \cos \theta$  is a circle of diameter  $a$ , touching  $Oy$  at  $O$ ; also that  $\psi = \theta + \pi/2$ .

2. Show that  $r = a \sin \theta$  is a circle of diameter  $a$ , touching  $Ox$  at  $O$ ; also that  $\psi = \theta$ .

3. Prove that for any curve  $r = a (\sin k\theta)^{1/k}$ ,  $\psi = k\theta$ . Draw the curve for the case  $k = 2$ .

4. Trace the curve  $r = a \sin 3\theta$  and find  $\tan \psi$  when  $\theta = \pi/3$ .

5. Trace the curve  $r = a \cos 2\theta$ , showing that it has four loops. How many loops has  $r = \cos n\theta$ ?

6. The curve  $r = l/(1 - e \cos \theta)$  is a conic of eccentricity  $e$  with a focus at  $O$  and with its axis on  $Ox$ ; trace it for the case  $e = 1$ , showing that  $\psi = \pi - \theta/2$ .

7. Trace the spiral of Archimedes  $r = a\theta$ . Find  $\psi$ .

8. Trace the limaçon  $r = c - a \cos \theta$  for the cases  $c = 2a$  and  $c = a/2$ .

9. Trace the curves (1)  $r = a \sin^3(\theta/3)$  (2)  $r = \cos \theta + \sin \theta$ .

10. Find the points where the slope of  $r = a \cos 2\theta$  is 0.

11. Find the points where the slope of the curve  $r = a(1 - \cos \theta)$  is  $\infty$ .

12. In Fig. 49,  $PT$  and  $PN$  are the tangent and normal to  $C$  at  $P$ , and  $TON$  is at right angles to  $OP = r$ . Prove that  $OT = r^2 / \frac{dr}{d\theta}$ ,  $ON = dr/d\theta$ .

If  $f(\theta)$  is  $\infty$  for  $\theta = \theta_1$ , then  $C$  has an infinitely distant point  $P_\infty(\theta_1, \infty)$ , the lines  $OP_\infty$  and  $TP_\infty$  are parallel, and  $TP_\infty$  is an asymptote to  $C$ . Find the asymptotes of the curve  $r = a \tan \theta$ ; also trace the curve.

13. Show that  $r \cos(\theta - \alpha) = p$  is the equation of a line whose perpendicular distance from  $O$  is  $p$ ,  $\alpha$  being the angle which the perpendicular makes with  $Ox$ .

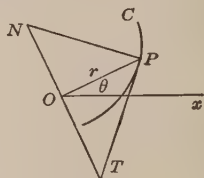


FIG. 49.



## IX. CURVATURE

**77. Differential of arc.** Let  $s$  denote the length of the arc of the curve  $S$  between the fixed point  $A$  and the variable point  $P(x, y)$ . Evidently  $s$  is a function of the  $x$  of  $P$ ; its derivative  $ds/dx$  may be found as follows:

Let  $Q$  be the point  $(x + \Delta x, y + \Delta y)$ , and  $\Delta s$  the length of the arc  $PQ$ . The length of  $PQ$  is  $[(\Delta x)^2 + (\Delta y)^2]^{1/2}$ ; hence

$$\frac{\Delta s}{\Delta x} = \frac{\Delta s}{PQ} \frac{PQ}{\Delta x} = \frac{\Delta s}{PQ} \left[ 1 + \left( \frac{\Delta y}{\Delta x} \right)^2 \right]^{1/2}$$

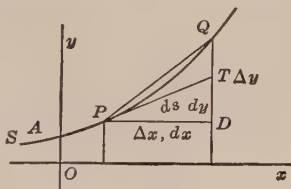


FIG. 50.

Let  $\Delta x \rightarrow 0$ . Then  $\Delta s/PQ \rightarrow 1$ , and therefore

$$\frac{ds}{dx} = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} \quad (1)$$

Multiply both members of (1) by  $dx$ ; we get, § 34,

$$ds = [(dx)^2 + (dy)^2]^{1/2} \quad (2)$$

Since  $dx = PD$  and  $dy = DT$ , (2) shows that  $ds = PT$ . Since  $dx, dy, ds$  are the sides of the triangle  $PDT$  and  $DPT$  is the slope angle  $\phi$ ,

$$dx = \cos \phi \, ds \quad dy = \sin \phi \, ds \quad (3)$$

**78. Curvature.** Let  $PQ$  be a convex arc, § 57, of a given curve  $S$ . Let  $\Delta s$  be the length of the arc  $PQ$ , and  $\Delta \phi$  the circular measure of the angle which the tangent at  $Q$  makes with that at  $P$ .

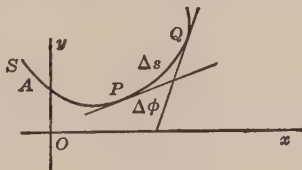


FIG. 51.

The ratio  $\Delta \phi / \Delta s$  is called the *mean curvature* of the arc

$PQ$ , expressed in radians per unit of arc. When  $Q$  is made to move along  $S$  into coincidence with  $P$ ,  $\Delta\phi/\Delta s$  approaches the limit  $d\phi/ds$ . This limit is called the *curvature* of  $S$  at  $P$ , and is denoted by  $K$ . Hence, by definition,

$$K = \frac{d\phi}{ds} \quad (1)$$

We may interpret  $s$  as the length of the arc from any fixed point  $A$  on  $S$  to  $P$ ,  $\phi$  as the angle which the tangent at  $P$  makes with  $Ox$ , and  $K = d\phi/ds$  as the rate of change of  $\phi$  with respect to  $s$  (in radians per unit of arc).

When no convention is made as to the positive sense of motion along  $S$ , the sign of  $K$  is ignored and  $|K|$  is called the *curvature* of  $S$  at  $P$ .

*The curvature of a circle is the same at every point and is the reciprocal of the radius.*

For since the angles  $PDE$  and  $PCQ$  are equal, we have  $\Delta s = r \Delta\phi$ . Hence

$$\frac{\Delta\phi}{\Delta s} = \frac{1}{r} \quad \text{and therefore} \quad \frac{d\phi}{ds} = \frac{1}{r}$$

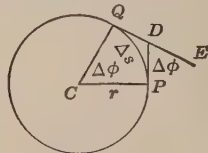


FIG. 52.

**79. Radius of curvature.** The reciprocal of the curvature  $K$  is called the *radius of curvature* of  $S$  at  $P$ , and is denoted by  $R$ . Hence, by definition

$$R = \frac{ds}{d\phi} \quad (1)$$

When the equation of  $S$  is given in rectangular coordinates,  $R$  may be found as follows: Let  $\phi$  be the circular measure of the positive or negative acute angle which the tangent at  $P$  makes with  $Ox$ ; then  $-\pi/2 < \phi < \pi/2$ , and therefore, § 65,  $\phi = \tan^{-1}(dy/dx)$ .

We have, § 55 (3),  $R = \frac{ds}{d\phi} = \frac{ds}{dx} \bigg/ \frac{d\phi}{dx}$

But, § 77 (1),  $\frac{ds}{dx} = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2}$

and  $\frac{d\phi}{dx} = \frac{d}{dx} \tan^{-1} \left( \frac{dy}{dx} \right) = \frac{d^2y}{dx^2} / \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]$

Hence  $R = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2} / \frac{d^2y}{dx^2}$  (2)

The reciprocal of the fraction (2) is the curvature  $K$  at  $P$ . It is positive or negative according as  $d^2y/dx^2$  is  $+$  or  $-$  at  $P$ , that is, according as  $S$  is concave upward or downward. At a point  $P$  where  $dy/dx$  is 0, we have  $K = d^2y/dx^2$ .

EXAMPLE 1. Find  $R$  at the point  $(2, -1)$  on the curve  $x^3 + y^3 = 7$ .

$$x^3 + y^3 \frac{dy}{dx} = 0 \quad 2x + 2y \left( \frac{dy}{dx} \right)^2 + y^2 \frac{d^2y}{dx^2} = 0$$

Substituting  $x, y = 2, -1$ , we find

$$\frac{dy}{dx} = -4, \quad \frac{d^2y}{dx^2} = 28, \quad \therefore \text{by (2), } R = 17^{3/2}/28.$$

EXAMPLE 2. A locomotive is running on a track which has the shape of the parabola  $y = x^2$ , a mile being the unit of length. At what rate per mile is its direction changing when  $x = 1$ ?

Here  $dy/dx = 2x$ ,  $d^2y/dx^2 = 2$ .  $\therefore K = 2/(1 + 4x^2)^{3/2}$ . Hence when  $x = 1$ ,  $K = 2/5^{3/2}$  radians/mile.

EXAMPLE 3. Find the following radii of curvature.

- |                                      |                                      |
|--------------------------------------|--------------------------------------|
| 1. $y = 1/x^2$ at $x = -1$           | 2. $y = \sin x$ at $x = \pi/2$       |
| 3. $y = \tan x$ at $x = \pi/4$       | 4. $y = e^x$ at $x = 0$              |
| 5. $y^2 - y + x = 0$ at $(0, 0)$     | 6. $y^2 - 2xy + x = 0$ at $(4/3, 2)$ |
| 7. $x^3 + y^3 = 3xy$ at $(3/2, 3/2)$ |                                      |

### 80. Center of curvature.

On the normal to  $S$  at  $P$  and to the concave side of  $S$ , take  $PC$  equal to the radius of curvature  $R$  at  $P$ . Then with  $C$  as center and  $CP$  as radius describe the circle  $S_1$ . We call  $S_1$  the

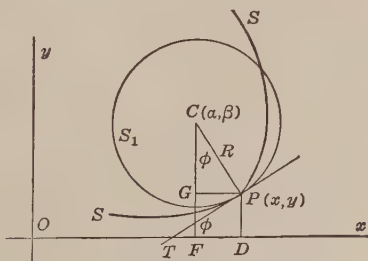


FIG. 53.

circle of curvature, and  $C$  the center of curvature of  $S$  at  $P$ . The circle  $S_1$  touches (and ordinarily crosses)  $S$  at  $P$ . Its curvature  $1/R$  is that of  $S$  at  $P$ .

Let the coordinates of  $P$  and  $C$  be  $(x, y)$  and  $(\alpha, \beta)$ . We can find  $\alpha, \beta$  in terms of  $x, y$  as follows:

$$\begin{aligned}\text{Since } GCP &= DTP = \phi \\ \alpha &= OF = OD - GP = x - R \sin \phi \\ \beta &= FC = DP + GC = y + R \cos \phi\end{aligned}\quad (3)$$

Hence, expressing  $\cos \phi$  and  $\sin \phi$  in terms of  $\tan \phi = dy/dx$  and using the formula for  $R$ , § 79 (2),

$$\alpha = x - \frac{dy}{dx} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right] / \frac{d^2y}{dx^2}, \quad \beta = y + \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right] / \frac{d^2y}{dx^2} \quad (4)$$

These formulas (3), (4) will be found true for all cases of the figure if it be remembered that since  $-\pi/2 < \phi < \pi/2$ ,  $\cos \phi$  is always  $+$  and  $\text{sgn} \sin \phi = \text{sgn} (dy/dx)$ , and that, when  $S$  is concave downward,  $d^2y/dx^2$  is  $-$ .

EXAMPLE. For  $y = x^3$  at  $(1, 1)$  we have  $dy/dx = 3x^2 = 3$ ,  $d^2y/dx^2 = 6x = 6$ . Hence, by (2) and (4),  $\alpha = -4$ ,  $\beta = 8/3$ ,  $R = 10^{3/2}/6$ , and the circle of curvature is  $(x + 4)^2 + (y - 8/3)^2 = 10^3/6^2$ .

**81. The evolute.** When  $P$  moves along  $S$ ,  $C$  traces a curve  $S'$ . This curve  $S'$  is called the *evolute* of  $S$ . If the equation of  $S$  be given in the form  $y = f(x)$ , we can express the second members of (4) in terms of the  $x$  of  $P$ ; the equations (4) then become parametric equations of  $S'$  in terms of the parameter  $x$ .

The normal  $PC$  to  $S$  at  $P$  is the tangent to  $S'$  at  $C$ . And if  $P_1, P_2$  be two points of  $S$ , and  $C_1, C_2$  the corresponding points of  $S'$ , and if  $PC$  increases only or decreases only as  $P$  moves from  $P_1$  to  $P_2$ , then  $\text{arc } C_1C_2 = |P_2C_2 - P_1C_1|$ .

For, differentiating the equations (3) with respect to  $x$ , we get

$$\frac{d\alpha}{dx} = 1 - \frac{dR}{dx} \sin \phi - R \cos \phi \frac{d\phi}{dx} \quad \frac{d\beta}{dx} = \frac{dy}{dx} + \frac{dR}{dx} \cos \phi - R \sin \phi \frac{d\phi}{dx}$$

But, § 77 (3), § 79 (1),

$$R \cos \phi \frac{d\phi}{dx} = \frac{ds}{d\phi} \frac{dx}{ds} \frac{d\phi}{dx} = 1, \quad R \sin \phi \frac{d\phi}{dx} = \frac{ds}{d\phi} \frac{dy}{ds} \frac{d\phi}{dx} = \frac{dy}{dx}.$$

$$\text{Hence} \quad \frac{d\alpha}{dx} = -\frac{dR}{dx} \sin \phi \quad (a) \quad \frac{d\beta}{dx} = \frac{dR}{dx} \cos \phi \quad (b)$$

1 Dividing (b) by (a) gives  $\frac{d\beta}{d\alpha} = -\cot \phi = -1 \left/ \frac{dy}{dx} \right.$ , which shows that  $PC$  touches  $S'$  at  $C$ .

$$2. \text{ Squaring (a) and (b) and adding gives } \left( \frac{d\alpha}{dx} \right)^2 + \left( \frac{d\beta}{dx} \right)^2 = \left( \frac{dR}{dx} \right)^2$$

Therefore  $(ds')^2 = (dR)^2$ , where  $ds'$  is the differential of arc of  $S'$ , § 77 (2). Hence if  $R = PC$  increases as  $C$  moves from  $C_1$  to  $C_2$ , and  $s'$  is the length of the arc  $C_1C$ , we have  $ds' = dR$  and therefore  $s' = R + c$ , where  $c$  is a constant, § 104. When  $C$  is at  $C_1$  this equation becomes  $0 = P_1C_1 + c$ , and when  $C$  is at  $C_2$  it becomes arc  $C_1C_2 = P_2C_2 + c$ ; hence, eliminating  $c$ , arc  $C_1C_2 = P_2C_2 - P_1C_1$ .

If  $R$  decreases as  $C$  moves from  $C_1$  to  $C_2$ , it increases as  $C$  moves from  $C_2$  to  $C_1$ ; hence arc  $C_2C_1 = P_1C_1 - P_2C_2$ .

EXAMPLE. Find the evolute of the parabola  $y^2 = 4x$ .

Substituting  $y = 2x^{1/2}$  ( $OA$ ) in (4) gives

$$\alpha = 2 + 3x, \beta = -2x^{3/2} \quad (C_0B')$$

Substituting  $y = -2x^{1/2}$  ( $OB$ ) in (4) gives

$$\alpha = 2 + 3x, \beta = 2x^{3/2} \quad (C_0A')$$

Eliminating the parameter  $x$ , gives  $\beta^2 = \frac{4}{27}(\alpha - 2)^3$  ( $A'C_0B'$ )

$$\text{Arc } C_0C = PC - OC_0 = PC - 2.$$

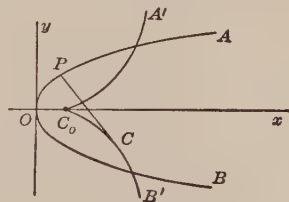


FIG. 54.

### EXERCISE XIII

1. Show that if  $x = \phi(t)$ ,  $y = \psi(t)$ , then  $ds = [\phi'^2(t) + \psi'^2(t)]^{1/2} dt$ .
2. For the ellipse  $x = a \cos \theta$ ,  $y = b \sin \theta$ , p. 63, Ex. 19, show that  $ds = a[1 - e^2 \cos^2 \theta]^{1/2} d\theta$ , where  $e$  denotes the eccentricity, so that  $a^2e^2 = a^2 - b^2$ .
3. For the cycloid  $x = a(\theta - \sin \theta)$ ,  $y = a(1 - \cos \theta)$ , § 94, show that  $ds = 2a \sin(\theta/2) d\theta$ .

4. For the parabola  $y^2 = 4ax$  find  $ds$  in terms of  $x$  and  $dx$ , also in terms of  $y$  and  $dy$ .

5. Show that the curvature at a point of inflection is 0. What is the curvature of a straight line?

6. Draw the parabola  $y^2 = 4x$  and its circle of curvature at the point where  $y = 2$ .

7. Find the following circles of curvature:

1.  $y = \cos x$  where  $x = \pi$     2.  $y^2 + xy - 15 = 0$  at  $(2, 3)$

3.  $y^3 + x - 2y = 0$  at  $(1, 1)$

8. For the equilateral hyperbola  $xy = a^2$ , show that

$$R = (x^2 + y^2)^{3/2} / 2a^2$$

9. For the ellipse  $x^2/a^2 + y^2/b^2 = 1$ , show that

$$R = (a^4y^2 + b^4x^2)^{3/2} / a^4b^4 = (a^2 - e^2x^2)^{3/2} / ab$$

10. At what points of the curve  $y = x^3$  is  $|R|$  a minimum?

11. Show that  $\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2} \bigg/ \frac{d^2x}{dy^2} = - \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \bigg/ \frac{d^2y}{dx^2}$ .

12. By aid of Ex. 11, find  $R$  for the curve  $x = y + y^2 + y^3$  at the point where  $y = -1$ .

13. For the curve  $x = \phi(t)$ ,  $y = \psi(t)$ , show that

$$R = [\phi'^2(t) + \psi'^2(t)]^{3/2} / [\phi'(t)\psi''(t) - \phi''(t)\psi'(t)]$$

14. Show for the cycloid (Ex. 3) that  $R = -4a \sin(\theta/2)$

15. For a curve  $r = f(\theta)$  in polar coordinates, show that

$$R = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2} \bigg/ \left[r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2r}{d\theta^2}\right]$$

16. Find  $R$  for the curves  $r = e^{k\theta}$  and  $r = a(1 - \cos \theta)$ .

17. Show that the circle  $(x - a)^2 + (y - b)^2 = r^2$  which passes through the point  $P(x, y)$  of the curve  $S$  and has at  $P$  the same values of  $dy/dx$  and  $d^2y/dx^2$  that  $S$  has is the circle of curvature of  $S$  at  $P$ .

18. Show that the evolute of the ellipse  $x^2/a^2 + y^2/b^2 = 1$  has the equation  $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$ .

19. Let  $N$  be the point where the normal to the curve  $S$  at the point  $P$  is met by the normal at  $Q$ . We have  $PN = PQ \sin PQN / \sin PNQ$ . Show that  $\lim_{Q \rightarrow P} PN = R$ .

20. A point  $P$  is moving away from  $O$  on the upper half of the parabola  $y^2 = x$ . Its rate of motion along the curve is constant, 2 ft./sec. How fast is its direction changing, and how fast are its projections on  $Ox$  and  $Oy$  moving, when its  $x$  is 3 ft.?

## X. CURVILINEAR MOTION

**82. Vectors.** The displacement of a point from the position  $A$  to the position  $B$  may be represented by the directed line segment  $AB$ . Such a directed line segment is called a *vector*. In this chapter  $AB$  will mean vector  $AB$ .

Two vectors  $AB$  and  $CD$  are *equal* when and only when they have the same length and direction.

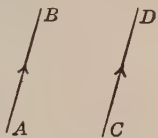


FIG. 55.

If  $k$  be any real number,  $kAB$  denotes the vector whose length is  $|k|$  times the length of  $AB$  and which has the same direction as  $AB$  or the opposite direction according as  $k$  is positive or negative. In particular,

$$-AB = BA \quad (1)$$

**83. Vector addition.** The displacement of a point from  $A$  to  $B$ , followed by its displacement from  $B$  to  $C$  is equivalent to the single displacement from  $A$  to  $C$ . Corresponding to this, the vector  $AC$  is called the *sum* of the vectors  $AB$  and  $BC$ .

Hence, by definition,

$$AB + BC = AC \quad (2)$$

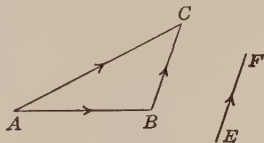


FIG. 56.

To add  $EF$  to  $AB$ , take  $BC = EF$  and then apply (2).

If  $a, b, c$  are vectors, and  $k$  any real number, then (as with numbers)

$$1. \ a + b = b + a \qquad 2. \ (a + b) + c = a + (b + c)$$

$$3. \ k(a + b) = ka + kb$$



The proofs of 1. and 3. are as follows; that of 2. is left to the reader.

$$\begin{aligned}
 a + b &= AB + BC \\
 &= AC \\
 &= AD + DC \\
 &= b + a
 \end{aligned}$$

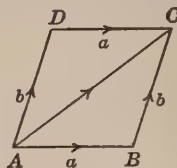


FIG. 57.

$$\begin{aligned}
 \text{Take } A'B &= kAB, BC' = kBC \\
 \text{Then } ka + kb &= A'B + BC' \\
 &= A'C' = kAC \\
 &= k(a + b)
 \end{aligned}$$

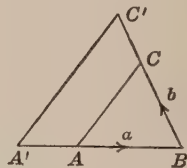


FIG. 58.

We define  $AC - AB$  (Fig. 56) as  $AC + (-AB)$  or as  $AC + BA = BA + AC = BC$ . Hence

$$AC - AB = BC \quad (3)$$

**84. Vector components.** The relation  $AC = AB + BC$  is also read:  $AB$  and  $BC$  are *components* of  $AC$ , and  $AC$  is the *resultant* of  $AB$  and  $BC$ . When  $AB$ ,  $BC$  are parallel to  $Ox$ ,  $Oy$ , they are called the  $x$ - and  $y$ -components of  $AC$ .

**85. Velocity.** A point  $P$  is supposed to be moving on a curve  $C$ . Let  $P$  and  $P'$  be its positions at the instants  $t$  and  $t + \Delta t$ , and let  $\Delta s$  denote the length of the arc  $PP'$ . Produce the vector  $PP'$  to a point  $U$  such that

$$\frac{PP'}{\Delta t} = PU$$

When  $\Delta t \rightarrow 0$ ,  $PU$  will approach as limit a vector  $PV$  tangent to  $C$  at  $P$ . This vector

$$PV = \lim_{\Delta t \rightarrow 0} \frac{PP'}{\Delta t} \quad (1)$$

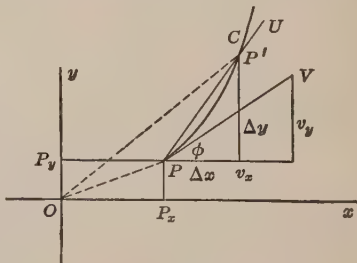


FIG. 59.

is called the *velocity* of the moving point  $P$  at the instant  $t$ ; for it represents in magnitude and direction the time rate of displacement of  $P$  at the instant  $t$ .

Let  $v$  denote the length of  $PV$ , and  $\overline{PP'}$  the length of  $PP'$ ; then

$$\lim_{\Delta t \rightarrow 0} \overline{PP'}/\Delta t = \lim_{\Delta t \rightarrow 0} \Delta s/\Delta t, \quad \text{and therefore}$$

$$v = \frac{ds}{dt} \quad (2)$$

Let the coordinates of  $P$  be  $x, y$ , and those of  $P'$ ,  $x + \Delta x, y + \Delta y$ . Then, § 83, 3.,

$$\frac{\overline{PP'}}{\Delta t} \text{ has the } x\text{- and } y\text{-components } \frac{\Delta x}{\Delta t} \text{ and } \frac{\Delta y}{\Delta t}$$

Hence the  $x$ - and  $y$ -components of  $PV$ , denoted by  $v_x$  and  $v_y$ , are

$$v_x = \frac{dx}{dt} \quad \text{and} \quad v_y = \frac{dy}{dt} \quad (3)$$

the same, therefore, as the velocities of  $P_x$  and  $P_y$ , the projections of  $P$  on  $Ox$  and  $Oy$ .

If  $\phi$  is the angle which  $PV$  makes with the direction  $Ox$ ,

$$\begin{aligned} v_x &= v \cos \phi & v_y &= v \sin \phi \\ v &= (v_x^2 + v_y^2)^{1/2} & \tan \phi &= v_y/v_x \end{aligned} \quad (4)$$

Evidently  $PV$  is determined when  $v$  and  $\phi$  are known. We find  $v$  and  $\phi$  by substituting in (4) the values of  $v_x, v_y$  obtained by (3).<sup>1</sup>

We call  $v$  the *speed* and  $\phi$  the *direction angle* of the motion at  $P$ .

EXAMPLE. If the equations of  $C$  in terms of  $t$  are  $x = t^2, y = 2t$ , then  $v_x = 2t, v_y = 2$ ; hence at  $t = 2$  we have  $v_x = 4, v_y = 2 \therefore v = 2\sqrt{5}$  and  $\tan \phi = 1/2$ ; also  $v_x, v_y$  are  $+$   $\therefore 0 < \phi < \pi/2$ .

**86. Derivative of a vector.** We may call  $OP$  the *position vector* of  $P$  referred to  $O$ . We have  $\overline{PP'} = \overline{OP'} - \overline{OP}$ ,

<sup>1</sup> The signs of  $v_x$  and  $v_y$  indicate which of the angles given by  $\tan \phi = v_y/v_x$  to take as the  $\phi$  of  $PV$ .

§ 83 (3), so that  $PP'$  is the increment of  $OP$  corresponding to the  $t$ -increment  $\Delta t$  and may be represented by  $\Delta(OP)$ . Hence if we extend to vectors the notion of derivative as defined in § 23, we have from § 85 (1)

$$PV = \lim_{\Delta t \rightarrow 0} \frac{\Delta(OP)}{\Delta t} = \frac{d}{dt} OP \quad (5)$$

*The velocity at  $P$  is the  $t$ -derivative of the position vector of  $P$ .*

**87. Acceleration.** 1. As in the case of rectilinear motion, the time rate of change of velocity is called *acceleration*. Hence by definition

$$\text{Acceleration at } P = \frac{d}{dt}(PV) \quad (6)$$

Take  $O_1\bar{x}$ ,  $O_1\bar{y}$  parallel to  $Ox$ ,  $Oy$  in Fig. 59, also  $O_1Q = PV$ . The coordinates of  $Q$  are  $\bar{x} = v_x$ ,  $\bar{y} = v_y$ . As the point  $P$  in Fig. 59 traces the curve  $C$ , the point  $Q$  traces a curve  $C'$ , which is called the *hodograph* or *velocity curve* of  $P$ .

Since  $d(PV)/dt = d(O_1Q)/dt$ , it follows from §§ 85, 86, that the acceleration at  $P$  is a vector  $QA$  tangent to the curve  $C'$  at  $Q$  and having the  $\bar{x}$ -,  $\bar{y}$ -, or  $x$ -,  $y$ -components

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \quad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \quad (7)$$

Let  $a$  denote the length of  $QA$ , and  $\phi'$  the angle which  $QA$  makes with  $O_1\bar{x}$  or  $Ox$ ; then

$$\begin{aligned} a_x &= a \cos \phi' & a_y &= a \sin \phi' \\ a &= (a_x^2 + a_y^2)^{1/2} & \tan \phi' &= a_y/a_x \end{aligned} \quad (8)$$

Evidently the acceleration  $QA$  is determined when its *magnitude*  $a$  and its *direction angle*  $\phi'$  are known; we find  $a$  and  $\phi'$  by substituting in (8) the values of  $a_x$ ,  $a_y$  given by (7).<sup>1</sup>

<sup>1</sup> The signs of  $a_x$  and  $a_y$  indicate which of the angles given by  $\tan \phi' = a_y/a_x$  is the  $\phi'$  of  $QA$ .

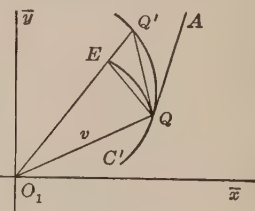


FIG. 60.

2. Let  $a_T$  and  $a_N$  denote the lengths of the components of  $QA$  in the directions of the tangent and normal to the curve  $C$  at  $P$ , that is, parallel and perpendicular to  $O_1Q$ . They can be found as follows:

Let  $O_1Q'$  correspond to  $t + \Delta t$ . With  $O_1$  as center and  $v$  as radius draw the arc  $QE$ . We have  $QQ' = QE + EQ'$ ; hence

$$\frac{QQ'}{\Delta t} \text{ has the components } \frac{EQ'}{\Delta t} \text{ and } \frac{QE}{\Delta t}$$

The directions approached by these components when  $\Delta t \rightarrow 0$  are those of the tangent and normal to  $C$  at  $P$ . Also the length of  $EQ'$  is  $\Delta v$  and that of arc  $QE$  is  $v\Delta\phi$ . Hence

$$a_T = \frac{dv}{dt} \qquad a_N = v \frac{d\phi}{dt} \quad (9)$$

Again if  $R$  be the radius of curvature of  $C$  at  $P$ , we have, by § 79 (1),

$$v \frac{d\phi}{dt} = v \frac{d\phi}{ds} \frac{ds}{dt} = \frac{v^2}{R}; \text{ hence } a_N = \frac{v^2}{R} \quad (10)$$

In the case of rectilinear motion, and in that case only,  $a_N$  is 0 at every instant (since  $d\phi/dt$  is then 0), and  $a_T$  is the entire acceleration.

When  $P$  describes equal arcs of  $C$  in equal intervals of time, its speed  $v = ds/dt$  is constant, hence  $dv/dt = 0$  and therefore  $a_N$  is the entire acceleration.

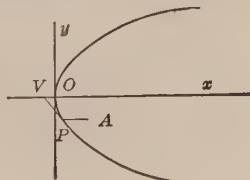


FIG. 61.

EXAMPLE 1. For the motion  $x = t^2/2$ ,  $y = t$ , (clockwise on  $y^2 = 2x$ ), we have  $v_x = t$ ,  $v_y = 1$ ,  $v = (t^2 + 1)^{1/2}$ ,  $\tan \phi = 1/t$ ,  $a_x = 1$ ,  $a_y = 0$ ,  $a = 1$ ,  $\tan \phi' = 0$ ,  $a_T = t/(t^2 + 1)^{1/2}$ ,  $a_N = -1/(t^2 + 1)^{1/2}$ .

The hodograph is  $y = 1$ .

At  $t = -1$ , we have (Fig. 61)

$$v = \sqrt{2}, \quad \phi = 3\pi/4 \text{ (PV)}; \text{ also } a = 1, \quad \phi' = 0 \text{ (PA)}.$$

EXAMPLE 2. A point  $P$  is moving away from  $O$  on the upper half of  $y^2 = 4x$  with the constant speed 10. What is the acceleration at  $(4, 4)$ ?

We have  $y^2 - 4x = 0$  (1)  $y \frac{dy}{dt} - 2 \frac{dx}{dt} = 0$  (2)

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 100 \quad (3)$$

Differentiating (2),  $\left(\frac{dy}{dt}\right)^2 + y \frac{d^2y}{dt^2} - 2 \frac{d^2x}{dt^2} = 0$  (4)

Differentiating (3),  $\frac{dx}{dt} \frac{d^2x}{dt^2} + \frac{dy}{dt} \frac{d^2y}{dt^2} = 0$  (5)

Setting  $x, y = 4, 4$  in (2), (3) gives  $\frac{dx}{dt} = 4\sqrt{5}$ ,  $\frac{dy}{dt} = 2\sqrt{5}$ . Hence, by (4), (5), we have  $a_x = 2$ ,  $a_y = -4$ ,  $a = 2\sqrt{5}$ .

**88. Rotation.** When the path of  $P$  is a circle about  $O$  as center, the motion is called a *rotation about  $O$* . Let  $r$  be the radius, and  $\theta$  the circular measure of  $xOP$ . Evidently the motion is completely determined when  $\theta$  is a known function of  $t$ .

$\frac{d\theta}{dt}$  is the *angular velocity*:  $\omega$

$\frac{d^2\theta}{dt^2}$  is the *angular acceleration*:  $\alpha$

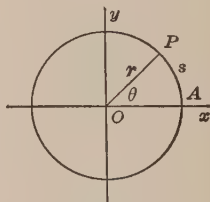


FIG. 62.

The length of the arc  $AP$  is  $r\theta$ , so that  $ds/dt = r d\theta/dt$ . The direction angle  $\phi$  of the velocity vector of  $P$  is  $\theta + \pi/2$ , so that  $d\phi/dt = d\theta/dt$ . Hence for the motion of  $P$

$$v = r \frac{d\theta}{dt} \quad a_T = r \frac{d^2\theta}{dt^2} \quad a_N = r \left(\frac{d\theta}{dt}\right)^2 \quad (11)$$

We may obtain  $v_x, v_y, a_x, a_y$  from  $x = r \cos \theta$ ,  $y = r \sin \theta$ .

EXAMPLE. A fly wheel of 4 ft. radius is making 150 revolutions a minute. (1) For a point  $P$  on its rim find  $v, a_T, a_N$  in feet per sec. (2) What constant angular acceleration would bring the wheel from rest to its present angular velocity in 1 min.?

(1) Since 1 revolution =  $2\pi$  radians,  $\omega = d\theta/dt = 150 \cdot 2\pi/60 = 5\pi$  rad./sec.

Hence  $v = 4 \cdot 5\pi = 20\pi$  ft./sec.  $a_T = 0$   $a_N = 4(5\pi)^2 = 100\pi^2$  ft./sec<sup>2</sup>.

(2) Calling the required acceleration  $k$ , we have  $d\omega/dt = k \therefore \omega = kt + \text{some constant } C$  (§ 104). But  $\omega = 0$  when  $t = 0$ ; hence  $C = 0$ , and  $\omega = kt$ . Therefore  $5\pi = k/60$ , that is,  $k = \pi/12$  rad./sec<sup>2</sup>.

#### EXERCISE XIV

1. Prove by vector addition that the diagonals of a parallelogram bisect each other.

2. Show that when the speed of a motion is constant the hodograph is a circle or part of one.

3. A fly wheel of 5 ft. radius is making 240 revolutions per minute. For a point  $P$  on its rim find  $v$  and  $a$  in feet and seconds. If the center be at  $O$ , find  $v_x, v_y, a_x, a_y$  when  $xOP = \pi/6$ . How many feet will  $P$  travel in 3 minutes? What constant retardation (negative angular acceleration) would bring the wheel to rest in 3 minutes?

4. The equations of motion of  $P$  are  $x = r \cos kt, y = r \sin kt$ ,  $k$  being positive. Show that  $P$  moves counterclockwise on the circle  $x^2 + y^2 = r^2$  with the constant speed  $kr$ , and with the constant acceleration  $k^2r$  directed toward  $O$ .

5. The equations of motion of  $P$  are  $x = 3 \cos t, y = 2 \sin t$ . Show that  $P$  moves counterclockwise on the ellipse  $x^2/9 + y^2/4 = 1$ . Find  $v, a_T, a_N, a$  when  $t = 0$ ; when  $t = \pi/4$ .

6. For each of the following motions find the path; also the vectors  $PV, PA$  when  $t = 1$ .

1.  $x = t + 2, y = 1 - t/2$       2.  $x = t + 1, y = t^2 + 2t$
3.  $x = 2t, y = 3 \sin t$ .

7. A point is moving on the line  $3x + 4y - 12 = 0$  in such a way that  $x = 4t^2$ . Find  $PV$  and  $PA$  when  $t = 1/2$ . Also find the hodograph.

8. Discuss the motion  $x = \dot{2} \sin t, y = \cos 2t$ , showing that it is oscillatory on the arc of the parabola  $y = 1 - x^2/2$  between the points  $(-2, -1)$  and  $(2, -1)$ , and that its hodograph is the curve  $y^2 = x^2(4 - x^2)$ .

9. A point  $P$  is moving away from  $O$  on the upper half of the curve  $3y^2 = 2x^3$  with the constant speed 1 in./sec. Find  $PV$  and  $PA$  at  $(6, 12)$ .

10. A point  $P$  is moving on the hyperbola  $xy = 4$  in such a manner that  $v_x = 6$  in./sec. Express  $v_y$  and  $a_y$  in terms of  $x$ . Draw  $PV$  and  $PA$  when  $P$  is at  $(2, 2)$ .

11. A train is running toward a station at  $O$  on a track  $y = x^2$  (1 m. being unit). If its speed at  $(1, 1)$  is 50 m.h., how rapidly is it then approaching  $Ox$ ,  $Oy$ , and  $O$ ?

12. A point makes a circuit of the ellipse  $x^2 - 4xy + 6y^2 = 34$ . Prove that there are certain positions in which the  $y$ -component of its velocity is double the  $x$ -component, and find these positions.

13. A projectile  $P$  starts from  $O$  with the speed  $v_0$  ft./sec. in a direction which makes the angle  $\alpha$  with the horizontal  $Ox$ . The resistance of the air being neglected, the position of  $P$  at the time  $t$  (in ft., sec.) is

$$x = v_0 t \cos \alpha \qquad y = v_0 t \sin \alpha - \frac{1}{2} g t^2 \qquad (1)$$

where  $g = 32$  is the acceleration of gravity. The equations (1) represent a parabola.

(1) Let  $R$  denote the *range*, that is, the distance from  $O$  to the point of  $Ox$  to which  $P$  falls;  $T$ , the time of flight;  $H$ , the greatest height. Prove that

$$R = \frac{v_0^2 \sin 2\alpha}{g} \qquad T = \frac{2 v_0 \sin \alpha}{g} \qquad H = \frac{v_0^2 \sin^2 \alpha}{2g} \qquad (2)$$

(2) Show that  $R$  is greatest when  $\alpha = \pi/4$ , and  $R = v_0^2/g$ . Show also that there are two angles  $\alpha$  which will give any range less than  $v_0^2/g$ .

(3) Find the speed  $v$ ; also when and where it is least.

(4) What is the smallest  $v_0$  that will give a range of ten miles?

14. A fielder is standing 300 ft. from the home plate when a ball is hit toward him for which  $v_0 = 120$  ft./sec. and  $\alpha = \pi/6$ . How fast must he run to catch the ball?

15. A man can throw a ball with the speed 100 ft./sec. What elevation must he give the ball if he is to hit a mark whose coordinates, referred to his throwing hand as origin, are  $(100 \text{ ft.}, 50 \text{ ft.})$ ?



## XI. CURVE TRACING

**89. Equations of the type  $y^2 = f(x)$ .** The graph of an equation of this type is confined to portions of the plane in which  $f(x)$  is positive or 0; for when  $f(x)$  is negative,  $y$  is imaginary. To each positive value of  $f(x)$  correspond the values  $\pm\sqrt{f(x)}$  of  $y$ ; hence the curve is symmetric with respect to  $Ox$ . In what follows,  $f(x)$  is a rational function, integral or fractional.

**EXAMPLE 1.** Find the graph of  $4y^2 = x^3 - 3x$  or  $2y = \pm(x^3 - 3x)^{1/2}$   
 $4f(x) = x^3 - 3x = (x + \sqrt{3})x(x - \sqrt{3})$

Hence  $f(x)$  is 0 at  $x = -\sqrt{3}, 0, \sqrt{3}$ , and is + for  $-\sqrt{3} < x < 0$  and for  $x > \sqrt{3}$ , and then only. The corresponding parts of the curve are an oval and an infinite branch, both symmetric to  $Ox$ . For great values of  $x$  the dominant term of  $x^3 - 3x$  is  $x^3$ ; hence for such values of  $x$  the graph approximates the curve  $4y^2 = x^3$  (in shape). Similarly for small values of  $x$  it approximates the curve  $4y^2 = -3x$ .

The turning points  $M, M'$  and the points of inflexion  $I, I'$  may be found as follows:

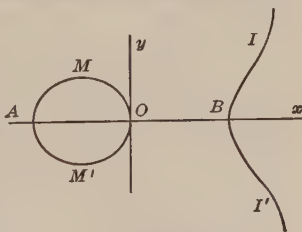


FIG. 63.

Differentiating the equation  $4y^2 = x^3 - 3x$  (1) twice gives

$$8y \frac{dy}{dx} = 3x^2 - 3 \quad (2) \qquad 8y \frac{d^2y}{dx^2} + 8\left(\frac{dy}{dx}\right)^2 = 6x \quad (3)$$

By (2),  $dy/dx = 0$  when  $^1 3x^2 - 3 = 0 \therefore$  by (1) at the points  $M(-1, 1/\sqrt{2}), M'(-1, -1/\sqrt{2})$

By (3),  $d^2y/dx^2 = 0$  when  $^1 4(dy/dx)^2 = 3x \therefore$  by (2), (1), when  $x^4 - 6x^2 - 3 = 0$  (4)

The equations (4) and (1) give the points  $I(2.5, 1.4), I'(2.5, -1.4)$ .

<sup>1</sup> Since, by (1),  $y \neq 0$  at these points.

**EXAMPLE 2.** Show that from the equation (2) in Ex. 1 it follows that the curve cuts  $Ox$  at right angles at the points  $A$ ,  $O$ , and  $B$ .

**90. Points on  $Ox$ .** The curve  $y^2 = f(x)$ , call it  $C$ , meets  $Ox$  at points where  $f(x)$  is 0. When  $f(a)$  is 0,  $f(x)$  can be expressed in the form  $(x - a)^r \phi(x)$ , where  $r \geq 1$  and  $\phi(a) \neq 0$  or  $\infty$ , § 50, 2.

1. If, as is ordinarily the case,  $r = 1$ , the curve  $C$  cuts  $Ox$  at right angles at  $A(a, 0)$ . For let  $P(x, y)$  be a point of  $C$  near  $A$ . The slope of  $AP$  is  $y/(x - a)$ ; hence the slope of  $C$  at  $A$  is  $\lim_{P \rightarrow A} [y/(x - a)]$ . But since  $y^2 = (x - a)\phi(x)$ , and  $\phi(a) \neq 0$ , we have

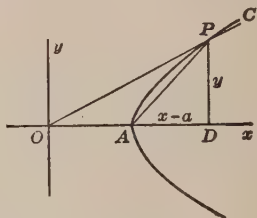


FIG. 64.

$$\lim_{P \rightarrow A} \left[ \frac{y}{x - a} \right]^2 = \lim_{P \rightarrow A} \frac{\phi(x)}{x - a} = \infty$$

2. When  $r > 1$ ,  $A$  is a *singular point*<sup>1</sup> of  $C$ . In particular, [see Fig. 65]

(1) When  $r = 2$  so that  $y^2 = (x - a)^2 \phi(x)$ , and  $\phi(a)$  is  $+$ , then  $\lim_{P \rightarrow A} [y/(x - a)]^2 = \phi(a)$ . Hence two branches of  $C$  pass through  $A$  with the slopes  $\pm \sqrt{\phi(a)}$ . We call  $A$  a *node*.

(2) When  $r = 3$  or a greater odd number,  $A$  is the end point of two branches of  $C$  both of which touch  $Ox$  at  $A$  and extend to one side only of  $A$ . We call  $A$  a *cusp*.

(3) When  $r = 4$  or a greater even number and  $\phi(a)$  is  $+$ , two branches of  $C$  touch  $Ox$  at  $A$  and extend to both sides of  $A$ . We call  $A$  a *point of osculation*.

(4) When  $r$  is even and  $\phi(a)$  is  $-$ ,  $A$  is an isolated point of  $C$ . It is called a *conjugate point*.

<sup>1</sup> A point of any curve  $F(x, y) = 0$  where both  $\partial F/\partial x$  and  $\partial F/\partial y$  are 0 is called a singular point (§ 42, 4.). For the case in which  $F(x, y) \equiv y^2 - f(x)$ , we have  $\partial F/\partial x = -f'(x)$ ,  $\partial F/\partial y = 2y$ ; and both  $f'(x)$  and  $y$  are 0 at  $A$  when  $r > 1$ .

**EXAMPLE 1.** Find the graph of  $y^2 = (x + 1)^2 x^3 (x - 1)^2 (x - 2)^4$ , showing that

$A(-1, 0)$  is a *conjugate point*

$O(0, 0)$  is a *cusp*

$B(1, 0)$  is a *node*, slopes  $\pm 2$

$C(2, 0)$  is a *point of osculation*

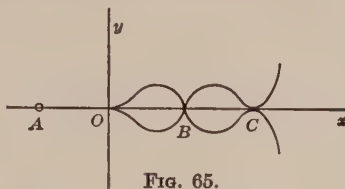


FIG. 65.

**EXAMPLE 2.** Find the graphs of the following equations:

1.  $y^2 = x^2(a^2 - x^2)$

2.  $y^2 = ax^3 - x^4$

3.  $y^2 = x^4(a^2 - x^2)$

4.  $y^2 = x^2(x^2 - a^2)$

5.  $y^2 = x^2(x - a)$

6.  $y^2 = x^3 + 1$

### 91. Infinite branches. Asymptotes. Call the curve $C$ .

1. If  $f(x) \rightarrow +\infty$  when  $x \rightarrow b$  or when  $x \rightarrow b$ , then  $x = b$  is an asymptote of a branch of  $C$  above  $Ox$  and of the corresponding branch below. Thus,  $x = 0$  is an asymptote of  $y^2 = x + 1/x$ .

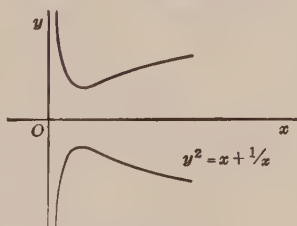


FIG. 66.

2. The slope of  $OP$  in Fig. 64 is  $y/x$ , and  $(y/x)^2 = f(x)/x^2$ . Let  $x \rightarrow +\infty$ ; if  $f(x)$  remains positive as  $x$  increases, the upper and lower halves of  $C$  extend infinitely far to the right, and according as  $f(x)/x^2 \rightarrow \infty$ ,  $0$ , or  $l(>0)$ , their directions will approach those of  $Oy$ ,  $Ox$ , or the lines  $y = \pm l^{1/2}x$ . Similarly when  $x \rightarrow -\infty$ .

3. The curve  $C$  has oblique or horizontal asymptotes if  $f(x)$  can be reduced to the form  $u^2 + v$ , where  $u$  is real and of the first degree in  $x$  or constant, and  $v/u \rightarrow 0$  when  $x \rightarrow \infty$ , the asymptotes being  $y = u$  and  $y = -u$ .

$$\begin{aligned} \text{For } \pm y &= (u^2 + v)^{1/2} = |u| + [(u^2 + v)^{1/2} - |u|] \\ &= |u| + \frac{v}{[(u^2 + v)^{1/2} + |u|]} \end{aligned}$$

and the fractional term  $\rightarrow 0$  when  $v/u \rightarrow 0$ .  $\therefore$  when  $x \rightarrow \infty$ . Also  $\lim_{x \rightarrow \infty} y/x = \pm \lim_{x \rightarrow \infty} u/x =$  the slope of  $y = \pm u$ . Hence the lines  $y = u$  and  $y = -u$  are asymptotes, § 51, 2.

4. This reasoning also shows that the graph of

$$(y - mx - c)^2 = u^2 + v \quad \text{or} \quad y = mx + c \pm (u^2 + v)^{1/2}$$

has the asymptotes  $y = mx + c + u$  and  $y = mx + c - u$ .

EXAMPLE 1. Find the asymptotes of the hyperbola  $y^2 = x^2 + 2x$ .

$$x^2 + 2x = (x + 1)^2 - 1, \text{ and } \lim_{x \rightarrow \infty} [1/(x + 1)] = 0$$

Hence the asymptotes are  $y = x + 1$ ,  $y = -(x + 1)$ .

EXAMPLE 2. Solving  $y^2 - 2xy + 1 = 0$  for  $y$  gives

$$y = x \pm (x^2 - 1)^{1/2}; \text{ and } \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

Hence the graph (an hyperbola) has the asymptotes  $y = x \pm x$ , or  $y = 2x$  and  $y = 0$ .

EXAMPLE 3. Show that  $y^2 = (x + 1)/x = 1 + 1/x$  has the asymptotes  $y = 1$ ,  $y = -1$  and  $x = 0$ .

EXAMPLE 4. Trace the curve  $y^2 = (x^3 + x^2)/(x - 1)$ . The points on  $Ox$ , given by  $(x^3 + x^2) = (x + 1)x^2 = 0$ , are  $A(-1, 0)$  and  $O$ ; and  $O$  is a conjugate point since  $f(x)$  is  $+$  only when  $x < -1$  or  $x > 1$ . The line  $x - 1 = 0$  is an asymptote. Also, dividing by  $x - 1$ ,

$$\begin{aligned} f(x) &= x^2 + 2x + 2 + \frac{2}{x - 1} \\ &= (x + 1)^2 + \left(1 + \frac{2}{x - 1}\right), \end{aligned}$$

and

$$\left(1 + \frac{2}{x - 1}\right)/(x + 1) \rightarrow 0 \quad \text{when } x \rightarrow \infty.$$

Hence the lines  $y = x + 1$  and  $y = -(x + 1)$  are asymptotes.

EXAMPLE 5. Show that the curve  $y^2 = x^3(x - 1)^2/(x + 1)^3$  has a cusp at  $O$ , a node at  $(1, 0)$ , and the asymptotes  $x + 1 = 0$ ,  $y = x - 5/2$ ,  $y = -x + 5/2$ . Trace it.

EXAMPLE 6. Trace the following curves:

- |                                |                            |                            |
|--------------------------------|----------------------------|----------------------------|
| 1. $y^2 = (x - 1)/x$           | 2. $y^2 = (1 - x)/x$       | 3. $y^2 = (1 - x)^3/x$     |
| 4. $y^2 = (x - 1)^3/x$         | 5. $y^2 = x^2/(a^2 - x^2)$ | 6. $y^2 = x^2/(x^2 - a^2)$ |
| 7. $y^2 = (x^2 + x^3)/(1 - x)$ | 8. $y^2 = x^2/(x^2 + 1)$   | 9. $y^2 = x^4/(x^2 + 1)$   |

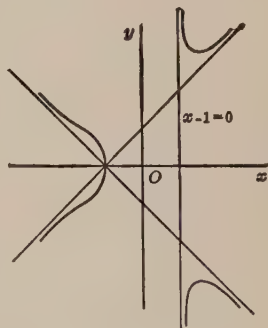


FIG. 67.

**92. Conics.** The general equation of the second degree in  $x, y$  is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

where  $a, h, \dots$  denote constants. The examples  $x^2 - y^2 = 0$ ,  $x^2 + y^2 = 0$ ,  $x^2 + y^2 + 1 = 0$  show that such an equation may represent a pair of straight lines, or a conjugate point, or may have no real solution and therefore no graph. In all other cases it represents a conic, and this conic is an

ellipse, parabola, or hyperbola according as  $h^2 - ab \lessgtr 0$

**EXAMPLE 1.** Trace the curve  $x^2 - 2xy + 2y^2 - 2x = 0$ .

Solving for  $y$ , 
$$y = \frac{x}{2} \pm \frac{(4x - x^2)^{1/2}}{2}$$

$4x - x^2 = x(4 - x)$  is 0 at  $x = 0$  and 4, and  $+$  when  $0 < x < 4$

Hence the curve lies between the lines  $x = 0$  and  $x = 4$ . Draw these lines and also the "guide line"  $y = x/2$ . Then for any  $x = OX$  between  $x = 0$  and  $x = 4$  we get two curve points  $P_1$  and  $P_2$  by decreasing and increasing the ordinate  $XQ$  of  $y = x/2$  by the value of  $(4x - x^2)^{1/2}/2$  for  $x = OX$ . When  $X$  is at  $O$  or  $A$ , the points  $P_1, P_2$  coincide; the curve touches  $Oy$  at  $O$  and  $AB$  at  $B$ . Since the curve is a conic and is closed it is an *ellipse*; observe that  $h^2 - ab = 1^2 - 1 \cdot 2 = -1 < 0$ .

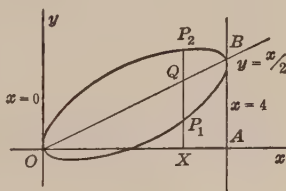


FIG. 68.

Since  $OB$  bisects a system of parallel chords, it is a *diameter*. Its mid-point is the *center* of the ellipse. The highest and lowest curve points may be found by solving the given equation  $f(x, y) = 0$  and  $df/\partial x = 2x - 2y - 2 = 0$  for  $x, y$ .

**EXAMPLE 2.** Trace the curve  $(y - x)^2 = x + 2$ .

Solving for  $y$ , 
$$y = x \pm (x + 2)^{1/2}$$

The curve extends indefinitely to the right of the line  $x + 2 = 0$  and tends to parallelism with the guide line  $y = x$ . As the curve is a conic and consists of a single infinite part it is a *parabola*. It cuts  $Ox$  at  $x = -1$  and 2; and  $Oy$  at  $y = \pm\sqrt{2}$ .

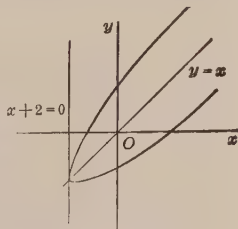


FIG. 69.

EXAMPLE 3. Trace the curve  $y^2 + xy - 2x^2 - 4 = 0$ .

Solving for  $y$ ,  $y = -\frac{x}{2} \pm \frac{(9x^2 + 16)^{1/2}}{2}$ .

The radicand is of the form  $u^2 + v$  where  $\frac{v}{u} = \frac{16}{3x} \rightarrow 0$  when  $x \rightarrow \infty$ . Hence, § 91, 3., the

curve has the asymptotes  $y = -\frac{x}{2} \pm \frac{3x}{2}$ , that is,

$y = x$  and  $y = -2x$ ; and as  $(9x^2 + 16)^{1/2} > 3|x|$ , it lies with respect to these asymptotes as in

Fig. 70. Since it is a conic with asymptotes, it is an *hyperbola*.

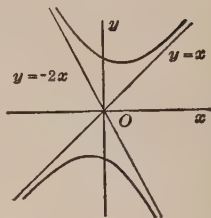


FIG. 70.

EXAMPLE 4. Trace the following:

1.  $y^2 - x^2 - 2y = 0$

2.  $x^2 - xy + y^2 - 3 = 0$

3.  $4y^2 + 4xy + x^2 + x = 0$

4.  $y^2 + xy - 2x^2 + 4 = 0$

5.  $y^2 - 2xy + 2x = 0$

6.  $y^2 + 4x^2 - 2x + 2y = 0$

7.  $y^2 - 2xy + 2x^2 - x - 2 = 0$

8.  $y^2 - 2xy - 3x^2 + 2x - 2y + 1 = 0$

**93. Other algebraic curves.** The graphs of algebraic equations  $f(x, y) = 0$  of a degree higher than two in both  $y$  and  $x$  will be considered later. The following example illustrates a method applicable to equations of degree  $n > 1$  in  $x, y$ , which contain terms of degrees  $n$  and  $n - 1$  only.

EXAMPLE 1. Trace the curve  $x^3 + y^3 - 3axy = 0$  (1) [*Folium of Descartes*]

The equation  $y = tx$  (2) represents the line  $OP$  through  $O$  with the slope  $t$ .

Solving (1), (2) for  $x, y$  shows that  $OP$  cuts the curve at  $O$  twice and at the point

$$P: x = \frac{3at}{t^3 + 1} \quad y = \frac{3at^2}{t^3 + 1} \quad (3)$$

From (3) it follows that when  $t \rightarrow 0$ , then  $P \rightarrow O$ , and  $OP \rightarrow Ox$ ; also when  $t \rightarrow \infty$ , then  $P \rightarrow O$  and

$OP \rightarrow Oy$ . Hence  $Ox$  and  $Oy$  are tangents to the curve at  $O$ .

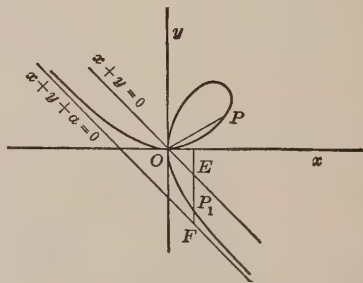


FIG. 71.

When  $t \rightarrow -1$ ,  $x$  and  $y$  become infinite. This leads us to look for an asymptote parallel to the line  $y + x = 0$ . We find this asymptote to be the line  $y + x + a = 0$  (4). For [footnote, p. 53]

$$EP_1 = y + x = \frac{3at^2 + 3at}{t^3 + 1} = \frac{3at}{t^2 - t + 1}, \text{ which } \rightarrow -a \text{ when } t \rightarrow -1.$$

Hence  $FP_1 \rightarrow 0$  when  $t \rightarrow -1$ : that is,  $y + x + a = 0$  is an asymptote. Trace the curve as  $t$  increases from  $-\infty$  to  $\infty$ .

EXAMPLE 2. Show that the curve  $y^3 - x^3 = 3x^2$  has the asymptote  $y = x + 1$  and a cusp at  $O$  with the tangent  $x = 0$ , and also meets  $Ox$  at  $x = -3$ . Trace the curve.

**94. The cycloid.** Suppose a circle to roll, without sliding, on a straight line; the curve traced by any point on its cir-

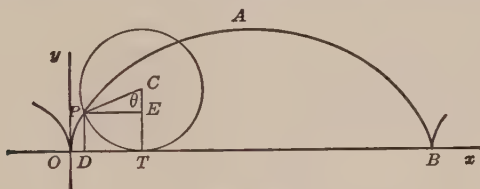


FIG. 72.

cumference is called a *cycloid*. In Fig. 72 the line is  $Ox$ , the circle rolls to the right and the initial position of  $P(x, y)$  is  $O$ .

Let  $a$  = radius,  $\theta = TCP$ . Then

$$OT = \text{arc } TP = a\theta,$$

$$x = OD = OT - PE = a\theta - a \sin \theta$$

$$y = DP = TC - EC = a - a \cos \theta.$$

Hence the equations of this cycloid are

$$x = a(\theta - \sin \theta) \qquad y = a(1 - \cos \theta) \qquad (1)$$

As  $\theta$  increases clockwise from 0 to  $2\pi$ ,  $P$  traces the arch  $OAB$ ;  $O$  and  $B$  are cusps.

**95. Epicycloid, hypocycloid.** 1. As the circle  $CP$  rolls on the circle  $OA$ , Fig. 73, the point  $P(x, y)$ , starting at  $A$ , traces an *epicycloid*.



Let  $a = OT$ ,  $b = TC$ ,  $\theta = AOT$ ,  $\phi = TCP$ . Then

$$x = OD = OF + EP = (a + b) \cos \theta + b \sin ECP$$

$$y = DP = FC - EC = (a + b) \sin \theta - b \cos ECP$$

But arc  $AT = \text{arc } TP \therefore a\theta = b\phi$ ;

$$\text{also } ECP = \phi - \left(\frac{\pi}{2} - \theta\right).$$

Hence

$$x = (a+b) \cos \theta - b \cos \frac{a+b}{b} \theta,$$

$$y = (a+b) \sin \theta - b \sin \frac{a+b}{b} \theta \quad (1)$$

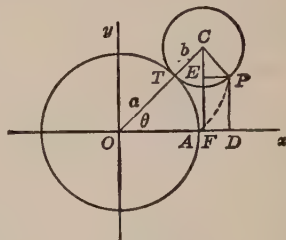


FIG. 73.

2. When the rolling circle is inside, the point  $P$  traces a *hypocycloid* whose equations—got by replacing  $b$  by  $-b$  in the equations (1)—are

$$x = (a-b) \cos \theta + b \cos \frac{a-b}{b} \theta, \quad y = (a-b) \sin \theta - b \sin \frac{a-b}{b} \theta \quad (2)$$

### EXERCISE XV

1. Show that the hypocycloid for which  $a = 2b$  is the horizontal diameter of the fixed circle counted twice. Prove this also by elementary geometry.

2. Show that when  $a = 4b$ , the equations of § 95 (2) reduce to  $x = a \cos^3 \theta$ ,  $y = a \sin^3 \theta \therefore$  to  $x^{2/3} + y^{2/3} = a^{2/3}$ . The curve is the *hypocycloid of four cusps*. Trace it.

3. Show that the epicycloid for which  $b = a$  is a cardioid (p. 78, Fig. 45) whose polar equation referred to the point  $A$  in Fig. 73 as origin is  $r = 2a(1 - \cos \theta)$ .

4. Show that the slope of the cycloid at the point  $P$ , Fig. 72, is  $\cot \theta/2$ ; hence that the tangent at  $P$  passes through the highest point  $F$  of the rolling circle, and that  $PT$  is the normal<sup>1</sup> at  $P$  and has the length  $2a \sin \theta/2$ .

<sup>1</sup> This may also be shown as follows: When the circle, starting from the position in Fig. 72, turns through the angle  $\delta\theta$ , the point  $T$  of the circle moves a distance  $\epsilon$  which is very small as compared with  $\delta\theta$  and such that  $\epsilon/\delta\theta \rightarrow 0$  when  $\delta\theta \rightarrow 0$ . Hence we may regard the motion of the circle as at every instant one of rotation about its point of tangency  $T$ , which means that at the instant  $P$  is moving in a direction perpendicular to  $TP$ . Similarly, in Fig. 73,  $TP$  is normal to the curve that is being traced by  $P$ .

5. Prove that the radius of curvature of the cycloid at  $P$  is  $4a \sin \theta/2 = 2PT$ . Hence show geometrically that the evolute of the cycloid  $OAB$  is an equal cycloid generated by a point  $Q$  of the circumference of a circle of radius  $a$  which rolls on the line  $y = -2a$ .

6. Show that a point  $Q$  on the radius  $CP$  of the rolling circle in Fig. 72, such that  $CQ = b$ , generates the curve  $x = a\theta - b \sin \theta$ ,  $y = a - b \cos \theta$  (a *trochoid*).

7. How many cusps has an epicycloid for which  $b/a = 2/3$ ?  $\sqrt{2}$ ?

8. Show that when  $d\theta/dt$  is constant the speed of  $P$  in Fig. 72 is proportional to  $PT$ .

9. A rolling wheel of 2 ft. radius moves along the track at the rate of 60 ft./sec. What is its angular velocity  $d\theta/dt$ ? Find, in magnitude and direction, the velocity and acceleration of a point  $P$  on the rim when  $\theta = 0, \pi/2, 3\pi/4, \pi, 3\pi/2$ .

10. Show that  $8y^2 + 12xy + 17x^2 = 20$  (1) represents an ellipse with its center at  $O$ . Show that its axes can be determined by finding the maximum and minimum values of  $u = x^2 + y^2$  (2) subject to the condition (1) by the method of § 49, and that the major axis is on the line  $2y - x = 0$ , the minor axis on the line  $y + 2x = 0$ , their lengths being 4 and 2. Trace the curve.

11. Trace the curves :

1.  $x^3 = y(y-1)^2(y-2)^3$

2.  $y^2 - 2xy + x^4 = 0$

3.  $y^3 - x^2y - x^4 = 0$

4.  $x^2y - 2xy^2 + x^2 - y^2 = 0$

## XII. MEAN VALUE THEOREM. INDETERMINATE FORMS

**96. Rolle's Theorem.** *Let  $f(x)$  be a function which vanishes at  $x = a$  and at  $x = b$ , and has a finite derivative at all points in  $(a, b)$ . Then  $f'(x)$  vanishes at some point  $x_1$  between (and distinct from)  $a$  and  $b$ .*

The geometrical meaning of the theorem is that between the points  $x = a$  and  $x = b$  where the curve  $y = f(x)$  meets  $Ox$  it has at least one turning point. Its arithmetical proof is as follows:

Since  $f(x)$  is continuous in  $(a, b)$ , § 25, 2., it takes least and greatest values,  $m$  and  $M$ , in  $(a, b)$ , § 18, 1. And unless  $f(x)$  is 0 throughout  $(a, b)$  — in which case  $f'(x)$  also is 0 throughout  $(a, b)$  and the theorem is true — at least one of  $m$  and  $M$  is different from 0 and therefore corresponds to a value of  $x$  distinct from  $a$  and  $b$ , where  $f(x)$  is

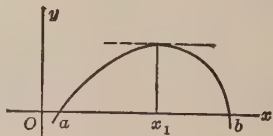


FIG. 74.

0. Let this value of  $x$  be  $x_1$ . Then  $f'(x_1)$  is 0. For  $f'(x_1)$  is finite by hypothesis; and were it positive or negative,  $f(x_1)$  could not be  $m$  or  $M$ , since in  $(a, b)$ , near  $x_1$ , there would be values of  $f(x)$  less, and values greater, than  $f(x_1)$ , § 31.

**97. Mean value theorem.** *Let  $f(x)$  be a function which has a finite derivative at all points of the interval  $(a, b)$ . There exists a value  $x_1$  of  $x$  between (and distinct from)  $a$  and  $b$  such that*

$$f(b) - f(a) = f'(x_1)(b - a) \quad (1)$$

For there is a number  $k$  such that  $f(b) - f(a) = k(b - a)$ , or that

$$f(b) - f(a) - k(b - a) = 0 \quad (2)$$

If we replace  $b$  by  $x$  in the first member of (2), we obtain a function

$$F(x) = f(x) - f(a) - k(x - a) \quad (3)$$

which satisfies the conditions of Rolle's theorem, § 96. For  $F(a)$  is 0 identically,  $F(b)$  is 0 by (2), and  $F'(x) = f'(x) - k$  is finite throughout  $(ab)$ . Hence there exists a value  $x_1$  of  $x$  between, and distinct from,  $a$  and  $b$  such that

$$F'(x_1) = f'(x_1) - k = 0 \therefore \text{that } f'(x_1) = k \quad (4)$$

But the substitution of  $f'(x_1)$  for  $k$  in (2) gives (1); as was to be proved.

Any number between  $a$  and  $b$  may be expressed in the form  $a + \theta(b - a)$  where  $\theta$  denotes some number between 0 and 1. Hence (1) may be written

$$f(b) - f(a) = f'[a + \theta(b - a)](b - a) \quad (5)$$

If in (5) we replace  $a$  by  $x$  and  $b$  by  $x + \Delta x$ ,  $b - a$  becomes  $\Delta x$  and  $f(b) - f(a)$  becomes  $f(x + \Delta x) - f(x)$ , or  $\Delta y$  if  $y = f(x)$ . We therefore have

$$\Delta y = f'(x + \theta \Delta x) \Delta x \quad (6)$$

Respecting the graph of  $y = f(x)$ , the mean value theorem asserts that on the arc between the points  $A[a, f(a)]$  and  $B[b, f(b)]$  there is a point  $P[x_1, f(x_1)]$  where the tangent is parallel to the chord  $AB$ ; for by (1)

$$\frac{f(b) - f(a)}{b - a} = f'(x_1) \quad (7)$$

The first member of (7) is the slope of  $AB$ ; the second that of the tangent at  $P$ .

**EXAMPLE 1.** Find  $x_1$  such that  $f(b) - f(a) = f'(x_1)(b - a)$  in the following cases:

$$1. f(x) = x^2, a = 2, b = 5 \quad 2. f(x) = x^3, a = 1, b = 2$$

$$3. f(x) = x^{1/2}, a = 1, b = 4$$

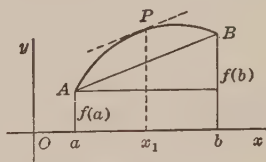


FIG. 75.

**EXAMPLE 2.** Given a curve  $C: y = f(x)$ , and three points  $P_1, P_2, P_3$  on  $C$ . Ordinarily when  $P_2, P_3 \rightarrow P_1$  on  $C$ , the circle determined by  $P_1, P_2, P_3$  approaches a definite circle  $S$  as limit. This circle  $S$  is called the *osculating circle* to  $C$  at  $P_1$ . We are to prove that it is the circle of curvature of  $C$  at  $P_1$ . We suppose  $f''(x)$  continuous between  $P_1, P_2, P_3$ .

Represent the equation of the circle  $P_1P_2P_3$  by

$$(x - a)^2 + (y - b)^2 - r^2 = 0 \quad (1)$$

and consider the function  $F(x) = [x - a]^2 + [f(x) - b]^2 - r^2$  (2)

Since  $P_1(x_1, y_1), P_2(x_2, y_2), P_3(x_3, y_3)$  are on the circle (1),  $F(x)$  vanishes at  $x = x_1, x_2, x_3$ .

Hence, by Rolle's theorem, § 96,  $F'(x)$  vanishes at a point  $x_1'$  between  $x_1$  and  $x_2$  and also at a point  $x_2'$  between  $x_2$  and  $x_3$ . And since  $F'(x)$  vanishes at  $x_1'$  and  $x_2'$ ,  $F''(x)$  vanishes at a point  $x_1''$  between  $x_1'$  and  $x_2'$ . When  $x_2, x_3 \rightarrow x_1$ , then  $x_1', x_2', x_1'' \rightarrow x_1$ .

Hence when  $x_2, x_3 \rightarrow x_1$  and (1) becomes the equation of the osculating circle at  $P_1$ , the constants  $a, b, r^2$  satisfy the equations  $F(x_1) = 0$ ,  $F'(x_1) = 0$ ,  $F''(x_1) = 0$ , that is, the equations:

$$\begin{aligned} (x_1 - a)^2 + [f(x_1) - b]^2 - r^2 &= 0, & (x_1 - a) + [f(x_1) - b] f'(x_1) &= 0, \\ 1 + f'(x_1)^2 + [f(x_1) - b] f''(x_1) &= 0 \end{aligned}$$

But these equations when solved for  $a, b$ , and  $r$  give the values found in § 80, (4) and § 79, (2) for the center  $(\alpha, \beta)$  and the radius  $R$  of the circle of curvature.

Hence the circle of curvature at  $P_1$  crosses  $C$  at  $P_1$ , as in Fig. 53.

**98. Corollary.** If  $f'(x)$  is 0 throughout  $(a, b)$ , then  $f(x)$  is constant throughout  $(a, b)$ .

For if  $x$  denote any number in  $(a, b)$ , we have, by § 97, (1),

$$f(x) - f(a) = f'(x_1)(x - a) \quad \text{where} \quad a < x_1 < x$$

But  $f'(x_1) = 0 \therefore f(x) - f(a) = 0 \therefore f(x)$  has the constant value  $f(a)$  in  $(a, b)$ .

## INDETERMINATE FORMS

**99. The form 0/0.** If  $F(x)$  takes an indeterminate form when  $x = a$ , but  $\lim_{x \rightarrow a} F(x)$  exists, it is customary to assign to  $F(a)$  the value  $\lim_{x \rightarrow a} F(x)$ , § 13.

1. If the fraction  $f(x)/\phi(x)$  takes the form 0/0 when  $x = a$ , it is often possible, as in § 13, to discover a factor common to

$f(x)$  and  $\phi(x)$  which vanishes when  $x = a$ , and after the removal of this factor to find  $\lim_{x \rightarrow a} f(x)/\phi(x)$  directly.

EXAMPLE. When  $x = 0$ ,  $\sin 2x/\tan x$  becomes  $0/0$ ; find

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{\tan x} = \lim_{x \rightarrow 0} \frac{2 \sin x \cos x}{\sin x / \cos x} = 2 \cos^2 x; \text{ hence } \lim_{x \rightarrow 0} \frac{\sin 2x}{\tan x} = \lim_{x \rightarrow 0} 2 \cos^2 x = 2.$$

2. A more general method is supplied by the following theorems:

If  $f(x)$  and  $\phi(x)$  have finite derivatives throughout the interval  $(a, b)$ , and if  $\phi'(x)$  does not vanish in  $(a, b)$  except perhaps at  $a$  or  $b$ , then there exists in  $(a, b)$  a value  $x_1$  of  $x$ , distinct from  $a$  and  $b$ , such that

$$\frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(x_1)}{\phi'(x_1)}. \quad (1)$$

For let  $k$  be the value of  $[f(b) - f(a)]/[\phi(b) - \phi(a)]$ , so that

$$f(b) - f(a) - k[\phi(b) - \phi(a)] = 0 \quad (2)$$

If we replace  $b$  by  $x$  in the first member of (2), we obtain a function

$$F(x) = f(x) - f(a) - k[\phi(x) - \phi(a)] \quad (3)$$

which satisfies the conditions of Rolle's theorem; for  $F(a) = 0$  identically,  $F(b) = 0$  by (2), and  $F'(x) = f'(x) - k\phi'(x)$  is finite throughout  $(a, b)$ . Hence, by § 96, there exists a value  $x_1$  of  $x$ , between and distinct from  $a$  and  $b$ , such that  $F'(x_1) = f'(x_1) - k\phi'(x_1) = 0$ , therefore, since  $\phi'(x_1) \neq 0$ , such that  $f'(x_1)/\phi'(x_1) = k$ ; which proves (1).

If  $f(a) = 0$  and  $\phi(a) = 0$ , and  $\lim_{x \rightarrow a} \frac{f'(x)}{\phi'(x)}$  exists, then

$$\lim_{x \rightarrow a} \frac{f(x)}{\phi(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{\phi'(x)}$$

For in (1) replace  $b$  by  $x$ . Then since  $f(a) = 0$ ,  $\phi(a) = 0$ , (1) becomes  $\frac{f(x)}{\phi(x)} = \frac{f'(x_1)}{\phi'(x_1)}$ , where  $x_1$  is between  $a$  and  $x$ .

When  $x \rightarrow a$ , then  $x_1 \rightarrow a$ . Hence  $\lim_{x \rightarrow a} \frac{f(x)}{\phi(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{\phi'(x)}$ .

Therefore  $\lim_{x \rightarrow a} f(x)/\phi(x) = f'(a)/\phi'(a)$  or  $\infty$ , unless both  $f'(a)$  and  $\phi'(a)$  are 0.

If  $f'(a) = 0$  and  $\phi'(a) = 0$  but  $\lim_{x \rightarrow a} f''(x)/\phi''(x)$  exists, then  $\lim_{x \rightarrow a} f'(x)/\phi'(x) = \lim_{x \rightarrow a} f''(x)/\phi''(x)$ , from which it follows that  $\lim_{x \rightarrow a} f(x)/\phi(x) = f''(a)/\phi''(a)$  or  $\infty$ , unless both  $f''(a)$  and  $\phi''(a)$  are 0. And so on.

Before each step in this process the fraction under consideration should be reduced to its simplest form. Sometimes a transformation is necessary in order to avoid a never ending sequence of indeterminate forms.

EXAMPLE 1. At  $x = 1$ ,  $(\log x)/(x^2 - x)$  becomes  $0/0$ ; find its limiting value when  $x \rightarrow 1$ .

$$\frac{f'(x)}{\phi'(x)} = \frac{1/x}{2x-1} \quad \therefore \frac{f'(1)}{\phi'(1)} = \frac{1}{2-1} = 1, \text{ the limiting value required.}$$

EXAMPLE 2. At  $x = 0$ ,  $(e^x + e^{-x} - 2)/x^2$  becomes  $0/0$ ; find its limiting value when  $x \rightarrow 0$ .

$$\begin{aligned} \frac{f'(x)}{\phi'(x)} &= \frac{e^x - e^{-x}}{2x} \quad \therefore \frac{f'(0)}{\phi'(0)} = \frac{0}{0} \\ \frac{f''(x)}{\phi''(x)} &= \frac{e^x + e^{-x}}{2} \quad \therefore \frac{f''(0)}{\phi''(0)} = 1, \text{ the limiting value required.} \end{aligned}$$

EXAMPLE 3. Show that the following are indeterminate and find their limiting values.

1.  $\frac{\log x}{x-1}$  at  $x = 1$
2.  $\frac{\tan 2\theta - 2 \tan \theta}{2 \sin \theta - \sin 2\theta}$  at  $\theta = 0$
3.  $\frac{e^x - e^{-x} - 2 \sin x}{4x^3}$  at  $x = 0$



**100. The form  $\infty/\infty$ .** It can also be proved<sup>1</sup> that when  $f(a)/\phi(a)$  has the form  $\infty/\infty$ , but  $\lim_{x \rightarrow a} f'(x)/\phi'(x)$  exists, then  $\lim_{x \rightarrow a} f(x)/\phi(x) = \lim_{x \rightarrow a} f'(x)/\phi'(x)$ .

EXAMPLE. When  $x = 0$ ,  $\log x/\cot x$  becomes  $\infty/\infty$ ; find its limiting value when  $x \rightarrow 0$ .

$$\frac{f'(x)}{\phi'(x)} = \frac{1/x}{-\operatorname{cosec}^2 x} = -\frac{\sin^2 x}{x} \quad \therefore \frac{f'(0)}{\phi'(0)} = \frac{0}{0}; \quad \frac{f''(x)}{\phi''(x)} = -\frac{2 \sin x \cos x}{1}$$

$$\therefore \frac{f''(0)}{\phi''(0)} = 0, \text{ the limiting value required.}$$

**101. The forms  $0 \cdot \infty$  and  $\infty - \infty$ .** The form  $0 \cdot \infty$  can be reduced to either of the forms  $0/0$  or  $\infty/\infty$ . The same is often true of the form  $\infty - \infty$ .

EXAMPLE 1. At  $x = \pi/2$ ,  $(1 - \sin x) \tan x$  becomes  $0 \cdot \infty$ ; find its limiting value when  $x \rightarrow \pi/2$ .

$$(1 - \sin x) \tan x = \frac{1 - \sin x}{\cot x} = \frac{0}{0} \text{ when } x = \frac{\pi}{2};$$

$$\text{but } \lim_{x \rightarrow \pi/2} \frac{1 - \sin x}{\cot x} = \lim_{x \rightarrow \pi/2} \frac{\cos x}{\operatorname{cosec}^2 x} = \frac{0}{1} = 0.$$

<sup>1</sup> 1. When  $a$  is  $\infty$ . Take an arbitrary constant  $c$  and suppose  $x > c$ . In  $(c, x)$  there is an  $x_1$  such that (§ 99(1))

$$\frac{f(x) - f(c)}{\phi(x) - \phi(c)} = \frac{f'(x_1)}{\phi'(x_1)}, \text{ and therefore } \frac{f(x)}{\phi(x)} = \frac{f'(x_1)}{\phi'(x_1)} \cdot \frac{1 - \phi(c)/\phi(x)}{1 - f(c)/f(x)}$$

Let  $l$  denote  $\lim_{x \rightarrow \infty} \frac{f'(x)}{\phi'(x)}$ , and  $\epsilon, \epsilon'$  positive numbers as small as we please. We

can take  $c$  large enough to make  $\left| \frac{f'(x_1)}{\phi'(x_1)} - l \right| < \epsilon$ , and then, keeping  $c$

fixed, we can take  $x$  large enough to make  $\left| \frac{1 - \phi(c)/\phi(x)}{1 - f(c)/f(x)} - 1 \right| < \epsilon'$ ; hence

$$\lim_{x \rightarrow \infty} f(x)/\phi(x) = \lim_{x \rightarrow \infty} f'(x)/\phi'(x) = l.$$

If  $\lim_{x \rightarrow \infty} f'(x)/\phi'(x) = \infty$ , then  $\lim_{x \rightarrow \infty} f(x)/\phi(x) = \infty$ .

2. When  $a$  is finite. Set  $x = a + 1/z$ , so that when  $x \rightarrow a$  then  $z \rightarrow \infty$ . Then if  $f(x), \phi(x)$  become  $F(z), \Phi(z)$ , we have  $f'(x) = -F'(z)z^2$ , and  $\phi'(x) = -\Phi'(z)z^2$   $\therefore f'(x)/\phi'(x) = F'(z)/\Phi'(z)$ .

Let  $\lim_{x \rightarrow a} f'(x)/\phi'(x) = l$ . Then  $\lim_{z \rightarrow \infty} F'(z)/\Phi'(z) = l$  and therefore, by 1.,  $\lim_{z \rightarrow \infty} F(z)/\Phi(z) = l$ . Hence  $\lim_{x \rightarrow a} f(x)/\phi(x) = l$ .

EXAMPLE 2. At  $x = \pi/2$ ,  $\sec x - \tan x$  becomes  $\infty - \infty$ ; find its limiting value when  $x \rightarrow \pi/2$ .

$$\sec x - \tan x = \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \frac{1 - \sin x}{\cos x} = \frac{0}{0} \text{ when } x = \frac{\pi}{2};$$

but 
$$\lim_{x \rightarrow \pi/2} \frac{1 - \sin x}{\cos x} = \lim_{x \rightarrow \pi/2} \frac{\cos x}{\sin x} = 0.$$

**102. The forms  $1^\infty$ ,  $0^0$ ,  $\infty^0$ .** A function of the type  $f(x)^{\phi(x)}$  may take one of the indeterminate forms  $1^\infty$ ,  $0^0$ ,  $\infty^0$ . For if  $u = f(x)^{\phi(x)}$ , then  $\log u = \phi(x) \log f(x)$ , and at the point  $x = a$  this product may take the form  $0 \cdot \infty$  in the following cases:

(1) When  $\phi(a) = \infty$  and  $f(a) = 1$ ,  $u$  then having the form  $1^\infty$ .

(2) When  $\phi(a) = 0$  and  $f(a) = 0$  or  $\infty$ ,  $u$  then having the form  $0^0$  or  $\infty^0$ .

In all these cases we find  $\lim_{x \rightarrow a} \log u$  by the method of § 101. Then if  $b$  denote this limit, we have, by § 67, 2,  $\lim_{x \rightarrow a} u = e^b$ .

EXAMPLE. If  $u = (1 + a/x)^x$ , find  $\lim_{x \rightarrow \infty} u$  [the case  $1^\infty$ ].

$$\log u = x \log (1 + a/x) = \frac{\log (1 + a/x)}{1/x}, \text{ which becomes } \frac{0}{0} \text{ when } x = \infty.$$

$$\text{Hence } \lim_{x \rightarrow \infty} \frac{\log (1 + a/x)}{1/x} = \lim_{x \rightarrow \infty} \frac{a/x^2}{(1 + a/x)(1/x^2)} = \lim_{x \rightarrow \infty} \frac{a}{1 + a/x} = a.$$

Therefore  $\lim_{x \rightarrow \infty} u = e^a$ .

## EXERCISE XVI

Find the following limiting values 1-21:

1.  $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{\sin 2x}$

2.  $\lim_{x \rightarrow a} \frac{\sin x - \sin a}{x - a}$

3.  $\lim_{x \rightarrow \pi/2} \frac{\sec x + 1}{\tan x}$

4.  $\lim_{x \rightarrow 0} x \log x$

5.  $\lim_{x \rightarrow \infty} \frac{x^n}{e^x}$

6.  $\lim_{x \rightarrow \infty} [\log (1 + x) - \log x]$

7.  $\lim_{x \rightarrow 0} \frac{\tan x - x}{x - \sin x}$

8.  $\lim_{x \rightarrow 0} \frac{\sin x + \tan x}{e^x + e^{-x} - 2}$

9.  $\lim_{x \rightarrow 0} \frac{xe^{4x} - x}{1 - \cos 2x}$
10.  $\lim_{x \rightarrow 5} \frac{2 - \sqrt{x-1}}{x^2 - 25}$
11.  $\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - \sin 2x}{x^3 + x^4}$
12.  $\lim_{x \rightarrow 1} \frac{\sin \pi x}{x - 1}$
13.  $\lim_{x \rightarrow 0} \frac{\log \sin x}{\log \tan x}$
14.  $\lim_{x \rightarrow 0} \frac{\tan x}{\tan 4x}$
15.  $\lim_{x \rightarrow 0} \sin x \log 2x$
16.  $\lim_{x \rightarrow 1/2} \frac{\log(1-2x)}{\tan \pi x}$
17.  $\lim_{x \rightarrow 0} \left[ \frac{1}{\sin x} - \frac{1}{x} \right]$
18.  $\lim_{x \rightarrow 0} \left[ \frac{2}{\sin^2 x} - \frac{1}{1 - \cos x} \right]$
19.  $\lim_{x \rightarrow 0} x^x$
20.  $\lim_{x \rightarrow \infty} (1+x)^{1/x}$
21.  $\lim_{x \rightarrow 0} (e^{2x} + x)^{1/x}$

22. By aid of the mean value theorem, prove that if  $f'(x)$  is positive throughout  $(a, b)$  and  $x', x''$  are any two numbers in  $(a, b)$  such that  $x' < x''$ , then  $f(x') < f(x'')$ . Compare § 32.

23. Prove that if  $f'(x)$  is continuous in  $(a, b)$  and  $M$  denote the greatest value of  $|f'(x)|$  in  $(a, b)$ , then in every part of  $(a, b)$  of length  $< \epsilon/M$ , the difference between the greatest and least values of  $f(x)$  is less than  $\epsilon$ .

### XIII. NEWTON'S METHOD OF APPROXIMATION

**103. Newton's method of approximation.** An equation  $f(x) = 0$  being given, suppose that two numbers  $b$  and  $c$  have been found such that

- (1)  $f(b)$  and  $f(c)$  have opposite signs,
- (2)  $f'(x)$  and  $f''(x)$  are continuous and  $\neq 0$  in  $(b, c)$ .

The equation  $f(x) = 0$  then has one and but one root  $\beta$  in  $(b, c)$ , §§ 18, 32, and we can find as close an approximation to  $\beta$  as we please by the following method, due to Newton.

Since by (2) the signs of  $f'(x)$  and  $f''(x)$  are constant in  $(b, c)$ , it follows from §§ 32, 46 that the graph of  $y = f(x)$  in  $(b, c)$  must be of the type of the curve arc in Fig. 76, 1. or 2., or of an arc symmetric to 1. or 2. with respect to  $Ox$ .

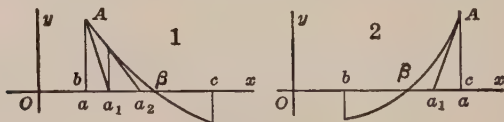


FIG. 76.

Hence if  $a$  denote that one of  $b$  or  $c$  for which  $f(x)$  has the sign of  $f''(x)$ , the tangent at the corresponding curve point  $A$  will cut  $Ox$  at a point  $a_1$  between  $a$  and  $\beta$  — which means that  $a_1$  is a closer approximation to  $\beta$  than  $a$  is.

The equation of the tangent at  $A$  is  $y - f(a) = f'(a)(x - a)$ . Hence we can find  $a_1$  by setting  $y = 0$  in this equation and then solving for  $x$ . We thus get

$$a_1 = a - \frac{f(a)}{f'(a)} \quad (1)$$

Next, using  $a_1$  as we have just used  $a$ , we get a second and closer approximation to  $\beta$ , namely  $a_2 = a_1 - f(a_1)/f'(a_1)$ ; and so on.

As  $n$  increases, either  $a_n \rightarrow \beta$  as in Fig. 76, 1., or  $a_n \searrow \beta$  as in Fig. 76, 2.<sup>1</sup>

EXAMPLE 1. Find the real roots of  $e^x + x - 2 = 0$  ( $e = 2.71828 \dots$ ).

$$f(x) = e^x + x - 2 \quad f'(x) = e^x + 1 \quad f''(x) = e^x$$

Since  $f(0) = -1$  and  $f(1) = 1.718$  have opposite signs, and  $f'(x)$  is always  $+$ , there is one and but one real root  $\beta$ , and it is in the interval  $(0, 1)$ . And since  $f''(x)$  is always  $+$  and therefore  $\text{sgn } f(1) = \text{sgn } f''(1)$ , we may begin the reckoning for  $\beta$  by setting  $a = 1$  in the formula (1). But the reckoning will be shortened if we start with a smaller value of  $a$ . We find  $f(.5) = .1487$ , which is also  $+$ , and therefore take  $a = .5$ . The 1st and 2nd approximations to  $\beta$  are then

$$a_1 = .5 - \frac{f(.5)}{f'(.5)} = .4439 \quad a_2 = .4439 - \frac{f(.4439)}{f'(.4439)} = .442884.$$

EXAMPLE 2. Solve the equations (1)  $2 \sin x = x$  (2)  $e^x - 3x = 0$ .

EXAMPLE 3. Show that  $x^3 - 2x - 5 = 0$  has the root 2.09455 [Newton's example].

EXAMPLE 4. From the graphs of  $y = \tan x$  and  $y = x$  show that the equation  $\tan x = x$  has infinitely many roots; and find the smallest positive root to the fourth decimal figure.

<sup>1</sup> The equation  $f(x) = 0$  may be written in the form  $x = \phi(x)$ , where  $\phi(x) = x - f(x)/f'(x)$ . By (1) we then have  $a_1 = \phi(a)$   $\therefore$   $a_2 = \phi(a_1)$ ,  $\dots$ ,  $a_n = \phi(a_{n-1})$ . As  $n$  increases,  $a_n$  steadily increases or decreases but remains finite and therefore approaches a limit  $l$ , § 5. Therefore, since  $\phi(x)$  is continuous in  $(b, c)$ ,  $l = \phi(l)$ , that is,  $l = \beta$ .

## XIV. INTEGRATION

**104. Integrals.** Let  $f(x)$  denote a given function of  $x$ ; a function  $F(x)$  which has  $f(x)$  for its derivative, or  $f(x)dx$  for its differential, is called an *integral* of  $f(x)$  or of  $f(x)dx$ . Thus  $x^2$  is an integral of  $2x$  or of  $2x dx$ .

The derivative of any constant  $C$  being 0, if  $F(x)$  is an integral of  $f(x)$ , so also is  $F(x) + C$ . Furthermore

*If  $f(x)$  is continuous and has the integral  $F(x)$  in the interval  $(a, b)$ , then every integral of  $f(x)$  in  $(a, b)$  is of the form  $F(x) + C$ .*

For let  $G(x)$  denote another integral of  $f(x)$  in  $(a, b)$ . Then

$$\frac{d}{dx} [G(x) - F(x)] = f(x) - f(x) = 0$$

Hence in  $(a, b)$  the function  $G(x) - F(x)$  has a zero derivative and is therefore a constant, § 98. Let  $c$  denote this constant. Then  $G(x) - F(x) = c$  and therefore  $G(x) = F(x) + c$ .

Hence if  $F(x)$  be any *particular integral* of  $f(x)$ , the *general integral* of  $f(x)$  is  $F(x) + C$ , where  $C$  denotes an arbitrary constant. A general integral is also called an *indefinite integral*. Thus the general integral of  $2x$  is  $x^2 + C$ .

The symbol for the general integral of  $f(x)$  or  $f(x)dx$  is

$$\int f(x)dx \quad (1)$$

read "integral  $f(x)dx$ ." Hence if  $F(x)$  be a particular integral, then

$$\int f(x)dx = F(x) + C \quad (2)$$

To *integrate*  $f(x)$  is to express  $\int f(x)dx$  in terms of known functions. We may interpret  $\int$  as the symbol for this opera-

tion and call  $f(x)$  the *integrand* to which it is applied. By the definition of  $\int f(x)dx$ ,

$$\frac{d}{dx} \int f(x)dx = f(x) \quad \text{or} \quad d \int f(x)dx = f(x)dx \quad (3)$$

Hence integration is the *inverse* of differentiation, that is, it is the operation which differentiation undoes. To effect it, one must express  $f(x)dx$  in terms of differentials whose integrals, like that of  $2x dx$ , are known from the formulas of differentiation.<sup>1</sup>

**105. Two properties of integrals.** From the definition of integral it follows that if  $u$  and  $v$  denote functions of  $x$ , and  $k$  a constant, then

$$\int (u + v)dx = \int u dx + \int v dx + C \quad (1)$$

$$\int ku dx = k \int u dx + C \quad (2)$$

For  $\int (u + v)dx$  and  $\int u dx + \int v dx$  have the same derivative,  $u + v$ ; and  $\int ku dx$  and  $k \int u dx$  have the same derivative,  $ku$ .

We may extend (1) to a sum of any finite number of functions.

By (2) it is allowable to shift a constant factor  $k$  from one side to the other of the integral sign; but not so a variable factor.

**EXAMPLE.** Thus

$$\int (x^2 + 3x + 5)dx = \int x^2 dx + \int 3x dx + \int 5 dx = \frac{x^3}{3} + \frac{3}{2}x^2 + 5x + C$$

**106. The integral  $\int x^n dx$ .** 1. Let  $n$  denote any constant except  $-1$ . Since

$$\frac{d}{dx} \frac{x^{n+1}}{n+1} = x^n, \quad \text{we have} \quad \int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (1)$$

<sup>1</sup>Not every  $f(x)dx$  can be so expressed. Thus  $(1+x^2)^{1/2}dx$  cannot. Hence, though  $\int (1+x^2)^{1/2}dx$  exists (in a sense explained in § 126), it cannot be expressed in terms of rational, irrational, trigonometric, exponential, and logarithmic functions.



To integrate  $x^n$ ,  $n \neq -1$ , increase the exponent by 1 and then divide by the new exponent.

EXAMPLE 1. Thus, omitting the arbitrary constant  $C$ , we have

$$\int x^5 dx = \frac{x^6}{6}, \quad \int x^{3/4} dx = \frac{4}{7} x^{7/4}, \quad \int \sqrt{x} dx = \int x^{1/2} dx = \frac{2}{3} x^{3/2},$$

$$\int \frac{dx}{x^3} = \int x^{-3} dx = -\frac{x^{-2}}{2} = -\frac{1}{2x^2}.$$

2. Let  $n = -1$ . Since

$$\frac{d}{dx} \log x = \frac{1}{x}, \quad \text{we have} \quad \int \frac{dx}{x} = \log x + C \quad (2)$$

EXAMPLE 2. Find  $\int y^{-1} dy$ ,  $\int y^{-1/3} dy$ ,  $\int t^3 dt$ ,  $\int \sqrt[3]{x^5} dx$ ,  
 $\int (1/x^4) dx$ ,  $\int (1/\sqrt{x}) dx$ .

The usefulness of (1) and (2) is greatly increased by the following theorem:

**107. Theorem.** Let  $u$  be a function of  $x$ , and  $\phi(u)$  a function of  $u$ ; then

$$\int \phi(u) \frac{du}{dx} dx = \int \phi(u) du + C \quad (1)$$

For the integrals in (1) have the same  $x$ -derivative.

Thus 
$$\frac{d}{dx} \int \phi(u) \frac{du}{dx} dx = \phi(u) \frac{du}{dx}$$

and 
$$\frac{d}{dx} \int \phi(u) du = \left[ \frac{d}{du} \int \phi(u) du \right] \frac{du}{dx} = \phi(u) \frac{du}{dx}$$

Therefore under the integral sign, as elsewhere, when  $u$  is a function of  $x$  we may replace  $(du/dx) dx$  by  $du$ , and vice versa.

EXAMPLE 1. Find  $\int 2x\sqrt{x^2+1} dx$ .

The factor  $\sqrt{x^2+1}$  is a function of  $u = x^2+1$ , and the other factor,  $2x$ , is  $du/dx$ . Hence

$$\int 2x\sqrt{x^2+1} dx = \int u^{1/2} \frac{du}{dx} dx = \int u^{1/2} du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (x^2+1)^{3/2} + C.$$

Or more briefly, since  $2x dx = d(x^2+1)$  we have at once by the theorem,

$$\int 2x\sqrt{x^2+1} dx = \int (x^2+1)^{1/2} d(x^2+1) = \frac{2}{3} (x^2+1)^{3/2} + C$$

EXAMPLE 2. Find  $\int \frac{x+1}{x^2+2x} dx$

Since  $d(x^2+2x) = 2(x+1)dx$ , we may replace  $(x+1)dx$  by  $d(x^2+2x)$  if at the same time we put the "compensating factor"  $1/2$  before the integral. Hence

$$\int \frac{x+1}{x^2+2x} dx = \frac{1}{2} \int \frac{d(x^2+2x)}{x^2+2x} = \frac{1}{2} \log(x^2+2x) + C$$

EXAMPLE 3. Find  $\int (x^3+5)^{2/3} x^2 dx$ ,  $\int \frac{x dx}{1-x^2}$ ,  $\int \frac{x dx}{\sqrt{1-x^2}}$ ,  
 $\int \frac{x^2+2x}{x^3+3x^2+1} dx$

In particular, since  $d(ax+b) = a dx$ , we have

$$\int \phi(ax+b) dx = \frac{1}{a} \int \phi(ax+b) d(ax+b). \quad (2)$$

Thus  $\int \frac{dx}{x+1} = \int \frac{d(x+1)}{x+1} = \log(x+1) + C$

$$\int \sqrt{1-2x} dx = -\frac{1}{2} \int (1-2x)^{1/2} d(1-2x) = -\frac{1}{3} (1-2x)^{3/2} + C.$$

EXAMPLE 4. Find  $\int \sqrt{4x+3} dx$ ,  $\int \frac{dx}{\sqrt{2-5x}}$ ,  $\int (2x+7)^{10} dx$ ,  
 $\int \frac{\log^3(2x)}{2x} dx$ ,  $\int \frac{x^3+5x}{x-1} dx$   $\left[ \frac{x^3+5x}{x-1} = x^2+x+6+\frac{6}{x-1} \right]$

**108. Integration by substitution.** An integral  $\int \phi(x) dx$  may often be reduced to an integrable form by expressing  $x$  as a function of another variable  $t$ . It follows from the theorem of § 107 that if  $x = f(t)$ , the substitution  $x = f(t)$ ,  $dx = f'(t) dt$  will transform  $\int \phi(x) dx$  into an equivalent integral in  $t$ . We integrate this  $t$ -integral and then express  $t$  in terms of  $x$  in the result.

EXAMPLE 1. Find  $\int \frac{dx}{x^{1/2}+1}$ .

Set  $x = t^2$  and therefore  $dx = 2t dt$ . Then

$$\begin{aligned} \int \frac{dx}{x^{1/2}+1} &= 2 \int \frac{t}{t+1} dt = 2 \int \left[ 1 - \frac{1}{t+1} \right] dt \\ &= 2t - 2 \log(t+1) + C = 2x^{1/2} - 2 \log(x^{1/2}+1) + C. \end{aligned}$$

When  $\phi(x)$  is rational with respect to  $x$  and  $(bx + c)^{1/q}$ , the substitution  $bx + c = t^q$  will transform  $\int \phi(x) dx$  into a  $t$ -integral whose integrand is rational with respect to  $t$ .

EXAMPLE 2. Find  $\int \frac{dx}{x^{1/2} - 1}$ ,  $\int \frac{x dx}{(x + 1)^{1/2}}$ ,  $\int \frac{dx}{2 + x^{1/3}}$ .

### EXERCISE XVII

Integrate the following :

1.  $\int (x^2 + ax + a^2) dx$
2.  $\int x^3(2x^2 + 5) dx$
3.  $\int (x^{1/2} + a^{1/2})^2 dx$
4.  $\int \left(x + \frac{1}{x}\right)^2 dx$
5.  $\int \frac{dx}{2x - 1}$
6.  $\int \frac{dx}{(x - 1)^2}$
7.  $\int \frac{1 + x}{x^2} dx$
8.  $\int \frac{dx}{\sqrt{4 - 5x}}$
9.  $\int \frac{(x - 3)(2x - 1)}{x} dx$
10.  $\int \frac{1 + x}{1 - x} dx$
11.  $\int (x^4 + a^4)^{1/2} x^3 dx$
12.  $\int (ax^2 + b)^{2/3} x dx$
13.  $\int \frac{1 + x}{\sqrt{x}} dx$
14.  $\int \frac{x^2 dx}{x^3 + 2}$
15.  $\int \frac{x^2 dx}{\sqrt{x^3 + 2}}$
16.  $\int \frac{(1 + \log x)^{1/2}}{x} dx$
17.  $\int \frac{(x - 1)(x + 3)}{x + 2} dx$
18.  $\int \frac{x^3}{x - 3} dx$
19.  $\int (bx + c)^{1/3} dx$
20.  $\int \sin^2 x \cos x dx$
21.  $\int \frac{x dx}{(x^2 + 2)^3}$
22.  $\int \frac{x^2 dx}{(x + 1)^{1/2}}$
23.  $\int x(2x + 5)^{1/2} dx$
24.  $\int x^3(x - 1)^{1/2} dx$

### 109. Determination of a function from its derivative.

If  $y$  be such a function of  $x$  that  $dy/dx = f(x)$ , or  $dy = f(x)dx$ , and if  $F(x)$  be some particular integral of  $f(x)$ , then, § 104,  $y = F(x) + C$ . Hence, when the derivative of a function is known, the function itself is determined except for an arbitrary added constant. If the value of the function for any one value of  $x$  is also known, we can find the constant  $C$  and

so make the determination of the function complete. Many important problems of geometry and mechanics reduce to the determination of a function from its derivative. In dealing with such a problem one must bear in mind that, when nothing but the derivative is given, the solution is not a single function  $y = F(x)$  but the infinite set or "family" of functions  $y = F(x) + C$ .

**EXAMPLE 1.** The rate of change of  $y$  with respect to  $x$  is  $x^2$ , and  $y = 6$  when  $x = 3$ ; find  $y$  when  $x = 6$ .

Since  $dy/dx = x^2$ , we have  $y = x^3/3 + C$ . But since  $y = 6$  when  $x = 3$ ,  $C$  must be such that  $6 = 27/3 + C$ , which gives  $C = -3$ . Hence  $y = x^3/3 - 3$ , and when  $x = 6$ , then  $y = 69$ .

**EXAMPLE 2.** Find the curves which have the property that the slope at the point whose abscissa is  $x$  is  $2x - 1$ ; also that one of the curves which passes through  $(1, 1)$ .

Since  $dy/dx = 2x - 1$ , we have  $y = x^2 - x + C$  which represents a family of parabolas all having the line  $2x - 1 = 0$  for axis.

For the curve through  $(1, 1)$ ,  $1 = 1 - 1 + C$   
 $\therefore C = 1$ .

Hence the equation of this curve is

$$y = x^2 - x + 1.$$

**EXAMPLE 3.** A point is moving on a line with the acceleration  $6t$ . When  $t = 1$  and  $2$ , its distances from the origin are  $10$  and  $80$ . Find its velocity when  $t = 3$ .

$$\text{Here } \frac{d^2s}{dt^2} = 6t \quad \therefore \frac{ds}{dt} = 3t^2 + C_1 \quad \therefore s = t^3 + C_1t + C_2.$$

Subst'g the given values of  $s$ ,  $10 = 1 + C_1 + C_2$  and  $80 = 8 + 2C_1 + C_2$ ;  
 $\therefore C_1 = 63, C_2 = -54$ . Hence the equation of motion is  $s = t^3 + 63t - 54$ . Also  $v = 3t^2 + 63 \therefore$  when  $t = 3, v = 90$ .

**EXAMPLE 4.** Consider the path of a projectile  $P$  whose initial velocity is  $v_0$  ft./sec. in a direction which makes the angle  $\theta$  with the horizontal, the resistance of the air being disregarded. Taking the starting point as origin, and the horizontal and vertical as  $Ox$  and  $Oy$ , we have  $d^2x/dt^2 = 0, d^2y/dt^2 = -32$  ft./sec.<sup>2</sup>; also when  $t = 0$ , then  $x = 0, y = 0, dx/dt = v_0 \cos \theta, dy/dt = v_0 \sin \theta$ . Show that the equations of the path are

$$x = v_0 t \cos \theta \quad y = v_0 t \sin \theta - 16 t^2 \quad (1)$$

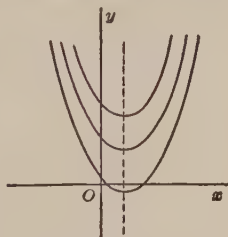


FIG. 77.

**110. Theorem.** If  $y$  be such a function of  $x$  that

$$\phi(x)dx = \psi(y)dy, \quad \text{then} \quad \int \phi(x)dx = \int \psi(y)dy + C$$

For since  $\phi(x) = \psi(y)dy/dx$ , we have by the theorem of § 107,

$$\int \phi(x)dx = \int \psi(y) \frac{dy}{dx} dx = \int \psi(y)dy + C$$

This theorem is applicable to any “ differential equation ” of the form  $X_1Y_1dx + X_2Y_2dy = 0$ , where  $X_1, X_2$  denote functions of  $x$  only, and  $Y_1, Y_2$  functions of  $y$  only. For when divided by  $Y_1X_2$  the equation becomes

$$(X_1/X_2)dx + (Y_2/Y_1)dy = 0,$$

which is of the form  $\phi(x)dx - \psi(y)dy = 0$ .

**EXAMPLE 1.** “ Integrate ” the equation

$$x(y^2 - 1)dx - y(x^2 + 1)dy = 0$$

$$\text{Dividing by } (y^2 - 1)(x^2 + 1), \quad \frac{x}{x^2 + 1}dx - \frac{y}{y^2 - 1}dy = 0$$

$$\text{Hence} \quad \int \frac{x dx}{x^2 + 1} - \int \frac{y dy}{y^2 - 1} = C$$

$$\therefore \log(x^2 + 1) - \log(y^2 - 1) = 2C \quad \therefore x^2 + 1 = C'(y^2 - 1).$$

**EXAMPLE 2.** In Fig. 78,  $PT$  and  $PN$  are the tangent and normal to the curve  $S$  at the point  $P(x, y)$ . Since

$$\frac{dy}{dx} = \tan DTP = \tan DPN,$$

$$1. \text{ Subtangent } DT = ydx/dy$$

$$2. \text{ Subnormal } DN = ydy/dx$$

Find the curves whose subnormal is the constant  $2a$ .

We have  $ydy/dx = 2a \quad \therefore ydy = 2adx \quad \therefore y^2/2 = 2ax + C$  or  $y^2 = 4ax + C'$ .

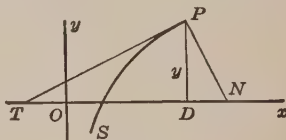


FIG. 78.

**EXAMPLE 3.** An orthogonal trajectory of a set of curves is a curve which cuts all the curves of the set at right angles. Find the orthogonal trajectories of the set of parabolas  $y = Cx^2$ .

Differentiating  $y = Cx^2$  (1) gives  $dy/dx = 2Cx$  (2), and eliminating  $C$  between (1) and (2) gives  $dy/dx = 2y/x$  (3). Hence the curve of the set (1) through the point  $P(x, y)$  has at  $P$  the slope  $2y/x \quad \therefore$  the orthogonal trajectory through  $P$  has the slope  $-x/2y$ . Therefore the differential equation of the set of orthogonal trajectories is  $dy/dx$

$= -x/2 y$  or  $x dx + 2 y dy = 0$  (4). The solution of (4) is  $x^2/2 + y^2 = C$ , which represents a family of ellipses.

EXAMPLE 4. To find  $y$ , a function of  $x$  whose rate of change with respect to  $x$  is proportional to  $y$  itself: so that  $dy/dx = ky$ , where  $k$  is a given constant.

$$dy/y = k dx \quad \therefore \log y = kx + C' \quad \therefore y = e^{kx+C'} = Ce^{kx}$$

If  $y = y_0$  when  $x = 0$ , then  $C = y_0$ , and we have  $y = y_0 e^{kx}$ .

### EXERCISE XVIII

1. Find that curve of the set defined by  $d^2y/dx^2 = 6x$  which passes through the points (1, 1) and (2, 4).

2. Find the curve which passes through the point (2, 3) and whose slope at every point  $(x, y)$  on it is (a)  $x^2$  (b)  $(1-x)(1+y)$  (c)  $-y/x$ .

3. Find the orthogonal trajectories of the following families of curves:  
(a)  $y = Cx$  (b)  $y^2 = Cx$  (c)  $xy = C$  (d)  $x^2 - y^2 = C$

4. Find the set of curves for which the subtangent at any point is  $x^{1/2}y^{1/2}$ .

5. Find that curve through (2, 3) whose subnormal at any point is  $x/y$ .

6. For the motion of a body which is sliding down a certain inclined plane we have  $ds/dt = 4s^{1/2}$ . Find the acceleration of the motion, and the inclination of the plane.

7. Assuming that the time rate of decomposition of a quantity  $y$  of radium is .038  $y$ , the time unit being 100 years, show how long it would take for one half of a given quantity  $y_0$  to be dissipated.

8. Assuming that the atmospheric pressure as indicated by the barometric reading,  $p$  in., decreases as the altitude,  $h$  ft., increases, at the rate  $dp/dh = -kp$ , and that  $p = 30$  when  $h = 0$  and  $p = 28.87$  when  $h = 1000$ , find  $h$  when  $p = 25$ .

**111. Formulas of Integration.** Every formula of differentiation, when inverted, yields a formula of integration. The student should familiarize himself with the following. In each of them  $u$  may represent the independent variable or a function of that variable; and in each, the constant of integration is to be added on the right.

1.  $\int u^n du = \frac{u^{n+1}}{n+1} \quad (n \neq -1)$
2.  $\int \frac{du}{u} = \log u$
3.  $\int e^u du = e^u$
4.  $\int \cos u du = \sin u$
5.  $\int \sin u du = -\cos u$
6.  $\int \sec^2 u du = \tan u$
7.  $\int \operatorname{cosec}^2 u du = -\cot u$
8.  $\int \sec u \tan u du = \sec u$
9.  $\int \operatorname{cosec} u \cot u du = -\operatorname{cosec} u$
10.  $\int \tan u du = -\log \cos u$
11.  $\int \cot u du = \log \sin u$
12.  $\int \sec u du = \log (\sec u + \tan u)$
13.  $\int \operatorname{cosec} u du = \log (\operatorname{cosec} u - \cot u)$
14.  $\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a}$
15.  $\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \log \frac{u-a}{u+a}$
16.  $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a}$
17.  $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log (u + \sqrt{u^2 \pm a^2})$

To prove these formulas, it is only necessary to show for each that the derivative of the right member is the integrand in the left member. This is obvious by inspection in the case of 1-9. We shall verify 12. and 17. The reader should verify the others.<sup>1</sup>

<sup>1</sup> When  $u < a$ , we may replace 15. by  $\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \log \frac{a+u}{a-u}$ .



$$\begin{aligned}\frac{d}{du} \log (\sec u + \tan u) &= \frac{1}{\sec u + \tan u} \frac{d}{du} (\sec u + \tan u) \\ &= \frac{\sec u \tan u + \sec^2 u}{\sec u + \tan u} = \sec u\end{aligned}$$

$$\begin{aligned}\frac{d}{du} \log (u + \sqrt{u^2 \pm a^2}) &= \frac{1}{u + \sqrt{u^2 \pm a^2}} \frac{d}{du} (u + \sqrt{u^2 \pm a^2}) \\ &= \frac{1 + (u^2 \pm a^2)^{-1/2}u}{u + \sqrt{u^2 \pm a^2}} = \frac{1}{\sqrt{u^2 \pm a^2}}\end{aligned}$$

The following integrations illustrate the application of these formulas.

1.  $\int e^{3x+4} dx = \frac{1}{3} \int e^{3x+4} d(3x+4) = \frac{1}{3} e^{3x+4} + C$
2.  $\int \left( \sin 5x + \cos \frac{x}{5} \right) dx = -\frac{\cos 5x}{5} + 5 \sin \frac{x}{5} + C$
3.  $\int \frac{\cos x}{3 + 4 \sin x} dx = \frac{1}{4} \int \frac{d(3 + 4 \sin x)}{3 + 4 \sin x} = \frac{1}{4} \log (3 + 4 \sin x) + C$
4.  $\int \frac{dx}{3x^2 + 2} = \frac{1}{3} \int \frac{dx}{x^2 + 2/3} = \frac{1}{3} \sqrt{\frac{3}{2}} \tan^{-1} \sqrt{\frac{3}{2}} x + C$
5.  $\int \frac{dx}{(3x^2 + 2)^{1/2}} = \frac{1}{\sqrt{3}} \int \frac{dx}{(x^2 + 2/3)^{1/2}}$   
 $= \frac{1}{\sqrt{3}} \log [x + (x^2 + 2/3)^{1/2}] + C$

### EXERCISE XIX

Perform the following integrations:

1.  $\int \sin (2 - x) dx$
2.  $\int x \sin x^2 dx$
3.  $\int \sec^2 4x dx$
4.  $\int \operatorname{cosec} 3x \cot 3x dx$
5.  $\int (\tan 2x + \cot 3x) dx$
6.  $\int e^{ax+b} dx$
7.  $\int e^{\cos x} \sin x dx$
8.  $\int [\sqrt{e^x} + 1/\sqrt{e^x}] dx$
9.  $\int (1 + e^x)^{1/2} e^x dx$
10.  $\int 2^x dx = \int e^{x \log 2} dx$
11.  $\int (e^x + e^{-x})^2 dx$
12.  $\int \cos^2 3x \sin 3x dx$
13.  $\int \frac{\sec^2 x}{\tan x + 1} dx$
14.  $\int (\tan x + \cot x)^2 dx$
15.  $\int \frac{dx}{3x^2 - 2}$

$$\begin{array}{lll}
16. \int \frac{dx}{4x^2 + 9} & 17. \int \frac{dx}{\sqrt{4 - x^2}} & 18. \int \frac{dx}{\sqrt{4 - 9x^2}} \\
19. \int \frac{dx}{\sqrt{x^2 - 4}} & 20. \int \frac{x dx}{x^4 + 1} & 21. \int \frac{x dx}{\sqrt{x^4 + 1}} \\
22. \int \frac{dx}{1 - \sin x} = \int \frac{1 + \sin x}{\cos^2 x} dx = \int (\sec^2 x + \sec x \tan x) dx \\
23. \int \frac{dx}{e^x + 1} = \int \left[ 1 - \frac{e^x}{e^x + 1} \right] dx & 24. \int \frac{dx}{1 + \cos x}
\end{array}$$

**112. Trigonometric integrals.** 1. Integrals of the type  $\int \sin^m x \cos^n x dx$ , where *either*  $m$  or  $n$  is an odd positive integer, may, by aid of the identity  $\sin^2 x + \cos^2 x = 1$ , be reduced to integrals with respect to  $\sin x$  or  $\cos x$  which can be found by § 111, Formula 1. or 2.

**EXAMPLE 1.** Find  $\int \sin^5 x dx$

Since  $\sin^5 x dx = \sin^4 x \cdot \sin x dx = -(1 - \cos^2 x)^2 \cdot d \cos x$ , we have

$$\begin{aligned}
\int \sin^5 x dx &= -\int (1 - 2 \cos^2 x + \cos^4 x) d \cos x \\
&= -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C.
\end{aligned}$$

**EXAMPLE 2.** Find the following:

$$\begin{array}{ll}
1. \int \sin x \cos x dx & 2. \int \sin^2 x \cos^3 x dx \\
3. \int \sin^{-2/3} x \cos x dx & 4. \int \frac{\sin^3 x}{\cos^2 x} dx
\end{array}$$

2. Products of the types  $\sin mx \cos nx$ ,  $\sin mx \sin nx$ , or  $\cos mx \cos nx$  may be integrated as in the following example.

**EXAMPLE 3.** Find  $\int \sin 3x \cos 2x dx$

Since  $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$ ,  
we have for  $A = 3x, B = 2x$ ,

$$\int \sin 3x \cos 2x dx = \frac{1}{2} \int (\sin 5x + \sin x) dx = -\frac{\cos 5x}{10} - \frac{\cos x}{2} + C$$

**EXAMPLE 4.** Find  $\int \sin 5x \cos x dx, \int \cos 3x \cos 2x dx,$   
 $\int \sin 3x \sin 2x dx$

3. Integrals of the type  $\int \sec^m x \tan^n x dx$  may be found *when*  $m$  is 0 or even by integrating with respect to  $\tan x$ , and *when*  $n$  is odd by integrating with respect to  $\sec x$ . The like is true of  $\int \operatorname{cosec}^m x \cot^n x dx$ . Thus

$$\begin{aligned}\int \tan^3 x \, dx &= \int \tan x (\sec^2 x - 1) \, dx = \int \tan x \cdot d \tan x - \int \tan x \, dx \\ &= \frac{\tan^2 x}{2} + \log \cos x + C\end{aligned}$$

$$\int \tan^3 x \sec x \, dx = \int (\sec^2 x - 1) d \sec x = \frac{\sec^3 x}{3} - \sec x + C$$

EXAMPLE 5. Find  $\int \sec^4 x \, dx$ ,  $\int \tan^3 x \sec^4 x \, dx$ ,  $\int \operatorname{cosec}^3 x \cot x \, dx$

EXAMPLE 6. Prove that  $\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$

**113. Integrands with quadratic denominators.** 1. Integrals of the types

$$1. \int \frac{dx}{x^2 + bx + c} \qquad 2. \int \frac{dx}{(\pm x^2 + bx + c)^{1/2}}$$

can be found by Formulas 14-17 of § 111. We complete the square of the  $x$  terms in  $\pm x^2 + bx + c$ , thus

$$\begin{aligned}x^2 + bx + c &= \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right) \\ -x^2 + bx + c &= \left(c + \frac{b^2}{4}\right) - \left(x - \frac{b}{2}\right)^2,\end{aligned}$$

and then set  $u = x + b/2$ , or  $u = x - b/2$ .

EXAMPLE 1. Find  $\int \frac{dx}{2x^2 - 6x + 5}$

$$2x^2 - 6x + 5 = 2(x^2 - 3x + 5/2) = 2[(x - 3/2)^2 + 1/4]. \quad \text{Hence}$$

$$\begin{aligned}\int \frac{dx}{2x^2 - 6x + 5} &= \frac{1}{2} \int \frac{d(x - 3/2)}{(x - 3/2)^2 + 1/4} = \tan^{-1} \frac{x - 3/2}{1/2} + C \\ &= \tan^{-1} (2x - 3) + C.\end{aligned}$$

EXAMPLE 2. Find  $\int \frac{dx}{(2ax - x^2)^{1/2}}$

$$2ax - x^2 = -(x^2 - 2ax) = -(x^2 - 2ax + a^2) + a^2 = a^2 - (x - a)^2.$$

Hence

$$\int \frac{dx}{(2ax - x^2)^{1/2}} = \int \frac{d(x - a)}{[a^2 - (x - a)^2]^{1/2}} = \sin^{-1} \frac{x - a}{a} + C.$$

EXAMPLE 3. Integrate the following:

$$1. \int \frac{dx}{x^2 + 4x + 5} \qquad 2. \int \frac{dx}{(x^2 + 4x + 5)^{1/2}} \qquad 3. \int \frac{dx}{(3 + 2x - x^2)^{1/2}}$$

2. Fractions with denominators like those in 1, 1. and 2., but having numerators of the first degree, may be integrated as in the following examples.

EXAMPLE 4. 
$$\int \frac{x+1}{x^2+1} dx = \int \frac{x dx}{x^2+1} + \int \frac{dx}{x^2+1}$$

$$= \frac{1}{2} \log(x^2+1) + \tan^{-1} x + C.$$

EXAMPLE 5. Find  $\int \frac{3x+2}{x^2+2x} dx$  and  $\int \frac{3x+2}{(x^2+2x)^{1/2}} dx$

First express the numerator  $3x+2$  in terms of the derivative of  $x^2+2x$ , that is, find the constants  $A$  and  $B$  such that

$$3x+2 \equiv A(2x+2) + B \equiv 2Ax + (2A+B).$$

We find  $A$  and  $B$  by equating coefficients of like powers of  $x$  in this supposed identity. Thus  $2A = 3$  and  $2A+B = 2 \therefore A = 3/2$ ,  $B = -1$ . Hence

$$\begin{aligned} \int \frac{3x+2}{x^2+2x} dx &= \frac{3}{2} \int \frac{d(x^2+2x)}{x^2+2x} - \int \frac{d(x+1)}{(x+1)^2-1} \\ &= \frac{3}{2} \log(x^2+2x) - \frac{1}{2} \log \frac{x}{x+2} + C. \end{aligned}$$

$$\begin{aligned} \int \frac{3x+2}{(x^2+2x)^{1/2}} dx &= \frac{3}{2} \int \frac{d(x^2+2x)}{(x^2+2x)^{1/2}} - \int \frac{d(x+1)}{[(x+1)^2-1]^{1/2}} \\ &= 3(x^2+2x)^{1/2} - \log[x+1+(x^2+2x)^{1/2}] + C. \end{aligned}$$

EXAMPLE 6. Find  $\int \frac{4x+7}{x^2+4x+5} dx$  and  $\int \frac{4x+7}{[x^2+4x+5]^{1/2}} dx$

### EXERCISE XX

Effect the following integrations :

- |                                       |                                          |                                           |
|---------------------------------------|------------------------------------------|-------------------------------------------|
| 1. $\int \sin(4-5x) dx$               | 2. $\int (x^e + e^x) dx$                 | 3. $\int \sqrt{\cos x} \sin^3 x dx$       |
| 4. $\int \cos^3 x / \sin^4 x dx$      | 5. $\int \tan x \sec^5 x dx$             | 6. $\int \tan^4 x dx$                     |
| 7. $\int \sin 10x \cos 5x dx$         | 8. $\int \frac{a+b \sin x}{\cos^2 x} dx$ | 9. $\int \frac{dx}{\sqrt{a^2-b^2x^2}}$    |
| 10. $\int \frac{dx}{x^2-x+1}$         | 11. $\int \frac{dx}{x^2-8x+12}$          | 12. $\int \frac{5x+6}{(x^2+4x)^{1/2}} dx$ |
| 13. $\int \frac{p+qx}{a^2+b^2x^2} dx$ | 14. $\int \frac{dx}{\sqrt{3x-x^2}}$      | 15. $\int \frac{dx}{\sqrt{x(1-x)}}$       |

$$16. \int \frac{3x - 4}{2x^2 - 3x + 1} dx$$

$$17. \int \frac{5 - 4x}{(1 + 4x - x^2)^{1/2}} dx$$

$$18. \int \frac{1 + 2x}{\sqrt{4 - x^2}} dx$$

$$19. \int \left( \frac{1-x}{1+x} \right)^{1/2} dx = \int \frac{1-x}{\sqrt{1-x^2}} dx$$

$$20. \int \left( \frac{a+x}{x} \right)^{1/2} dx$$

21. By aid of the trigonometric identities  $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ ,  $1 = \cos^2 \theta + \sin^2 \theta$ , show that

$$\int (1 + \cos x)^{1/2} dx = 2\sqrt{2} \sin \frac{x}{2}, \quad \int (1 - \cos x)^{1/2} dx = -2\sqrt{2} \cos \frac{x}{2}.$$

**114. Integration by parts.** Let  $u, v$  denote functions of  $x$ , and  $u', v'$  their derivatives. Since  $d(uv)/dx = uv' + vu'$ , we have

$$\int uv' dx = uv - \int vu' dx \quad (1)$$

Hence if  $v'$  denote an integrable factor of  $f(x)$ , or be 1, and if  $u$  be the complementary factor, so that  $f(x) = uv'$ , then the integral  $\int f(x) dx$ , or  $\int uv' dx$ , can be found when  $\int vu' dx$  can be found, the rule being:

*First integrate  $uv'$  as if  $u$  were a constant, then from the result,  $uv$ , subtract the integral of  $vu'$ .*

This process is called *integration by parts*. It is very useful. Since  $v' dx = dv$ ,  $u' dx = du$ , we may also write (1) in the form

$$\int u dv = uv - \int v du \quad (2)$$

**EXAMPLE 1.** Find  $\int x \tan^{-1} x dx$ .

The factor  $x$  being integrable, we have by the rule just given

$$\begin{aligned} \int \tan^{-1} x \cdot x dx &= \frac{x^2}{2} \tan^{-1} x - \int \frac{x^2}{2} \frac{1}{x^2 + 1} dx \\ &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} (x - \tan^{-1} x) + C. \end{aligned}$$

**EXAMPLE 2.** Find  $\int \log x dx$ .

There being no immediately integrable factor, we take  $v' = 1$  and have

$$\int \log x dx = x \log x - \int x \cdot \frac{1}{x} dx = x \log x - x + C.$$

EXAMPLE 3. Find  $\int x^2 \cos x \, dx$ .

Since the exponent of  $x^2$  would be increased by integration, take  $v' = \cos x$ ; then

$$\int x^2 \cdot \cos x \, dx = x^2 \sin x - 2 \int x \sin x \, dx$$

$$\int x \cdot \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C$$

$$\text{Hence } \int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

EXAMPLE 4. Show that

$$\int (x^2 + a^2)^{1/2} dx = \frac{1}{2} \{ x(x^2 + a^2)^{1/2} + a^2 \log [x + (x^2 + a^2)^{1/2}] \}.$$

$$\text{Int'g by parts, } \int (x^2 + a^2)^{1/2} dx = x(x^2 + a^2)^{1/2} - \int \frac{x^2 \, dx}{(x^2 + a^2)^{1/2}}$$

$$\begin{aligned} \text{But } \int \frac{x^2}{(x^2 + a^2)^{1/2}} \, dx &= \int \frac{(x^2 + a^2) - a^2}{(x^2 + a^2)^{1/2}} \, dx \\ &= \int (x^2 + a^2)^{1/2} dx - \int \frac{a^2 \, dx}{(x^2 + a^2)^{1/2}} \end{aligned}$$

$$\text{Hence } 2 \int (x^2 + a^2)^{1/2} dx = x(x^2 + a^2)^{1/2} + a^2 \log [x + (x^2 + a^2)^{1/2}].$$

EXAMPLE 5. Find 1.  $\int x e^x \, dx$ . 2.  $\int x^2 e^x \, dx$ . 3.  $\int x^3 e^{3x} \, dx$ .

$$4. \int x^3 \log x \, dx. \quad 5. \int \tan^{-1} x \, dx. \quad 6. \int \sin^{-1} x \, dx. \quad 7. \int x \sin 2x \, dx.$$

$$8. \int e^x \sin x \, dx.$$

EXAMPLE 6. Prove that

$$2 \int \sec^3 x \, dx = \sec x \tan x + \log (\sec x + \tan x).$$

115. Reduction formulas for  $\int \cos^n x \, dx$  and  $\int \sin^n x \, dx$ .

Integrating by parts,

$$\int \cos^n x \, dx = \int \cos^{n-1} x \cdot \frac{d}{dx} \sin x \, dx$$

$$= \sin x \cos^{n-1} x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx$$

$$= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

Hence, transposing the last term,

$$n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx \quad (1)$$

By the same method, or by replacing  $x$  by  $\pi/2 - x$  in (1), we find

$$n \int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx \quad (2)$$

These formulas enable one to find  $\int \cos^n x \, dx$  and  $\int \sin^n x \, dx$  when  $n$  is any given positive integer. Formula (1) is especially useful.

**EXAMPLE 1.** Find  $\int \cos^4 x \, dx$  and  $\int \cos^2 x \sin^2 x \, dx$ .

$$4 \int \cos^4 x \, dx = \cos^3 x \sin x + 3 \int \cos^2 x \, dx$$

$$2 \int \cos^2 x \, dx = \cos x \sin x + \int dx = \cos x \sin x + x + C$$

Hence  $\int \cos^4 x \, dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3}{8} x + C$

and  $\int \cos^2 x \sin^2 x \, dx = \int \cos^2 x \, dx - \int \cos^4 x \, dx$   
 $= -\frac{1}{4} \cos^3 x \sin x + \frac{1}{8} (\cos x \sin x + x) + C.$

**EXAMPLE 2.** Find  $\int \cos^6 x \, dx$ ,  $\int \sin^5 x \, dx$ ,  $\int \cos^3 2x \, dx$ ,  
 $\int \sin^2 x \cos^4 x \, dx.$

**116. Trigonometric substitutions.** Expressions involving  $\sqrt{a^2 - x^2}$  or  $\sqrt{x^2 \pm a^2}$  are often most easily integrated by aid of one of the following substitutions:

when  $\sqrt{a^2 - x^2}$  occurs, set  $x = a \sin \theta \therefore \sqrt{a^2 - x^2} = a \cos \theta$

when  $\sqrt{x^2 + a^2}$  occurs, set  $x = a \tan \theta \therefore \sqrt{x^2 + a^2} = a \sec \theta$

when  $\sqrt{x^2 - a^2}$  occurs, set  $x = a \sec \theta \therefore \sqrt{x^2 - a^2} = a \tan \theta$

**EXAMPLE 1.** Find  $\int \sqrt{a^2 - x^2} \, dx$ .

Let  $x = a \sin \theta \therefore \sqrt{a^2 - x^2} = a \cos \theta$  and  $dx = a \cos \theta \, d\theta$ . Then

$$\begin{aligned} \int \sqrt{a^2 - x^2} \, dx &= a^2 \int \cos^2 \theta \, d\theta \\ &= a^2 \frac{\sin \theta \cos \theta + \theta}{2} + C \\ &= \frac{1}{2} \left[ x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right] + C \end{aligned}$$

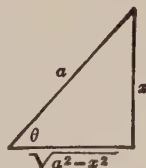


FIG. 79.

**EXAMPLE 2.** Find  $\int \frac{dx}{x \sqrt{x^2 - a^2}}$

Let  $x = a \sec \theta \therefore \sqrt{x^2 - a^2} = a \tan \theta$  and  $dx = a \sec \theta \tan \theta \, d\theta$ .

Then  $\int \frac{dx}{x \sqrt{x^2 - a^2}} = \frac{1}{a} \int d\theta = \frac{\theta}{a} + C = \frac{1}{a} \sec^{-1} \frac{x}{a} + C$



**EXAMPLE 3.** Find  $\int \frac{dx}{(x^2 + a^2)^{3/2}}$

Let  $x = a \tan \theta$   $\therefore (x^2 + a^2)^{1/2} = a \sec \theta$ , and  $dx = a \sec^2 \theta d\theta$ .

$$\begin{aligned} \text{Then } \int \frac{dx}{(x^2 + a^2)^{3/2}} &= \frac{a}{a^3} \int \frac{\sec^2 \theta}{\sec^3 \theta} d\theta = \frac{1}{a^2} \int \cos \theta d\theta \\ &= \frac{1}{a^2} \sin \theta = \frac{1}{a^2} \frac{x}{(x^2 + a^2)^{1/2}} \end{aligned}$$

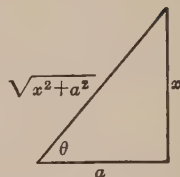


FIG. 80.

This method can be extended to expressions involving  $(ax^2 + bx + c)^{1/2}$ .

**EXAMPLE 4.** Find  $\int \frac{x dx}{\sqrt{4x - x^2}} = \int \frac{x dx}{[4 - (x - 2)^2]^{1/2}}$

Let  $x - 2 = 2 \sin \theta$   $\therefore \sqrt{4x - x^2} = 2 \cos \theta$  and  $dx = 2 \cos \theta d\theta$ .  
Then

$$\begin{aligned} \int \frac{x dx}{\sqrt{4x - x^2}} &= \int \frac{(2 + 2 \sin \theta) \cdot 2 \cos \theta}{2 \cos \theta} d\theta = 2 \int (1 + \sin \theta) d\theta \\ &= 2\theta - 2 \cos \theta = 2 \sin^{-1} \frac{x-2}{2} - \sqrt{4x - x^2} \end{aligned}$$

**EXAMPLE 5.** Find 1.  $\int \frac{(25 - x^2)^{1/2}}{x} dx$

2.  $\int \frac{dx}{x(3 - x^2)^{1/2}}$

3.  $\int (3 - 2x - x^2)^{1/2} dx$

4.  $\int \frac{dx}{(x^2 + 4)^{5/2}}$

5.  $\int \frac{dx}{x(x^2 + 4)^{1/2}}$

6.  $\int \frac{(x^2 - 1)^{1/2}}{x} dx$

7.  $\int \frac{dx}{x^3(x^2 - 1)^{1/2}}$

**117. The substitution  $x = 1/t$ .** Integrals of fractions whose denominators contain some power of  $x$  or of  $x - c$  as a factor may often be simplified by the substitution  $x = 1/t$  or  $x - c = 1/t$ ,  $dx = -dt/t^2$ .

**EXAMPLE 1.** Setting  $x = \frac{1}{t}$  gives

$$\int \frac{dx}{x^2(x^2 + 1)^{1/2}} = - \int \frac{t dt}{(1 + t^2)^{1/2}} = -(1 + t^2)^{1/2} = -\frac{(x^2 + 1)^{1/2}}{x}.$$

**EXAMPLE 2.** Find 1.  $\int \frac{dx}{x(x^2 + 1)^{1/2}}$

2.  $\int \frac{dx}{(x - 1)(2 - x^2)^{1/2}}$

3.  $\int \frac{dx}{x^2(x^2 + x + 1)^{1/2}}$

## EXERCISE XXI

Find the following integrals :

1.  $\int x \sin 3x \, dx$
2.  $\int x^2 \cos 2x \, dx$
3.  $\int x \sec^2 x \, dx$
4.  $\int \tan^{-1} \frac{x}{2} \, dx$
5.  $\int x^2 \tan^{-1} x \, dx$
6.  $\int \tan^{-1} \sqrt{x} \, dx$
7.  $\int x \sin^{-1} x \, dx$
8.  $\int x^2 \sin^{-1} x \, dx$
9.  $\int x^2 e^{4x} \, dx$
10.  $\int x^{-2} \log x \, dx$
11.  $\int \log (x^2 - 1) \, dx$
12.  $\int e^{3x} \cos 2x \, dx$
13.  $\int (4 - x^2)^{3/2} \, dx$
14.  $\int \frac{dx}{(x^2 + 1)^{5/2}}$
15.  $\int \frac{dx}{(x^2 - 1)^{5/2}}$
16.  $\int \frac{dx}{(x - 1)(x^2 + 1)^{1/2}}$
17.  $\int \frac{dx}{(2 + \tan^2 x)^{1/2}} = \int \frac{\cos x \, dx}{(2 - \sin^2 x)^{1/2}}$
18.  $\int \frac{x^2 \, dx}{(x^2 + 1)^{3/2}}$
19.  $\int \frac{dx}{(x^2 + 1)(x^2 + 2)^{1/2}}, [\text{set } x = \tan \theta.]$

20. Show that setting  $n = -m + 2$  in § 115 (1) gives the reduction formula

$$(m - 1) \int \sec^m x \, dx = \sec^{m-1} x \sin x + (m - 2) \int \sec^{m-2} x \, dx$$

**118. Integration of rational fractions.** Let  $f(x)/\phi(x)$  be a proper rational fraction with real coefficients, in its lowest terms. By the fundamental theorem of algebra, § 19, 3., the polynomial  $\phi(x)$  is a product of factors of one or both the types  $x - a$  and  $x^2 + bx + c$ , where  $a, b, c$  are real but the factors of  $x^2 + bx + c$  are imaginary. It can be proved<sup>1</sup> that  $f(x)/\phi(x)$  is equal to one and but one sum of fractions, called its *partial fractions*, related as follows to the factors of  $\phi(x)$  :

1. For each factor  $x - a$  occurring but once in  $\phi(x)$ , there is a single fraction of the form  $A/(x - a)$ , where  $A$  is a real constant, not 0.

2. For each factor  $x - a$  occurring  $r$  times in  $\phi(x)$ , there is a group of fractions of the form

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \dots + \frac{A_r}{(x - a)^r}$$

<sup>1</sup> See Fine's College Algebra, §§ 529-536.

where  $A_1, \dots, A_r$  are real constants, any of which except  $A_r$  may be 0.

3. For each factor  $x^2 + bx + c$  occurring but once in  $\phi(x)$ , there is a single fraction  $(Bx + C)/(x^2 + bx + c)$ , where  $B, C$  are real constants, not both 0.

4. For each factor  $x^2 + bx + c$  occurring  $r$  times in  $\phi(x)$ , there is a group of fractions

$$\frac{B_1x + C_1}{x^2 + bx + c} + \frac{B_2x + C_2}{(x^2 + bx + c)^2} + \dots + \frac{B_rx + C_r}{(x^2 + bx + c)^r}$$

where  $B_1, C_1, \dots$  are real constants of which not both  $B_r, C_r$  are 0.

Thus we may set

$$\frac{x^3 + x^2 - 5}{(2x - 1)x^3(3x^2 + 1)(x^2 + 1)^2} = \frac{A}{2x - 1} + \frac{B_1}{x} + \frac{B_2}{x^2} + \frac{B_3}{x^3} + \frac{Cx + D}{3x^2 + 1} + \frac{E_1x + F_1}{x^2 + 1} + \frac{E_2x + F_2}{(x^2 + 1)^2}$$

with assurance that one and but one set of values of the constants  $A, B_1, \dots, F_2$  exists for which this identity is true. The values of  $A, B_1, \dots, F_2$  may then be found by the methods used in computing undetermined coefficients.

The fractions in 1., 2., 3., 4. can be integrated (see Ex. 4 below). Hence we can integrate any given proper fraction  $f(x)/\phi(x)$ , if we can find the factors  $x - a$  and  $x^2 + bx + c$  of  $\phi(x)$ . To extend the method to an improper fraction, we express it as the sum of an integral expression and a proper fraction.

**EXAMPLE 1.** Find  $\int \frac{x^2 + 4x - 1}{x(2x - 1)(x + 2)} dx$

Set 
$$\frac{x^2 + 4x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$$

Then  $x^2 + 4x - 1 \equiv A(2x - 1)(x + 2) + B(x + 2)x + Cx(2x - 1)$

Since this is an identity, it is true for all values of  $x$ .

Let  $x = 0$  ; then  $-1 = A(-1) \cdot 2 \therefore A = 1/2$

Let  $2x - 1 = 0$  or  $x = 1/2$ ; then  $5/4 = B(1/2)(5/2) \therefore B = 1$

Let  $x + 2 = 0$  or  $x = -2$ ; then  $-5 = C(-2)(-5) \therefore C = -1/2$

Hence

$$\begin{aligned}\int \frac{x^2 + 4x - 1}{x(2x - 1)(x + 2)} dx &= \frac{1}{2} \int \frac{dx}{x} + \int \frac{dx}{2x - 1} - \frac{1}{2} \int \frac{dx}{x + 2} \\ &= \log \left[ \frac{x(2x - 1)}{x + 2} \right]^{1/2} + C\end{aligned}$$

EXAMPLE 2. Find  $\int \frac{x^3 + x^2 - 2x - 3}{x^3 - 1} dx$

$$\frac{x^3 + x^2 - 2x - 3}{x^3 - 1} = 1 + \frac{x^2 - 2x - 2}{x^3 - 1} = 1 + \frac{x^2 - 2x - 2}{(x - 1)(x^2 + x + 1)}$$

$$\text{Set } \frac{x^2 - 2x - 2}{(x - 1)(x^2 + x + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}$$

$$\begin{aligned}\text{Then } x^2 - 2x - 2 &= A(x^2 + x + 1) + (Bx + C)(x - 1) \\ &= (A + B)x^2 + (A - B + C)x + (A - C)\end{aligned}$$

Equating coefficients of like powers of  $x$ , we get

$$A + B = 1, A - B + C = -2, A - C = -2 \quad \therefore A = -1, B = 2, C = 1.$$

Hence

$$\begin{aligned}\int \frac{x^3 + x^2 - 2x - 3}{x^3 - 1} dx &= \int dx - \int \frac{dx}{x - 1} + \int \frac{2x + 1}{x^2 + x + 1} dx \\ &= x + \log \frac{x^2 + x + 1}{x - 1} + C\end{aligned}$$

EXAMPLE 3. Find  $\int \frac{x^2 - 4x + 7}{(x - 1)^2(x^2 + 1)} dx$

$$\text{Set } \frac{x^2 - 4x + 7}{(x - 1)^2(x^2 + 1)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{Cx + D}{x^2 + 1}$$

Then

$$x^2 - 4x + 7 = A(x - 1)(x^2 + 1) + B(x^2 + 1) + (Cx + D)(x - 1)^2 \quad (1)$$

Let  $x = 1$ ; then  $4 = 2B$  and therefore  $B = 2$

Transposing the term  $2(x^2 + 1)$  thus found to the first member of (1),

$$-x^2 - 4x + 5 = A(x - 1)(x^2 + 1) + (Cx + D)(x - 1)^2 \quad (2)$$

Dividing by  $x - 1$ ,

$$-x - 5 = A(x^2 + 1) + (Cx + D)(x - 1) \quad (3)$$

Again let  $x = 1$ . We get  $-6 = 2A$  and therefore  $A = -3$ . Also by equating the coefficients of  $x^2$  and  $x^0$  in (3) we get  $A + C = 0$ ,  $A - D = -5 \quad \therefore C = 3, D = 2$ . Hence

$$\begin{aligned}
& \int \frac{x^2 - 4x + 7}{(x-1)^2(x^2+1)} dx \\
&= -3 \int \frac{dx}{x-1} + 2 \int \frac{dx}{(x-1)^2} + \int \frac{(3x+2)}{x^2+1} dx \\
&= -3 \log(x-1) - \frac{2}{x-1} + \frac{3}{2} \log(x^2+1) + 2 \tan^{-1} x + C.
\end{aligned}$$

**EXAMPLE 4.** Find  $\int \frac{4x-1}{(x^2+2x+5)^2} dx$ .

Expressing  $4x-1$  in terms of  $\frac{d}{dx}(x^2+2x+5)$ , as in § 113, Ex. 5 we get

$$\begin{aligned}
\int \frac{4x-1}{(x^2+2x+5)^2} dx &= 2 \int \frac{2x+2}{(x^2+2x+5)^2} dx - 5 \int \frac{dx}{(x^2+2x+5)^2} \\
&= -\frac{2}{x^2+2x+5} - 5 \int \frac{d(x+1)}{[(x+1)^2+4]^2}
\end{aligned}$$

The substitution  $x+1 = 2 \tan \theta$  gives

$$\begin{aligned}
\int \frac{d(x+1)}{[(x+1)^2+4]^2} &= \frac{1}{8} \int \cos^2 \theta d\theta \\
&= \frac{1}{16} (\cos \theta \sin \theta + \theta) \\
&= \frac{1}{8} \frac{x+1}{x^2+2x+5} + \frac{1}{16} \tan^{-1} \frac{x+1}{2}
\end{aligned}$$

Hence

$$\int \frac{4x-1}{(x^2+2x+5)^2} dx = -\frac{1}{8} \frac{5x+21}{x^2+2x+5} - \frac{5}{16} \tan^{-1} \frac{x+1}{2} + C$$

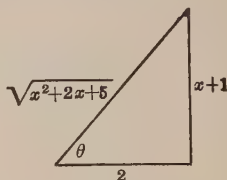


FIG. 81.

## EXERCISE XXII

Find the following integrals:

1.  $\int \frac{x^3}{x^2+3x+2} dx$
2.  $\int \frac{-4x+6}{(x-1)(x-2)(x-3)} dx$
3.  $\int \frac{2x+4}{x^3-x} dx$
4.  $\int \frac{x^2}{(x-a)(x-b)(x-c)} dx$
5.  $\int \frac{x^2+3x+3}{2x^3+5x^2-3x} dx$
6.  $\int \frac{x^2+1}{(x+2)^2} dx$
7.  $\int \frac{2x-1}{x(x-2)^2} dx$
8.  $\int \frac{3x^2+1}{x(x^2+x+1)} dx$
9.  $\int \frac{dx}{x^4-16}$

$$10. \int \frac{x^2 + 5x + 2}{x^4 - 5x^2 + 4} dx$$

$$12. \int \frac{dx}{(x^2 - 1)^2}$$

$$14. \int \frac{dx}{(x^2 + 9)^3}$$

$$11. \int \frac{2x^3 + 9}{x^4 + x^3 - 12x^2} dx$$

$$13. \int \frac{x^3 + 4}{(x^2 + 2x + 2)^2} dx$$

$$15. \int \frac{x^{1/3} - 1}{x^{3/2} - x} dx \text{ (set } x = t^6 \text{)}$$

**119. General conclusions.** Let  $R(u)$  or  $R(u, v)$  denote a rational function of  $u$  or of  $u, v$ .

1. It is shown in § 118 that every integral of the form  $\int R(x)dx$  is expressible in terms of rational, logarithmic and  $\tan^{-1}$  functions.

2. Any integral which has the form  $\int R(\sin \theta, \cos \theta)d\theta$  may be reduced by the substitution  $\theta = 2 \tan^{-1} t$  to the form  $\int R(t)dt$ . For if  $t = \tan \frac{\theta}{2}$ , then

$$\sin \theta = \frac{2t}{1+t^2}, \cos \theta = \frac{1-t^2}{1+t^2}, d\theta = \frac{2dt}{1+t^2} \quad (1)$$

all of which are rational with respect to  $t$ .

3. Any integral of the form  $\int R[x, (ax^2 + bx + c)^{1/2}]dx$  may be reduced by the trigonometric substitutions of § 116 to the form  $\int R(\sin \theta, \cos \theta)d\theta$ , and therefore, by 2., to the form  $\int R(t)dt$ .

**EXAMPLE.** By the substitution (1), show that

$$\int \operatorname{cosec} x \, dx = \log \tan \frac{x}{2}, \quad \int \sec x \, dx = \log \tan \left[ \frac{x}{2} + \frac{\pi}{4} \right].$$

$$\text{Also find } \int \frac{dx}{2 + \cos x} \quad \text{and} \quad \int \frac{dx}{3 \sin x + 4 \cos x}$$

## XV. DEFINITE INTEGRALS

**120. Second fundamental problem of the calculus.** The second principal problem of the calculus is to define and evaluate the numerical measures of continuous magnitudes, as for example geometric surfaces and solids. The method of definition is illustrated by § 121, and the theorems which justify this method are given in §§ 122, 123, 124.

**121. Area.** Consider the space  $R = abBA$  bounded by the lines  $Ox$ ,  $x = a$ ,  $x = b$ , and a continuous curve arc  $AB$  which does not cross  $Ox$ . The elementary definition of area applies to rectilinear figures only. It is desired to extend this notion to the space  $R$ .

Suppose the segment  $ab$  to be divided and redivided into parts in any manner such that as the process is continued the length of every part approaches the limit 0. At any stage of the process let  $h_1, h_2, \dots, h_n$  denote the parts and also their lengths. On each part  $h_i$  as base construct the rectangles whose altitudes are the lengths  $m_i$  and  $M_i$  of the least and greatest curve ordinates on  $h_i$ . The rectangles with least altitudes form a rectilinear figure contained in  $R$ , whose area is

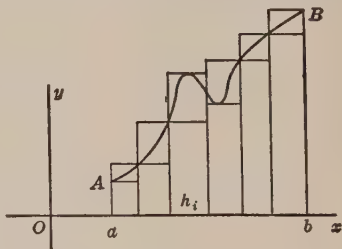


FIG. 82.

$$s_n = m_1 h_1 + m_2 h_2 + \dots + m_n h_n$$

The rectangles with greatest altitudes form a rectilinear figure containing  $R$ , whose area is

$$S_n = M_1 h_1 + M_2 h_2 + \dots + M_n h_n$$



In § 124 it is proved that as  $n \rightarrow \infty$  the sums  $s_n$  and  $S_n$  approach one and the same number  $l$  as limit. We call this limit  $l$  the *area of the space  $R$* .

Let  $y = f(x)$  be the equation of the arc  $AB$ . Then if  $x_i$  be any point in  $h_i$ , we shall have  $m_i \leq f(x_i) \leq M_i$  and therefore

$$\text{area } abBA = \lim_{n \rightarrow \infty} [f(x_1)h_1 + f(x_2)h_2 + \cdots + f(x_n)h_n]$$

**122. Uniform continuity.** Let  $f(x)$  be continuous in the interval  $(a, b)$ ; the difference between the greatest and least values of  $f(x)$  in any part of  $(a, b)$  is called the *oscillation* of  $f(x)$  in that part. Thus the oscillation of  $x^2$  in (1.2, 1.3) is  $1.3^2 - 1.2^2 = .25$ . It will be proved later that

*Corresponding to any assigned positive number  $\epsilon$  there is another positive number  $\delta$  such that in every part of  $(a, b)$  whose length is less than  $\delta$  the oscillation of  $f(x)$  is less than  $\epsilon$ .*

To indicate this, we say that if  $f(x)$  is continuous in  $(a, b)$ , it is *uniformly* continuous in  $(a, b)$ .

**123. Greatest and least values in an interval and its parts.** Evidently the greatest value of  $f(x)$  in a part of an interval cannot exceed its greatest value in the whole interval. Hence if an interval of length  $h$  be divided into parts of lengths  $h_1$  and  $h_2$ , and if  $M, M_1, M_2$  denote the greatest values of  $f(x)$  in the whole interval and these parts, then since  $M \geq M_1, M_2$ ,

$$\begin{aligned} Mh &= M(h_1 + h_2) \\ &= Mh_1 + Mh_2 \geq M_1h_1 + M_2h_2 \end{aligned} \quad (1)$$

Similarly, if  $m, m_1, m_2$  denote the least values of  $f(x)$  in the whole interval and the parts,

$$mh \leq m_1h_1 + m_2h_2 \quad (2)$$

The like is true when an interval is divided into any number of parts.

**124. Theorem.** Let  $f(x)$  be continuous in the interval  $(a, b)$ . Suppose  $(a, b)$  to be divided and redivided into parts in any manner such that as the process is continued the lengths of the parts all approach 0 as limit. At any stage of the process let  $h_1, h_2, \dots, h_n$  denote the parts and also their lengths; also let  $x_1$  denote any point in  $h_1$ ,  $x_2$  any point in  $h_2$ , and so on. Then, as  $n \rightarrow \infty$ , the sum of products

$$\Sigma f(x_i)h_i = f(x_1)h_1 + f(x_2)h_2 + \dots + f(x_n)h_n \quad (1)$$

approaches a limit, and the value of this limit is independent of the mode of division of  $(a, b)$  into parts  $h_i$  and of the choice of the  $x_i$ 's in the  $h_i$ 's.

For let  $M_i$  and  $m_i$  denote the greatest and least values of  $f(x)$  in the part  $h_i$ , and form the sums

$$S_n = M_1h_1 + M_2h_2 + \dots + M_nh_n \quad (2)$$

$$s_n = m_1h_1 + m_2h_2 + \dots + m_nh_n \quad (3)$$

Since  $s_n \leq \Sigma f(x_i)h_i \leq S_n$ , the theorem will be proved if it can be shown that  $\lim (S_n - s_n) = 0$  and that  $S_n$  approaches the same limit  $l$  for all modes of division of  $(a, b)$  into parts  $h_i$ .

1. The proof that  $\lim (S_n - s_n) = 0$ .

$$S_n - s_n = (M_1 - m_1)h_1 + (M_2 - m_2)h_2 + \dots + (M_n - m_n)h_n$$

But  $M_i - m_i$  is the oscillation of  $f(x)$  in the part  $h_i$ . Hence, § 122, we can take  $n$  great enough to make every  $M_i - m_i$  less than any assigned positive number  $\epsilon$  and therefore to make

$$S_n - s_n < \epsilon h_1 + \epsilon h_2 + \dots + \epsilon h_n = \epsilon(b - a) \quad (4)$$

We can take  $\epsilon(b - a)$  as small as we please; hence

$$\lim (S_n - s_n) = 0$$

2. The proof that  $S_n$  approaches a unique limit  $l$ .

First. Suppose that  $h_1, h_2, \dots, h_n$  are obtained by dividing  $(a, b)$  into certain parts, then subdividing these parts, and so on. Then, as  $n$  increases,  $S_n$  decreases (or remains un-

changed), § 123 (1), but remains always greater than any value of  $s_n$ . Hence in this case  $S_n$  approaches a limit  $l$ , § 5.

*Second.* Representing the points which divide  $(a, b)$  into the parts  $h_1, h_2, \dots, h_n$  just considered by  $t_1, t_2, \dots, t_{n-1}$ , let  $z_1, z_2, \dots, z_{m-1}$  denote any other set of points which divide  $(a, b)$  into parts whose lengths all  $\rightarrow 0$  when  $m \rightarrow \infty$ . Let  $S'_m$  and  $s'_m$  denote the sums like (2) and (3) which correspond to these  $z$ -parts. Then  $\lim_{m \rightarrow \infty} S'_m = \lim_{n \rightarrow \infty} S_n = l$ .

For let  $S''_p$  and  $s''_p$  denote the sums like (2) and (3) corresponding to the parts into which the  $t$ -points and the  $z$ -points when taken together divide  $(a, b)$ .

As already proved, (4), we can take  $n$  and  $m$  great enough to make

$$S_n - s_n < \epsilon(b - a) \quad \text{and} \quad S'_m - s'_m < \epsilon(b - a)$$

The  $z$ -points subdivide the  $t$ -parts; hence  $S_n \geq S''_p \geq s_n$ , and therefore

$$S_n - S''_p \leq S_n - s_n < \epsilon(b - a) \quad (5)$$

The  $t$ -points subdivide the  $z$ -parts; hence  $S'_m \geq S''_p \geq s'_m$ , and therefore

$$S'_m - S''_p \leq S'_m - s'_m < \epsilon(b - a) \quad (6)$$

But  $S_n - S'_m = (S_n - S''_p) - (S'_m - S''_p)$ ; hence

$$|S_n - S'_m| < \epsilon(b - a) \quad (7)$$

We can take  $\epsilon(b - a)$  as small as we please; hence

$$\lim S'_m = \lim S_n = l$$

We represent the limit  $l$  by the symbol  $\lim [\Sigma f(x_i)h_i]_a^b$ .

**125. Properties of  $\lim [\Sigma f(x_i)h_i]_a^b$ .** 1. If we replace the interval  $(a, b)$  by the interval  $(b, a)$ , we change the sign of every  $h_i$  in the sum  $\Sigma f(x_i)h_i$ , but leave the sum otherwise unchanged. Hence

$$\lim [\Sigma f(x_i)h_i]_b^a = - \lim [\Sigma f(x_i)h_i]_a^b \quad (1)$$

2. If we divide the interval  $(a, b)$  into two parts,  $(a, c)$  and  $(c, b)$ , we may break up the sum  $[\Sigma f(x_i)h_i]_a^b$  into the two sums  $[\Sigma f(x_i)h_i]_a^c$  and  $[\Sigma f(x_i)h_i]_c^b$ . Hence

$$\lim [\Sigma f(x_i)h_i]_a^b = \lim [\Sigma f(x_i)h_i]_a^c + \lim [\Sigma f(x_i)h_i]_c^b \quad (2)$$

By aid of (1) we can extend (2) to the case when  $c < a$  or  $c > b$ .

3. By § 123 the value of  $\lim [\Sigma f(x_i)h_i]_a^b$  is between  $m(b - a)$  and  $M(b - a)$ , where  $m$  and  $M$  denote the least and greatest values of  $f(x)$  in  $(a, b)$ . Hence, § 18, 2., it is equal to the value of  $f(x)(b - a)$  for some value  $\bar{x}$  of  $x$  in  $(a, b)$ . We have, therefore,

$$\lim [\Sigma f(x_i)h_i]_a^b = f(\bar{x})(b - a) \quad [a < \bar{x} < b] \quad (3)$$

**126. The function  $F(x) = \lim [\Sigma f(x_i)h_i]_a^x$ .** In  $[\Sigma f(x_i)h_i]_a^b$ , replace the constant  $b$  by a variable  $x$  whose values are in  $(a, b)$ . To each value of  $x$  corresponds a definite interval  $(a, x)$  and therefore, § 124, a definite number  $\lim [\Sigma f(x_i)h_i]_a^x$ .

Hence  $\lim [\Sigma f(x_i)h_i]_a^x$  is a function of  $x$  in  $(a, b)$ , § 14. Let  $F(x)$  denote this function:

$$F(x) = \lim [\Sigma f(x_i)h_i]_a^x \quad (1)$$

we shall prove that

$$dF(x)/dx = f(x) \quad (2)$$

For give to  $x$  the increment  $\Delta x$ . Then, § 125 (1), (2), (3),

$$\begin{aligned} F(x + \Delta x) - F(x) &= \lim [\Sigma f(x_i)h_i]_a^{x+\Delta x} - \lim [\Sigma f(x_i)h_i]_a^x \\ &= \lim [\Sigma f(x_i)h_i]_x^{x+\Delta x} = f(\bar{x})\Delta x \end{aligned}$$

where  $\bar{x}$  denotes some number between  $x$  and  $x + \Delta x$ .

Therefore, since  $\bar{x} \rightarrow x$  when  $\Delta x \rightarrow 0$ , we have

$$\frac{dF(x)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} f(\bar{x}) = f(x)$$

1. Since  $dF(x)/dx = f(x)$ , the function  $F(x)$  is an integral of  $f(x)$ , § 104. We represent this integral by the symbol  $\int_a^x f(x)dx$ , read "integral  $f(x)dx$ ,  $a$  to  $x$ ," and call  $a$  the *lower limit* and  $x$  the *upper limit* of the integral.

Hence, by definition,

$$\int_a^x f(x)dx = \lim [\Sigma f(x_i)h_i]_a^x \quad (3)$$

2. Let  $\phi(x)$  denote any *known* integral of  $f(x)$  in the interval  $(a, b)$ , that is, any function, found by inspection or the process of integration, whose derivative throughout  $(a, b)$  is  $f(x)$  and which therefore is continuous in  $(a, b)$ . From this integral  $\phi(x)$ , the integral  $\int_a^x f(x)dx$  can differ by a constant only, § 104, that is,

$$\int_a^x f(x)dx = \phi(x) + C \quad (4)$$

To find  $C$ , let  $x \rightarrow a$ . Then  $\int_a^x f(x)dx \rightarrow 0$  and  $\phi(x) \rightarrow \phi(a)$ . We therefore have  $0 = \phi(a) + C \therefore C = -\phi(a)$ . Therefore

$$\int_a^x f(x)dx = \phi(x) - \phi(a) \quad (5)$$

Hence  $\int_a^x f(x)dx$  is known when  $\phi(x)$  is known.

3. Since  $\lim [\Sigma f(x_i)h_i]_a^x$  is an integral of  $f(x)$ , *every continuous function  $f(x)$  has integrals.*

**127. Definite integrals.** Setting  $x = b$  in § 126 (3), we have

$$\int_a^b f(x)dx = \lim [\Sigma f(x_i)h_i]_a^b \quad (1)$$

The number  $\int_a^b f(x)dx$  which this formula defines is called the *definite integral* of  $f(x)$  between the limits  $a$  and  $b$ .

Again, setting  $x = b$  in § 126 (5), we have

$$\int_a^b f(x)dx = \phi(b) - \phi(a) \quad (2)$$

*If an integral  $\phi(x)$  of  $f(x)$  in  $(a, b)$  is known, then  $\int_a^b f(x)dx$  may be found by the rule: From  $\phi(b)$  subtract  $\phi(a)$ .*

**EXAMPLE 1.** Find  $\int_2^5 (2x + 5)dx$

Integrating  $2x + 5$ , we obtain  $\phi(x) = x^2 + 5x$ . Hence

$$\int_2^5 (2x + 5)dx = \phi(5) - \phi(2) = 50 - 14 = 36$$

For convenience we use  $[\phi(x)]_a^b$  as a symbol for  $\phi(b) - \phi(a)$  and write

$$\int_2^5 (2x + 5)dx = [x^2 + 5x]_2^5 = 50 - 14 = 36$$

**EXAMPLE 2.** The interval  $(1, 3)$  is divided into  $n$  equal parts, and the length of each part is multiplied by the square of any number in that part. Find the limit, when  $n \rightarrow \infty$ , of the sum of the products thus obtained.

$$\lim [\Sigma x_i^2 h_i]_1^3 = \int_1^3 x^2 dx = \left[ \frac{x^3}{3} \right]_1^3 = 9 - \frac{1}{3} = 8\frac{2}{3}$$

**EXAMPLE 3.** Find  $\int_{-1}^1 \frac{dx}{x^2 + 1}$

The simplest integral of  $1/(x^2 + 1)$  is  $\tan^{-1} x$ , that is, the angle between  $-\pi/2$  and  $\pi/2$  whose tangent is  $x$ . When  $x$  increases continuously from  $-1$  to  $1$ ,  $\tan^{-1} x$  increases continuously from  $-\pi/4$  to  $\pi/4$ . See Fig. 40, p. 66. Hence

$$\int_{-1}^1 \frac{dx}{x^2 + 1} = \tan^{-1}(1) - \tan^{-1}(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2} \quad (a)$$

Similarly

$$\int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3} \quad (b)$$

### EXERCISE XXIII

Evaluate the following definite integrals 1-12:

1.  $\int_{-2}^1 (2x^3 - x^2) dx$
2.  $\int_0^{\pi} \sin \theta d\theta$
3.  $\int_0^3 e^{2x-6} dx$
4.  $\int_0^{\pi/2} \sin^2 \theta d\theta$
5.  $\int_{-\pi/4}^{\pi/3} \tan^2 \theta d\theta$
6.  $\int_0^{\pi/2} x \cos x dx$
7.  $\int_0^2 \frac{dx}{x^2 + 4}$
8.  $\int_{-1}^{\sqrt{2}} \frac{dx}{\sqrt{4-x^2}}$
9.  $\int_{-1}^1 \frac{x dx}{x^2 + 1}$
10.  $\int_0^1 x(1-x)^{1/2} dx$
11.  $\int_1^4 x^2(1+\sqrt{x}) dx$
12.  $\int_{-1}^1 \tan^{-1} x dx$

13. Referring to the formula § 126 (5), show that

$$(1) \log x = \int_1^x \frac{dx}{x} \quad (2) \tan^{-1} x = \int_0^x \frac{dx}{x^2 + 1} \quad (3) \sin^{-1} x = \int_0^x \frac{dx}{\sqrt{1-x^2}}$$

14. By the definition  $\int_a^b f(x) dx = \lim [\Sigma f(x_i) h_i]_a^b$  and § 125, 3, show that

$$(1) \int_{-2}^2 \frac{x dx}{x^4 + 1} = 0 \quad (2) \int_0^2 \frac{dx}{(x^3 + 1)^{1/2}} \text{ is } > \frac{2}{3} \text{ and } < 2$$

**128. Computation of areas.** 1. By the definition of area in § 121, the area of the space  $R = abBA$  (Fig. 82) bounded by the lines  $Ox$ ,  $x = a$ ,  $x = b$  and an arc  $AB$  of a curve  $y = f(x)$  which lies above  $Ox$  is  $\lim [\Sigma f(x_i) h_i]_a^b$ .

Hence if  $A_a^b$  denote this area, we have, § 127 (1),

$$A_a^b = \int_a^b y \, dx \quad (1)$$

When the arc  $AB$  is below  $Ox$ , the integral  $\int_a^b y \, dx$  is negative; its numerical value is the area of  $R$ .

EXAMPLE 1. Find the areas of the spaces bounded by  $Ox$  and the curve  $y = x^3 - x^2 - 2x$ .

Since  $x^3 - x^2 - 2x = (x+1)x(x-2)$ , the curve cuts  $Ox$  at the points  $x = -1, 0, 2$ , is above  $Ox$  in  $(-1, 0)$ , below in  $(0, 2)$ .

$$\int_{-1}^0 y \, dx = \left[ \frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^0 = \frac{5}{12}$$

$$\therefore \text{area } ABO = \frac{5}{12}$$

$$\int_0^2 y \, dx = \left[ \frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_0^2 = -\frac{8}{3}$$

$$\therefore \text{area } OCD = \frac{8}{3}$$

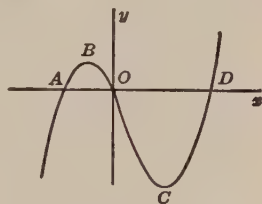


FIG. 83.

Observe that  $\int_{-1}^2 y \, dx$  is the *difference* of the areas  $ABO$  and  $OCD$ .

EXAMPLE 2. Draw the curve  $y = x^2 + 2x - 3$  and find the area of the space between it and  $Ox$ ; also the area of the space bounded by the curve and the lines  $Ox$ ,  $x = 2$ ,  $x = 4$ .

EXAMPLE 3. Show that if  $f(x)$  is  $+$  between  $x = a$  and  $x = b$ , the area of the space bounded by the curve  $y^2 = f(x)$  and the lines  $x = a$  and  $x = b$  is  $2 \int_a^b [f(x)]^{1/2} dx$ .

Find the area of the space between the parabola  $y^2 = 4x$  and the line  $x = 9$ .

2. Suppose that the curve  $y_1 = f(x)$  lies above the curve  $y_2 = \phi(x)$  throughout the  $x$ -interval  $(a, b)$ . Then the area of the space  $ABCDEF$  bounded by the two curves and the lines  $x = a$ ,  $x = b$  is

$$A_a^b = \int_a^b (y_1 - y_2) \, dx \quad (2)$$

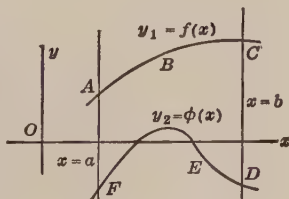


FIG. 84.



This follows from 1. and the identity

$$\int_a^b (y_1 - y_2) dx = \int_a^b y_1 dx - \int_a^b y_2 dx$$

**EXAMPLE 4.** Find area of the triangle whose sides are  $y = 2x$  (1),  $x + y = 6$  (2),  $y = x$  (3).

(1), (2) meet at  $A$  (2, 4); (2), (3) meet at  $B$  (3, 3).

$$OCA = \int_0^2 (2x - x) dx = \int_0^2 x dx = \left[ \frac{x^2}{2} \right]_0^2 = 2$$

$$ACB = \int_2^3 (6 - x - x) dx = \int_2^3 (6 - 2x) dx = 1$$

$$\text{Hence } OBA = 2 + 1 = 3$$

**EXAMPLE 5.** Find area bounded by the parabola  $(y - x)^2 = 4x$  and the line  $x = 5$ .

To find the curve, we proceed as in § 92, Ex. 2, and solving for  $y$  obtain

$$y_1 = x + 2x^{1/2}(OA) \text{ and } y_2 = x - 2x^{1/2}(OB)$$

$$\therefore OAB = \int_0^5 (y_1 - y_2) dx = 4 \int_0^5 x^{1/2} dx = \frac{40}{3} \sqrt{5}.$$

### EXERCISE XXIV

Find the areas bounded by the following curves, 1-9, and  $Ox$ , drawing the curves.

1.  $y = 9 - x^2$
2.  $y = x^2 - 3x - 4$
3.  $y = 3x^2 - x^3$
4.  $y = x(2x - 5)^2$
5.  $y = x^3 + 2x^2 - 3x$
6.  $y = (x^2 - 1)^2$
7.  $(y - 2x)^2 = 4x$
8.  $(y - x)^2 = x + 2$
9.  $y = (1 - x^2)/(x + 2)$

By the definition in § 121, the area bounded by the curve  $x = \phi(y)$  and the lines  $Oy$ ,  $y = c$ ,  $y = d$ , when  $\phi(y)$  is + in the  $y$ -interval  $(c, d)$ , is  $\int_c^d \phi(y) dy$ . Find the area bounded by each of the following curves, 10-12, and  $Oy$ .

10.  $x = 4y - y^2$
11.  $y^2 - 4x - 4 = 0$
12.  $y^2 - 4x - 4y = 0$

13. Find in two ways the area bounded by the curve  $y = x^3$ ,  $Oy$ , and the line  $y = 8$ .

Find the areas bounded by the following lines or curves, 14-21, drawing the graphs.

14.  $x - 2y + 4 = 0$ ,  $x = 0$ ,  $y = 3$ ,  $y = 5$
15.  $x - 2y + 6 = 0$ ,  $x + y = 0$ ,  $x - y = 0$

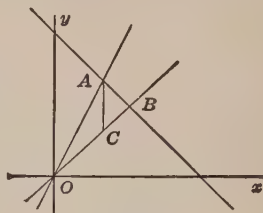


FIG. 85.

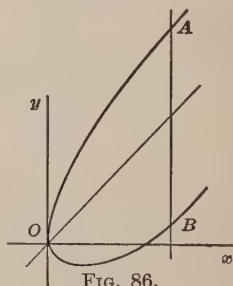


FIG. 86.



16.  $y = x^2, y = x, y = 2x$

17.  $y^2 = 4ax, y = 2x$

18.  $y^2 = 4ax, x^2 = 4ay$

19.  $y = x^2, x - y + 2 = 0$

20.  $y = x^2 + x, y = 2x^2 - 2$

21.  $y = x^3 - x, y = 3x - x^3$

22. Find the areas bounded by the loops of each of the curves:

(1)  $y^2 = x^2(a - x)$

(2)  $y^2 = x^2(a^2 - x^2)$

23. Find the area bounded by the hyperbola  $xy = 6$  and the lines  $y = 0, x = 1, x = e^2$ .

24. Show that the area bounded by the ellipse  $x^2/a^2 + y^2/b^2 = 1$  and the lines  $x = 0$  and  $x = x$  is  $(b/a)[x(a^2 - x^2)^{1/2} + a^2 \sin^{-1} x/a]$ ; hence that the area of the ellipse itself is  $ab\pi$ . See § 116, Ex. 1.

25. Show that the area bounded by the hyperbola  $x^2/a^2 - y^2/b^2 = 1$  and the line  $x = x$  is  $(b/a)[x(x^2 - a^2)^{1/2} - a^2 \log \{x + (x^2 - a^2)^{1/2}\}]$ . See § 114, Ex. 4.

26. Find the area bounded by the catenary  $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$  and the lines  $y = 0, x = 0, x = a$ .

27. Find the area bounded by the parabola  $x^{1/2} + y^{1/2} = a^{1/2}$  and the lines  $x = 0, y = 0$ .

**129. Properties of definite integrals.** For convenience represent  $\Sigma f(x_i)h_i$  by  $\Sigma f(x)\delta x$ , where  $\delta x$  stands for any one of the parts  $h_i$  into which the given  $x$ -interval  $(a, b)$  is supposed to be divided, and  $x$  for any point  $x$  in that part  $\delta x$ . We call  $f(x)\delta x$  an *element* of  $\Sigma f(x)\delta x$ . We then have

$$\int_a^b f(x)dx = \lim [\Sigma f(x)\delta x]_a^b \quad (1)$$

1. By (1), the value of  $\int_a^b f(x)dx$  depends solely on  $a, b$ , and the form of  $f(x)$ . Hence

$$\int_a^b f(x)dx = \int_a^b f(y)dy \quad (2)$$

2. The formulas (1) and (2) of § 125, give

$$\int_a^a f(x)dx = -\int_a^a f(x)dx \quad (3)$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx \quad (4)$$

3. If  $m$  and  $M$  denote the least and greatest values of  $f(x)$  in  $(a, b)$ , then, § 125, 3.,

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a) \quad (5)$$

and there is therefore a value  $\bar{x}$  of  $x$  in  $(a, b)$  such that

$$\int_a^b f(x) dx = f(\bar{x})(b-a) \quad (6)$$

For the  $y = f(x)$  whose graph in  $(a, b)$  is the arc  $APB$  in Fig. 87 we have  $m = aA$ ,  $M = bB$  and the geometric meaning of (5) is that in area

$$abCA < abBPA < abBD$$

and by (6) there is a point  $\bar{x}$  in  $(a, b)$  such that the rectangle  $aF'$  of altitude  $\bar{x}P = f(\bar{x})$  equals  $abBPA$  in area.

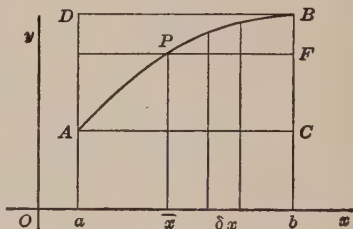


FIG. 87.

EXAMPLE 1. Show that  $\int_2^4 \frac{dx}{x^3 + 1}$  is between  $\frac{2}{65}$  and  $\frac{2}{9}$ . Locate its value more closely by dividing the interval  $(2, 4)$  into  $(2, 3)$  and  $(3, 4)$  and then applying (4) and (5).

EXAMPLE 2. Show that  $\int_1^3 (x^2 - 4x + 5)^{1/2} dx$  is between 2 and  $2\sqrt{2}$ .

EXAMPLE 3. Let  $\Delta S$  denote the area of the strip on  $\delta x$  in Fig. 87; show, by (6), that there is a point  $x'$  in  $\delta x$  such that  $\Delta S = f(x')\delta x$ .

EXAMPLE 4. Divide  $(a, b)$  into the parts,  $h_1, h_2, \dots, h_n$ , and show by (4) and (6) that points exist,  $x_1$  in  $h_1, x_2$  in  $h_2, \dots, x_n$  in  $h_n$ , such that

$$\int_a^b f(x) dx = f(x_1)h_1 + f(x_2)h_2 + \dots + f(x_n)h_n.$$

EXAMPLE 5. Show that (6) is the mean value theorem, § 97, for the function  $F(x) = \int_a^x f(x) dx$ .

EXAMPLE 6. By (1) and § 4, (3), show that

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

EXAMPLE 7. Show that if  $\phi(x) > 0$  throughout  $(a, b)$ , and  $m$  and  $M$  are the least and greatest values of  $f(x)$  in  $(a, b)$ , then

$$m \int_a^b \phi(x) dx \leq \int_a^b f(x) \phi(x) dx \leq M \int_a^b \phi(x) dx$$

**130. Related integrals.** 1. Suppose each of two equal  $x$ -intervals,  $(a, b)$  and  $(c, d)$ , to be divided into  $n$  equal parts  $\delta x$ . If, whatever the value of  $n$ , the sums  $[\Sigma f(x)\delta x]_a^b$  and  $[\Sigma f(x)\delta x]_c^d$  are equal, term to term, then  $\int_a^b f(x)dx = \int_c^d f(x)dx$ . If the terms of the first sum are the negatives of those of the second, then  $\int_a^b f(x)dx = -\int_c^d f(x)dx$ .

Hence in particular, by separating  $(-a, a)$  into  $(-a, 0)$  and  $(0, a)$ ,

$$\text{If } f(-x) \equiv f(x), \quad \text{then} \quad \int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx \quad (1)$$

$$\text{If } f(-x) \equiv -f(x), \quad \text{then} \quad \int_{-a}^a f(x)dx = 0 \quad (2)$$

**EXAMPLE 1.** Prove the following relations and interpret them geometrically.

$$\int_0^\pi \sin x dx = 2 \int_0^{\pi/2} \sin x dx, \quad \int_{-\pi/2}^{\pi/2} \sin x dx = 0$$

$$\int_{-\pi/2}^{\pi/2} \cos x dx = 2 \int_0^{\pi/2} \cos x dx, \quad \int_0^\pi \cos x dx = 0$$

**EXAMPLE 2.** Prove the following:

1.  $\int_0^\pi \sin^m x \cos^n x dx = 0$  when  $n$  is odd,  $\int_{-\pi/2}^{\pi/2} \sin^m x \cos^n x dx = 0$  when  $m$  is odd.

$$2. \int_0^\pi (\sin x + \cos x)^2 dx = \pi \quad 3. \int_{-1}^1 \frac{x^3 + x^2}{x^4 + 1} dx = 2 \int_0^1 \frac{x^2 dx}{x^4 + 1}.$$

2. If the interval  $(0, a)$  be divided into  $n$  equal parts  $\delta x$ , the terms of the sum  $[\Sigma f(x)\delta x]_0^a$  are the same as those of  $[\Sigma f(a-x)\delta x]_0^a$  in reverse order; hence

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx \quad (3)$$

$$\text{Thus, } \int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \sin^n(\pi/2 - x)dx = \int_0^{\pi/2} \cos^n x dx \quad (4)$$

**131. The integrals**  $\int_0^{\pi/2} \sin^n x dx$ ,  $\int_0^{\pi/2} \cos^n x dx$ . Let  $n$  be any positive integer. By § 115, (1),

$$n \int_0^{\pi/2} \cos^n x dx = [\cos^{n-1} x \sin x]_0^{\pi/2} + (n-1) \int_0^{\pi/2} \cos^{n-2} x dx$$

But  $[\cos^{n-1} x \sin x]_0^{\pi/2} = 0$ ; for  $\cos \pi/2 = 0$  and  $\sin 0 = 0$ .

Hence

$$\int_0^{\pi/2} \cos^n x dx = \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x dx$$

By repeated applications of this formula we find

$$1. \text{ For } n \text{ even, } \int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} \frac{n-3}{n-2} \dots \frac{1}{2} \cdot \frac{\pi}{2} \quad (1)$$

$$2. \text{ For } n \text{ odd, } \int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} \frac{n-3}{n-2} \dots \frac{2}{3} \quad (2)$$

These formulas apply also to  $\int_0^{\pi/2} \sin^n x \, dx$ , § 130 (4).

$$\text{Thus, } \int_0^{\pi/2} \cos^4 x \, dx = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3}{16} \pi, \quad \int_0^{\pi/2} \sin^5 x \, dx = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15},$$

$$\begin{aligned} \int_0^{\pi} \sin^2 x \cos^4 x \, dx &= 2 \int_0^{\pi/2} (\cos^4 x - \cos^6 x) \, dx \\ &= 2 \left[ \frac{3}{4} \frac{1}{2} - \frac{5}{6} \frac{3}{4} \frac{1}{2} \right] \frac{\pi}{2} = \frac{\pi}{16} \end{aligned}$$

**132. Substitutions.** The value of  $\int_a^b f(x) \, dx$  is sometimes found most readily by aid of a substitution  $x = \phi(t)$ . Let  $t_1$  and  $t_2$  be values of  $t$  such that (1)  $a = \phi(t_1)$  and  $b = \phi(t_2)$ , (2) when  $t$  changes continuously from  $t_1$  to  $t_2$ ,  $x$  increases continuously from  $a$  to  $b$ , (3)  $\phi'(t)$  is continuous in  $(t_1, t_2)$ . Then

$$\int_a^b f(x) \, dx = \int_{t_1}^{t_2} f[\phi(t)] \phi'(t) \, dt \quad (1)$$

For let  $\delta t$  denote any part of the  $t$ -interval  $(t_1, t_2)$ , and  $\delta x$  the corresponding part of the  $x$ -interval  $(a, b)$ . By the mean value theorem, § 97 (6),

$$\delta x = \phi'(t_i) \delta t$$

where  $t_i$  is some number in  $\delta t$ . Hence, if  $x_i = \phi(t_i)$ ,

$$f(x_i) \delta x = f[\phi(t_i)] \phi'(t_i) \delta t \quad (2)$$

where  $x_i$  is in  $\delta x$ . The members of (2) are corresponding elements of the integrals in (1). Hence these integrals are equal.

Under similar conditions, if  $x = \phi(t)$ ,  $y = \psi(t)$ , then

$$\int_a^b f(x, y) \, dx = \int_{t_1}^{t_2} f[\phi(t), \psi(t)] \phi'(t) \, dt \quad (3)$$

**EXAMPLE 1.** Find  $\int_0^a (a^2 - x^2)^{3/2} \, dx$

Set  $x = a \sin \theta$ . Then  $x = 0$ ,  $a$  correspond to  $\theta = 0, \pi/2$ ; when  $\theta$  increases from 0 to  $\pi/2$ ,  $x$  increases from 0 to  $a$ ; and  $d \sin \theta / d\theta = \cos \theta$  is continuous in  $(0, \pi/2)$ . Hence

$$\int_0^a (a^2 - x^2)^{3/2} \, dx = a^4 \int_0^{\pi/2} \cos^4 \theta \, d\theta = \frac{3}{16} a^4 \pi$$

EXAMPLE 2. Find  $\int_0^{2a} x(2ax - x^2)^{1/2} dx = \int_0^{2a} x[a^2 - (x - a)^2]^{1/2} dx$ .

Set  $x - a = a \sin \theta$ . Then  $x$  increases from 0 to  $2a$  when  $\theta$  increases from  $-\pi/2$  to  $\pi/2$ . Hence

$$\begin{aligned}\int_0^{2a} x[a^2 - (x - a)^2]^{1/2} dx &= a^3 \int_{-\pi/2}^{\pi/2} (1 + \sin \theta) \cos^2 \theta d\theta \\ &= 2a^3 \int_0^{\pi/2} \cos^2 \theta d\theta = \frac{a^3 \pi}{2}\end{aligned}$$

EXAMPLE 3. Find  $\int_{-1}^3 x(x+1)^{1/2} dx$  by the substitution  $x+1 = t^2$ .

EXAMPLE 4. Find the area between  $Ox$  and an arch of the cycloid  $x = a(\theta - \sin \theta)$ ,  $y = a(1 - \cos \theta)$ . [Fig. 72]

$$\text{Area} = 2 \int_{\theta=0}^{\theta=\pi} y dx = 2a^2 \int_0^{\pi} (1 - \cos \theta)^2 d\theta = 3a^2 \pi$$

EXAMPLE 5. Find the area of the ellipse  $x = a \cos \phi$ ,  $y = b \sin \phi$ .

EXAMPLE 6. Find the area bounded by the curve  $y^2 = x^4(a^2 - x^2)$ , tracing the curve.

**133. The integral  $\int_0^\infty f(x)dx$ .** Suppose that  $f(x)$  is continuous in the interval  $(a, x)$  and that  $\int_a^x f(x)dx$  approaches a finite limit when  $x \rightarrow \infty$ ; we then represent that limit by the symbol  $\int_a^\infty f(x)dx$ . Hence, by definition,

$$\int_a^\infty f(x)dx = \lim_{x \rightarrow \infty} \int_a^x f(x)dx \quad (1)$$

We define  $\int_{-\infty}^b f(x)dx$  in a similar manner, and  $\int_{-\infty}^\infty f(x)dx$  by aid of the formula  $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx$ . If  $\int_a^x f(x)dx$  does not approach a finite limit when  $x \rightarrow \infty$ ,  $\int_a^\infty f(x)dx$  is meaningless; similarly  $\int_{-\infty}^b f(x)dx$ .

EXAMPLE 1. We have

$$\int_0^x e^{-x} dx = [-e^{-x}]_0^x = 1 - e^{-x}.$$

$$\text{Hence } \int_0^\infty e^{-x} dx = \lim_{x \rightarrow \infty} (1 - e^{-x}) = 1$$

Observe that  $\int_0^\infty e^{-x} dx$  is the area between the curve  $y = e^{-x}$ , its asymptote  $y = 0$ , and  $Oy$ .

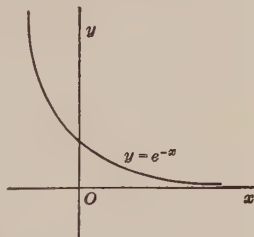


FIG. 88.

If  $\phi(x)$  is a known integral of  $f(x)$  in the interval  $(a, x)$ , and  $\phi(x)$  approaches a finite limit when  $x \rightarrow \infty$ , then

$$\int_a^\infty f(x)dx = \lim_{x \rightarrow \infty} \phi(x) - \phi(a) = \phi(\infty) - \phi(a) \quad (2)$$

EXAMPLE 2. Show that  $\int_1^\infty \frac{dx}{x^{3/2}} = 2$ , but that

$$\lim_{x \rightarrow \infty} \int_1^x \frac{dx}{x^{1/2}} = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} \int_{1/x}^x \frac{dx}{x} = \infty.$$

EXAMPLE 3. Show that  $\int_a^\infty \frac{dx}{(x-c)^n}$ ,  $a > c$ , exists when and only when  $n > 1$ .

EXAMPLE 4. Trace the curve  $y = 1/(x^2 + 1)$  and show that the area between the curve and its asymptote  $y = 0$  is  $\int_{-\infty}^\infty \frac{dx}{x^2 + 1} = \pi$ .

EXAMPLE 5. Show that  $\int_3^\infty \frac{dx}{x^2 - 2x} = \log \sqrt{3}$ . What area does this represent?

**134.** The integral  $\int_a^b f(x)dx$  when  $f(x)$  is  $\infty$  in  $(a, b)$ . Suppose that  $f(x)$  is continuous in  $(a, b)$  except at  $b$ , but that  $f(b) = \infty$ . If  $\int_a^x f(x)dx$  approaches a finite limit when  $x \rightarrow b$  we represent that limit by  $\int_a^b f(x)dx$  and write

$$\int_a^b f(x)dx = \lim_{x \rightarrow b} \int_a^x f(x)dx \quad (1)$$

Similarly, if  $f(a) = \infty$ , we have  $\int_a^b f(x)dx = \lim_{x \rightarrow a} \int_x^b f(x)dx$ .

If  $f(x)$  is  $\infty$  at some point  $c$  between  $a$  and  $b$ , but continuous elsewhere in  $(a, b)$ , we define  $\int_a^b f(x)dx$ , when it exists, by the formula

$$\int_a^b f(x)dx = \lim_{x \rightarrow c} \int_a^x f(x)dx + \lim_{x \rightarrow c} \int_x^b f(x)dx \quad (2)$$

EXAMPLE 1. Prove that  $\int_0^1 \frac{dx}{(1-x^2)^{1/2}} = \frac{\pi}{2}$ .

Here  $f(x) = \frac{1}{(1-x^2)^{1/2}}$  is  $\infty$  at  $x = 1$ ; but  $\int_0^x \frac{dx}{(1-x^2)^{1/2}} = \sin^{-1} x$ , and  $\lim_{x \rightarrow 1} \sin^{-1} x = \frac{\pi}{2}$ .

If  $\phi(x)$  be a known integral of  $f(x)$ , and if  $\phi(x)$  be continuous in  $(a, b)$ , then, even when  $f(x)$  is  $\infty$  at one or more points of  $(a, b)$ , the integral  $\int_a^b f(x)dx$  exists and is given by the formula

$$\int_a^b f(x)dx = \phi(b) - \phi(a) \quad (3)$$

Thus let  $f(x)$  be  $\infty$  at  $x = c$  in  $(a, b)$ . Since  $\phi(x)$  is continuous at  $x = c$ ,  $\lim_{x \rightarrow c} \phi(x) = \lim_{x \rightarrow c} \phi(x) = \phi(c)$ . Hence, by (2),

$$\int_a^b f(x) dx = [\phi(c) - \phi(a)] + [\phi(b) - \phi(c)] = \phi(b) - \phi(a)$$

But when  $\phi(x)$  becomes  $\infty$  in  $(a, b)$ , then (3) is false.

Thus  $\int_{-1}^1 \frac{dx}{x^{2/3}} = [3x^{1/3}]_{-1}^1 = 6$  is true though  $1/x^{2/3}$  is  $\infty$  at  $x = 0$ ; for  $3x^{1/3}$  is continuous in  $(-1, 1)$ . But  $\int_{-1}^1 \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^1 = -2$  is false,  $-\frac{1}{x}$  being  $\infty$  at  $x = 0$ ; in fact, by (2),  $\int_{-1}^1 \frac{dx}{x^2} = \infty$ .

EXAMPLE 2. Show that  $\int_1^2 \frac{dx}{(2-x)^{3/4}} = 4$ , but that

$$\lim_{x \rightarrow 2} \int_1^x \frac{dx}{2-x} = \infty, \quad \lim_{x \rightarrow 2} \int_1^x \frac{dx}{(2-x)^{4/3}} = \infty.$$

EXAMPLE 3. Show that  $\int_a^b \frac{dx}{(b-x)^n}$  and  $\int_a^b \frac{dx}{(x-a)^n}$  exist when and only when  $n < 1$ .

EXAMPLE 4. Show that  $\int_a^b \frac{dx}{(x-c)^{p/q}}$ ,  $a < c < b$ , exists when and only when  $q$  is odd and  $p/q < 1$ .

EXAMPLE 5. Draw the curve  $y^2 = 1/x$  and show that the area bounded by the curve, its asymptote  $x = 0$  and the line  $x = 1$  is 4.

EXAMPLE 6. Show that the area bounded by a branch of the hyperbola  $xy = 1$  and its asymptotes  $x = 0$  and  $y = 0$  is infinite.

### EXERCISE XXV

Verify the following, 1 - 11.

$$1. \int_{-1}^1 \frac{dx}{x^{4/5}} = 10 \quad 2. \int_0^\infty e^{-3x} dx = \frac{1}{3} \quad 3. \int_0^\infty \frac{dx}{x^2 + a^2} = \frac{\pi}{2a}$$

$$4. \int_{-a}^a \frac{dx}{\sqrt{a^2 - x^2}} = \pi \quad 5. \int_{-\infty}^\infty \frac{dx}{x^2 + 4x + 5} = \pi$$

$$6. \int_1^3 \frac{dx}{\sqrt{(x-1)(3-x)}} = \pi \quad 7. \int_1^\infty \frac{dx}{x\sqrt{2x^2-1}} = \frac{\pi}{4}$$

$$8. \int_2^\infty \frac{dx}{x^2-1} = \log 3^{1/2} \quad 9. \int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}} = \pi$$



$$10. \int_1^2 \frac{dx}{\sqrt{x^2 - 1}} = \log(2 + \sqrt{3})$$

$$11. \int_0^\infty \frac{dx}{(x^2 + 1)^n} = \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} \dots \frac{1}{2} \cdot \frac{\pi}{2} \quad [\text{let } x = \tan \theta].$$

Find the values of the following, 12-16.

$$12. \int_{-\pi/2}^{\pi/2} \cos^4 x \sin^3 x \, dx$$

$$13. \int_0^\pi (\sin x + \cos x)^3 \, dx$$

$$14. \int_0^{2\pi} \sin^2 x \cos^2 x \, dx$$

$$15. \int_0^{\pi/2} x \cos x \, dx$$

$$16. \int_0^1 \frac{x^3 \, dx}{\sqrt{1-x^2}} \quad [\text{substitute } x = \sin \theta].$$

17. Prove that  $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$  by aid of the substitution  $x = a - t$ .

18. Show that  $\int_0^{\pi/2} \sin^{n+1} x \, dx < \int_0^{\pi/2} \sin^n x \, dx$  and  $\int_0^{\pi/4} \tan^{n+1} x \, dx < \int_0^{\pi/4} \tan^n x \, dx$ .

19. Find the area between the "witch"  $y = 8a^3/(x^2 + 4a^2)$  and its asymptote.

Find the areas bounded by the following curves and lines, 20-25.

$$20. xy^2 = (x-1)^2, x=0$$

$$21. y^2 x(1-x) = 1, x=0, x=1$$

$$22. x(y-x)^2 = 1, x=0, x=1$$

$$23. (x+1)y^2 = 2-x, x=-1$$

$$24. (a^2 - x^2)y^2 = x^4, x=a, x=-a.$$

$$25. y^2(2a-x) = x^3, x=2a.$$

26. Prove the following theorems, where  $a > 0$ ,  $n > 1$ , and  $l$  denotes a given positive number:

(1)  $\int_a^x \frac{dx}{x^n}$  increases as  $x$  increases and  $\rightarrow 1/(n-1)a^{n-1}$  as  $x \rightarrow \infty$ .

(2) If  $f(x)$  is positive (or 0) and  $< l/x^n$  in  $(a, x)$ , then  $\int_a^x f(x) \, dx$  increases with  $x$ , but remains  $< l/(n-1)a^{n-1}$ ; it therefore approaches a limit, and  $\int_a^\infty f(x) \, dx$  exists.

(3) The integral  $\int_a^\infty f(x) \, dx$  also exists when  $|f(x)| < l/x^n$ ; for  $\int_a^x f(x) \, dx = \int_a^x \{f(x) + |f(x)|\} \, dx - \int_a^x |f(x)| \, dx$ , and these integrals are of the type (2),  $f(x) + |f(x)|$  being positive (or 0) and  $< 2l/x^n$ .

(4) The integral  $\int_a^\infty f(x) \, dx$  exists if  $f(x)$  is continuous in  $(a, x)$  and an exponent  $n > 1$  can be found such that  $x^n f(x)$  approaches a finite limit when  $x \rightarrow \infty$ .

27. Prove that  $\int_a^b f(x) \, dx$  exists when  $f(x)$  is  $\infty$  at  $x = a$  but is continuous elsewhere in  $(a, b)$ , in case an exponent  $n < 1$  can be found such that  $(x-a)^n f(x)$  approaches a finite limit when  $x \rightarrow a$ .

28. Prove the existence of the following integrals:

$$(1) \int_0^{\infty} \frac{x-1}{x^4+3x+3} dx \quad (2) \int_{-\infty}^{\infty} \frac{x^2}{x^4+3} dx$$

$$(3) \int_0^1 x^{m-1}(1-x)^{n-1} dx, \quad m, n > 0$$

29. Show by integration by parts and § 101 that, when  $n$  is a positive integer,  $\int_0^{\infty} x^n e^{-x} dx = n!$

**135. General definition of area.** The definition of area in § 121 is a particular case of the following more general definition.

Let  $S$  be a piece of the plane bounded by curve arcs or by curve arcs and line segments. Let  $E_n$  and  $I_n$  be the areas of two varying rectilinear figures of  $n$  sides, the first containing  $S$ , the second contained by  $S$ , and such that, as  $n \rightarrow \infty$ ,  $E_n$  and  $I_n$  approach the same limit  $l$ . We call  $l$  the area of  $S$ . The value of  $l$  is independent of the choice of the figures containing and contained by  $S$ . They may be any figures, rectilinear or not, whose areas  $E_n, I_n$  are known.<sup>1</sup>

**136. Circular sectors.** The area of a circular sector  $AOB$  of radius  $a$  and angle  $\theta$  is  $a^2\theta/2$ . For divide  $AOB$  into  $n$  equal parts and in and about each part  $AOP$  construct triangles  $AOP$  and  $A'OP'$  as in Fig. 89. Then, as  $n \rightarrow \infty$ ,

$$\Sigma AOP / \Sigma A'OP' = OQ^2 / OQ'^2 \rightarrow 1$$

and

$$\Sigma AOP = \frac{OQ}{2} \Sigma AP \rightarrow \frac{a}{2} \text{arc } AB = \frac{a^2\theta}{2}$$

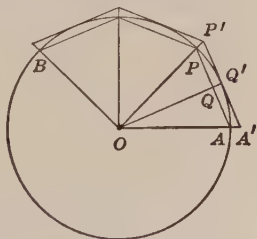


FIG. 89.

<sup>1</sup> For let  $E'_m, I'_m$  be the areas of any second pair of varying figures, the one containing and the other contained by  $S$ , and such that  $E'_m, I'_m$  approach the same limit  $l'$ . Then  $l' = l$ . For if  $l' \neq l$ , suppose  $l > l'$  and set  $l - l' = d$ . We then have

$E_n - I'_m > d$  (since  $E_n > l$  and  $I'_m < l'$ ), also  $E'_m - I_n > 0$  (since  $E'_m > I_n$ ) and therefore  $(E_n - I_n) + (E'_m - I'_m) > d$

which is impossible since  $E_n - I_n \rightarrow 0$  and  $E'_m - I'_m \rightarrow 0$ . Hence  $l' = l$ .

**137. Areas, polar coordinates.** Let  $AOB$  be the space bounded by a continuous curve arc  $AB$  whose equation in polar coordinates is  $r = f(\theta)$ , and the rays  $\theta = \alpha$  and  $\theta = \beta$  ( $OA$  and  $OB$ ). Divide  $AOB$  into  $n$  parts whose angles  $\delta\theta$  all  $\rightarrow 0$  when  $n \rightarrow \infty$ . In and about each part  $POQ$  construct circular sectors  $POR$  and  $QOT$  whose radii are the least and greatest values of  $r$  in that part. The space  $AOB$  contains the sum of the inner sectors and is contained by the sum of the outer sectors. The areas of both sector sums are sums of the form  $\Sigma r^2 \delta\theta / 2$  in which the  $r$  of each term is a value of  $r = f(\theta)$  in the  $\delta\theta$  of that term. Hence both areas  $\rightarrow \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$  when  $n \rightarrow \infty$ , § 124. Therefore the area of  $AOB$  is

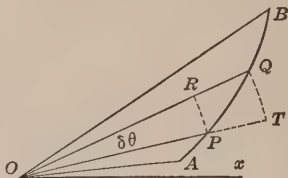


FIG. 90.

$$A_{\alpha}^{\beta} = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta \quad (1)$$

**EXAMPLE.** Find the area bounded by the cardioid  $r = a(1 - \cos \theta)$  (Fig. 45, p. 78).

Since  $\cos \theta = \cos(-\theta)$ , the curve is symmetric to  $Ox$ . Hence

$$\text{Area} = 2 \cdot \frac{a^2}{2} \int_0^{\pi} (1 - \cos \theta)^2 d\theta = a^2 \int_0^{\pi} (1 - 2 \cos \theta + \cos^2 \theta) d\theta = \frac{3a^2\pi}{2}$$

### EXERCISE XXVI

Find the areas bounded by the following curves and lines, drawing the graphs.

1.  $r = a(1 - \cos \theta)$ ,  $\theta = \pi/6$ ,  $\theta = \pi/3$
2.  $r = a \cos \theta$ ,  $\theta = 0$ ,  $\theta = \pi/4$
3.  $r = a \tan \theta$ ,  $\theta = 0$ ,  $\theta = \pi/6$
4.  $r = a\theta$ ,  $\theta = 0$ ,  $\theta = 2\pi$
5.  $r = e^{2\theta}$ ,  $\theta = 0$ ,  $\theta = \pi/2$
6.  $r^2 = a^2 \cos 2\theta$  [Fig. 47]
7. A loop of  $r = a \cos 3\theta$  [Fig. 46]
8. A loop of  $r = a \cos 2\theta$
9. The parabola  $r = a \sec^2 \frac{\theta}{2}$ ,  $\theta = 0$ ,  $\theta = \frac{\pi}{2}$
10.  $r = 2 - \cos \theta$
11.  $r = \cos \theta$  and  $r = \sin \theta$
12. The loop of  $r = \sin^3 \theta/3$ .

## VOLUMES OF SOLIDS

**138. Definition of volume.** The notion of volume may be extended from solids for which it is already defined (right prisms) to other solids by the following definition (see § 135).

Let  $S$  be a given solid and let  $E_n, I_n$  be the volumes of two varying solids of known volume, one containing  $S$ , the other contained by  $S$ , and such that, as  $n \rightarrow \infty$ ,  $E_n$  and  $I_n$  approach the same limit  $l$ . We call  $l$  the *volume of  $S$* .

The value of  $l$  is independent of the choice of the varying solids (see footnote p. 153).

**139. Right cylinders.** The volume of a right cylinder  $K$  of altitude  $a$  and base area  $B$  is  $aB$ . For in and about the base of  $K$  we can construct polygons whose areas approach  $B$  as limit. The right prisms of altitude  $a$  on these polygons include  $K$  between them, and their volumes both approach  $aB$  as limit.

1. The volume of a *right circular cylinder* of altitude  $a$  and radius  $r$  is  $ar^2\pi$ .

2. The area of a circular ring of inner radius  $r_1$  and outer radius  $r_2$  is  $\pi(r_2^2 - r_1^2) = \pi(r_2 + r_1)(r_2 - r_1)$ . Hence for the volume  $V$  of the *cylindrical shell* of altitude  $a$  on this ring, we have

$$V = a \, 2 \pi r \, \delta r, \text{ where } r = (r_1 + r_2)/2 \text{ and } \delta r = r_2 - r_1$$

**140. Solids of revolution.** 1. Let  $S_x$  denote the solid got by revolving the space  $abBA$  about  $Ox$ ,  $AB$  being a continuous arc of a curve  $y = f(x)$ .

By parallels to  $Oy$ , divide  $abBA$  into  $n$  strips  $PDEQ$  on parts  $\delta x$  of  $ab$  which  $\rightarrow 0$  when  $n \rightarrow \infty$ , and in and about each strip construct rectangles  $PE$  and  $QD$  whose altitudes are the least and

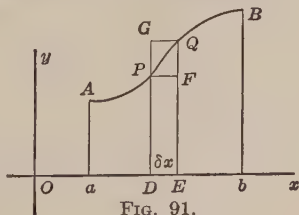


FIG. 91.

greatest curve ordinates in the strip. The part of  $S_x$  generated by  $PDEQ$  is included between the cylinders generated by  $PE$  and  $QD$ , and the volumes of these cylinders are  $DP^2\pi \delta x$  and  $EQ^2\pi \delta x$ . Hence  $S_x$  is included between two cylinder sums whose volumes are sums  $\pi \Sigma y^2 \delta x$  in both of which the  $y$  of each term is a value of  $y = f(x)$  in the  $\delta x$  of that term, and which, therefore,  $\rightarrow \pi \int_a^b y^2 dx$  when  $n \rightarrow \infty$ . Hence the volume  $V_x$  of  $S_x$  is

$$V_x = \pi \int_a^b y^2 dx \quad (1)$$

2. It being supposed that the arc  $AB$  does not cut  $Oy$ , let  $S_y$  be the solid got by revolving  $abBA$  about  $Oy$ . The part of  $S_y$  generated by  $PDEQ$  is included between the cylindrical shells generated by  $PE$  and  $QD$ , and by § 139, 2, the volumes of these shells are  $2\pi x' DP \delta x$  and  $2\pi x' EQ \delta x$  where  $x'$  is the midpoint of  $\delta x$ . Both of these products are of the form  $2\pi x' f(x) \delta x$  where  $x$  and  $x'$  are points of  $\delta x$ , and by a theorem to be proved later, § 145,  $\Sigma x' f(x) \delta x \rightarrow \int_a^b x f(x) dx$ . Hence the volume  $V_y$  of  $S_y$  is

$$V_y = 2\pi \int_a^b xy dx \quad (2)$$

3. Similarly the volumes of the solids got by revolving about  $Oy$  and  $Ox$  the space bounded by the curve  $x = \phi(y)$  and the lines  $x = 0$ ,  $y = c$ ,  $y = d$  is

$$V_y = \pi \int_c^d x^2 dy \quad (3) \quad V_x = 2\pi \int_c^d xy dy \quad (4)$$

EXAMPLE 1. For the sphere of radius  $a$  got by revolving the circle  $x^2 + y^2 = a^2$  about  $Ox$ ,

$$V_x = \pi \int_{-a}^a y^2 dx = 2\pi \int_0^a (a^2 - x^2) dx = \frac{4}{3} a^3 \pi.$$

EXAMPLE 2. Find the volumes of the solids got by revolving about  $Ox$  and  $Oy$  the space bounded by the lines  $y_1 = x$ ,  $y_2 = 2x$ ,  $x = 1$ ,  $x = 3$ .

$$V_x = \pi \int_1^3 (y_2^2 - y_1^2) dx = \pi \int_1^3 3x^2 dx = 26\pi$$

$$V_y = 2\pi \int_1^3 x(y_2 - y_1) dx = 2\pi \int_1^3 x^2 dx = \frac{5}{3} \pi$$

## EXERCISE XXVII

1. Find by integration the volumes of the cones got by revolving about  $Ox$  and  $Oy$  the triangle bounded by  $Ox$ ,  $Oy$ , and the line  $3x + 2y = 6$ .

2. Prove that the volume of a right circular cone of altitude  $a$  and base radius  $r$  is  $ar^2\pi/3$ .

3. Prove that the volumes of the *prolate spheroid* and the *oblate spheroid* got by revolving the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ,  $a > b$ , about its major and minor axes are  $\frac{4}{3}ab^2\pi$  and  $\frac{4}{3}a^2b\pi$ .

Find the volumes got by revolving about  $Ox$  the spaces bounded by the following curves:

4.  $y^2 = x^3$ ,  $x = 4$

5.  $y^2 = x^3$ ,  $y = x$

6.  $y^2 = x^3$ ,  $y = 4$ ,  $y = 2x$

7.  $y^2 = 4x$ ,  $2y = x$

8.  $y^2 = 4x$ ,  $x^2 + y^2 = 5x$

9.  $y = x^2$ ,  $y = x$ ,  $y = 2x$

10. An arch of  $y = \sin x$

11.  $y^2 = 1/(x^2 + 1)$ ,  $y = 0$

12.  $x^2 + y^2 = 9$ ,  $x = 2$ ,  $x = 3$

13.  $y = 8a^3/(x^2 + 4a^2)$ ,  $y = 0$

14.  $x = a \cos \theta$ ,  $y = a \sin \theta$ ,  $\theta = 0$  and  $\theta = \pi/4$

15. The loop of  $y^2 = x^2(3 - x)$

16. An arch of the cycloid  $x = a(\theta - \sin \theta)$ ,  $y = a(1 - \cos \theta)$

Find the volumes got by revolving about  $Oy$  the spaces bounded by the following curves:

17.  $x^2 + y - 2 = 0$ ,  $y = 0$

18.  $y^2 = 4x$ ,  $x = 0$ ,  $y = 4$

19.  $y = x^2$ ,  $y = x$ ,  $y = 2x$

20.  $xy = 1$ ,  $x = 0$ ,  $x = 1$

21.  $x^2 - y^2 = a^2$ ,  $y = -a$ ,  $y = a$

22.  $x^2 + 3y^2 - 6y = 0$

23.  $y = e^x$ ,  $x = 0$ ,  $y = 4$

24. Segment of  $x^2 + y^2 + 2x = 3$  to the right of  $Oy$

25.  $y = x$ ,  $y = x + 2$ ,  $x = 2$ ,  $x = 4$

26.  $3x - 2y = 1$ ,  $2x + y = 3$ ,  $5x - y = 11$

27.  $y^2/a^2 = (a - x)/x$ ,  $x = 0$

28.  $y^2 = (2 - x)^3/x$ ,  $x = 0$

29. Show that the volume of the solid got by revolving about the line  $x = 2$  the parabolic segment between  $y^2 = 4x$  and  $x = 2$  is  $4\pi \int_0^2 y(2 - x)dx$ , and find its value.

30. The space bounded by the circle  $x^2 + y^2 = a^2$  and its tangents  $x = a$  and  $y = a$  is revolved about  $x = a$ . Find the volume of the solid thus generated.

**31.** Prove that the volume of the *torus* [Fig. 98] got by revolving the circle  $x^2 + (y - b)^2 = a^2$ ,  $a < b$ , about  $Ox$  is  $2ba^2\pi^2$ .

**32.** The axis of a cylinder of radius  $4a$  passes through the center of a sphere of radius  $5a$ ; show that the volume of the part of the sphere exterior to the cylinder is  $36\pi a^3$ .

**33.** Find a formula for the volume of a spherical segment.

**141. Other solids.** Let  $A_x$  denote the section of a solid  $K$  by a plane perpendicular to  $Ox$  at the distance  $x$  from  $O$ . The area of  $A_x$  is a function of  $x$ ; represent it by  $A(x)$ , and suppose that  $A(x)$ , is continuous between  $x = a$  and  $x = b$ . Divide the segment  $ab$  of  $Ox$  into  $n$  parts  $\delta x$  which  $\rightarrow 0$  when  $n \rightarrow \infty$ . The sections  $A_x$  through the end points of any part  $\delta x$  cut out a part  $\Delta K$  of  $K$ ; suppose that the right cylinders of altitude  $\delta x$  on these sections include  $\Delta K$  between them; then since the volumes of both cylinders are of the form  $A(x)\delta x$ , where  $x$  is a point in  $\delta x$ , we may conclude, as in § 140, that the volume  $V_a^b$  of  $K$  between  $x = a$  and  $x = b$  is

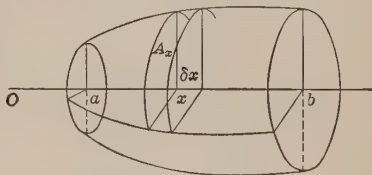


FIG. 92.

$$V_a^b = \int_a^b A(x) dx \quad (1)$$

**EXAMPLE 1.** The volume of the *ellipsoid*  $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$  is  $(4/3)abc\pi$ .

For the plane  $x = x'$  perpendicular to  $Ox$  at the distance  $x'$  from  $O$  cuts the surface in an ellipse whose equation in the plane  $x = x'$  is

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x'^2}{a^2}$$

The semiaxes of this ellipse are  $b\left(1 - \frac{x'^2}{a^2}\right)^{1/2}$ ,  $c\left(1 - \frac{x'^2}{a^2}\right)^{1/2}$ ; hence its area is  $bc\left(1 - \frac{x'^2}{a^2}\right)\pi$ , p. 145, Ex. 24.

Therefore

$$V = 2\int_0^a bc\left(1 - \frac{x^2}{a^2}\right)\pi dx = \frac{4}{3}abc\pi$$

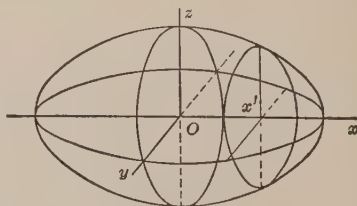


FIG. 93.



**EXAMPLE 2.** Find the volume of the piece cut from the part of a right cylinder standing on the quadrant  $OAB$  of radius  $a$  by the plane  $OAQC$ , the angle  $BOC$  being  $30^\circ$ .

Every section of  $OABC$  by a plane perpendicular to  $Oy$  is a triangle  $PDQ$  similar to  $OBC$ , and if  $OP = y$ , then  $PD = (a^2 - y^2)^{1/2}$ ,  $DQ = PD \tan 30^\circ$ ,

$$\therefore PDQ = \frac{\sqrt{3}}{6} (a^2 - y^2)$$

$$\text{Hence } V = \frac{\sqrt{3}}{6} \int_0^a (a^2 - y^2) dy = \frac{a^3}{3\sqrt{3}}$$

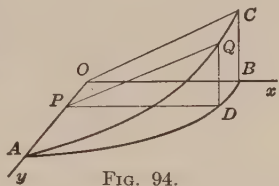


FIG. 94.

**EXAMPLE 3.** Find the volume bounded by the elliptic paraboloid  $4x = y^2 + 2z^2$  and the plane  $x = 5$ .

**EXAMPLE 4.** Solve Ex. 2, using sections perpendicular to  $Ox$ .

**EXAMPLE 5.** Let  $BC$  be the major axis of the ellipse  $x^2 + 2y^2 = 2$ ,  $PQ$  any chord perpendicular to  $BC$ , and  $A_x$  the equilateral triangle on  $PQ$  and perpendicular to  $BC$ . What is the volume of the solid generated when  $A_x$  moves from  $B$  to  $C$ ?

**EXAMPLE 6.** In the pyramid  $O-DEF$ , the angles  $ODE$ ,  $ODF$  are right angles. Show that the volume is  $1/3$  the altitude times the area of the base. Hence show that the volume of any pyramid or cone of altitude  $a$  and base area  $B$  is  $(1/3)aB$ .

## CURVE ARCS AND SURFACES OF REVOLUTION

**142. Length of a curve arc.** In an arc  $AB$  of a curve  $y = f(x)$  along which  $f'(x)$  is continuous, inscribe a polygon of  $n$  sides all of which  $\rightarrow 0$  when  $n \rightarrow \infty$ .

Let  $PQ$  be any side of this polygon and  $\delta x$  its projection on  $Ox$ . By § 97 (7), there is a point on the arc  $PQ$  where the slope angle equals  $DPQ$ ; let  $x$  denote the abscissa of this point. Then since  $\tan DPQ = f'(x)$ , we have

$$PQ = PD \sec DPQ = \delta x [1 + f'(x)^2]^{1/2}$$

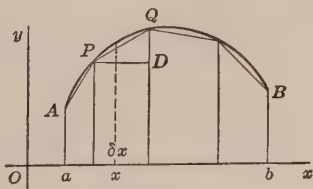


FIG. 95.



Hence the perimeter  $\Sigma PQ$  of the polygon can be expressed in the form  $[\Sigma(1 + f'(x)^2)^{1/2}\delta x]_a^b$ ; it therefore approaches a definite limit when  $n \rightarrow \infty$ , § 124. We call this limit, namely  $\int_a^b [1 + f'(x)^2]^{1/2} dx$ , the *length of the arc AB*; and representing it by  $s_a^b$ , have the formula

$$s_a^b = \int_a^b \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} dx \quad (1)$$

**EXAMPLE 1.** Find the length of the arc of  $y = x^{3/2}$  between  $x = 0$  and  $x = 4/3$ .

$$s = \int_0^{4/3} (1 + \frac{9}{4}x)^{1/2} dx = \frac{8}{27} [(1 + \frac{9}{4}x)^{3/2}]_0^{4/3} = \frac{8}{27} (8 - 1) = \frac{56}{27}.$$

Let  $s$  denote the length of the arc of  $y = f(x)$  between any fixed point  $P_0(x_0, y_0)$  on the curve and the variable point  $P(x, y)$ ,  $f'(x)$  being continuous along  $P_0P$ . Then

$$s = \int_{x_0}^x \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} dx, \quad \frac{ds}{dx} = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} \quad (2)$$

Multiplying both sides of (2) by  $dx$ , we obtain

$$ds = [(dx)^2 + (dy)^2]^{1/2} \quad (3)$$

When the curve is given by parametric equations  $x = \phi(t)$ ,  $y = \psi(t)$ , we have  $dx = \phi'(t)dt$ ,  $dy = \psi'(t)dt$ , and therefore by (3),

$$ds = [\phi'(t)^2 + \psi'(t)^2]^{1/2} dt \quad (4)$$

When the curve is given by a polar equation  $r = \phi(\theta)$ , referred to the same origin and  $Ox$  as in Fig. 95, we have  $x = r \cos \theta$ ,  $y = r \sin \theta$ , where  $r = \phi(\theta)$ ; hence

$$\frac{dx}{d\theta} = -r \sin \theta + \frac{dr}{d\theta} \cos \theta \quad \frac{dy}{d\theta} = r \cos \theta + \frac{dr}{d\theta} \sin \theta$$

In (4) replace  $dt$ ,  $\phi'(t)$ ,  $\psi'(t)$  by  $d\theta$ ,  $dx/d\theta$ ,  $dy/d\theta$ ; we obtain

$$ds = \left[ r^2 + \left( \frac{dr}{d\theta} \right)^2 \right]^{1/2} d\theta \quad (5)$$

We have identically,

$$s = \int_{P_0}^P ds \quad (6)$$

In (6) we may replace  $ds$  by its expression in terms of any variable if we also replace  $P_0, P$  by the appropriate limits. Hence, if  $t_0, t$  and  $\theta_0, \theta$  correspond to  $P_0, P$ , we have from (4) and (5)

$$s = \int_{t_0}^t [\phi'(t)^2 + \psi'(t)^2]^{1/2} dt \quad (7) \quad s = \int_{\theta_0}^{\theta} \left[ r^2 + \left( \frac{dr}{d\theta} \right)^2 \right]^{1/2} d\theta \quad (8)$$

EXAMPLE 2. Find the length of the hypocycloid of four cusps  $x = a \cos^3 \theta, y = a \sin^3 \theta$ .

Substituting in (7), we find for the length  $s$  of the first of the four equal quadrants of the curve

$$\begin{aligned} s &= 3a \int_0^{\pi/2} [\cos^4 \theta \sin^2 \theta + \sin^4 \theta \cos^2 \theta]^{1/2} d\theta \\ &= 3a \int_0^{\pi/2} \sin \theta \cos \theta d\theta = \frac{3a}{2}. \end{aligned}$$

EXAMPLE 3. Find the length of the cardioid  $r = a(1 - \cos \theta)$ . Substituting in (8), we find for the length  $s$  of half the curve

$$\begin{aligned} s &= a \int_0^{\pi} [(1 - \cos \theta)^2 + \sin^2 \theta]^{1/2} d\theta = a \int_0^{\pi} [2 - 2 \cos \theta]^{1/2} d\theta \\ &= 2a \int_0^{\pi} \sin \frac{\theta}{2} d\theta = 4a. \end{aligned}$$

### EXERCISE XXVIII

Find the following curve lengths, drawing the graphs.

- $9y^2 = 4x^3, x = 3 \text{ to } x = 8$
- $y = \log \cos x, x = 0 \text{ to } x = \pi/4$
- $x^2 + y^2 = 4, x = 1 \text{ to } x = 2$
- Arc of  $y^2 = 6x - x^2$  intercepted by  $y = x$
- $y = \log(1 - x^2), x = -\frac{1}{2} \text{ to } x = \frac{1}{2}$
- $y = 2 \sec x, x = 0 \text{ to } x = \pi/4$
- $y = ax^2, x = 0 \text{ to } x = x$
- $y^2 = 16x, x = 0 \text{ to } x = 9$
- $y = \frac{a}{2}(e^{x/a} + e^{-x/a}), x = 0 \text{ to } x = a$
- Hypocycloid  $x^{2/3} + y^{2/3} = a^{2/3}$
- $y = \log x, x = 1 \text{ to } x = 2$
- $9y^2 = 4(x^2 - 1)^3, x = 1 \text{ to } x = 3$
- Loop of  $9ay^2 = x^2(x + 3a)$
- Loop of  $y^2 = x^2(\frac{1}{3} - x)$
- $r = e^{\theta}, \theta = 0 \text{ to } \theta = 2\pi$
- $r = a \cos \theta, \theta = 0 \text{ to } \pi/4$
- $r = a \cos^2 \frac{\theta}{2}, \theta = 0 \text{ to } \theta = \pi$
- $r = a \cos \theta + b \sin \theta$

19.  $x = a(\sin \theta - \theta \cos \theta)$ ,  $y = a(\cos \theta + \theta \sin \theta)$ ,  $\theta = 0$  to  $\theta = \pi$

20. An arch of the cycloid  $x = a(\theta - \sin \theta)$ ,  $y = a(1 - \cos \theta)$

21. Prove respecting curves of the type  $x = \phi(t)$ ,  $y = \psi(t)$  that:

If  $\phi'(t) = f(t)(at + b)$ ,  $\psi'(t) = f(t)(a't + b')$ ,

then  $s_0^t = \int_0^t f(t)(At^2 + Bt + C)^{1/2} dt$

Hence show that an arc of  $x = t^n$ ,  $y = t^{n+1}$ , or  $y^n = x^{n+1}$ , can be found by the formulas of integration.

**143. Area of frustum of cone.** In the bases of a frustum of a right circular cone inscribe similarly placed regular polygons of  $n$  sides, and then form trapezoids, like  $PQP'Q'$ , by joining their corresponding angular points. The limit which the sum of the areas of these trapezoids approaches when  $n \rightarrow \infty$  is called the area of the lateral surface of the frustum. It is  $2\pi rh$ , where  $h$  is the slant height of the frustum and  $r$  is the radius of the midsection parallel to the bases.

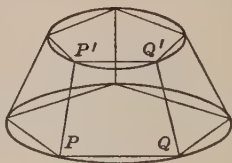


FIG. 96.

**144. Area of a surface of revolution.** When the arc  $AB$  of  $y = f(x)$  considered in § 142 is revolved about  $Ox$  it generates a surface. The area of this surface is defined as the limit of the area generated by the inscribed polygon of  $n$  sides described in § 142, when  $n \rightarrow \infty$ . Each side  $PQ$  of the polygon generates the lateral surface of the frustum of a right circular cone of which  $PQ$  is the slant height; its area is  $2\pi \cdot EM \cdot PQ$ , where  $M$  is the midpoint of  $PQ$ . As shown in § 142, there is a point  $x$  in  $\delta x$ , the projection of  $PQ$  on  $Ox$ , such that  $PQ = [1 + f'(x)^2]^{1/2} \delta x$ . And since  $EM$  is between the least and greatest values of  $f(x)$  in  $\delta x$ , there is also a point

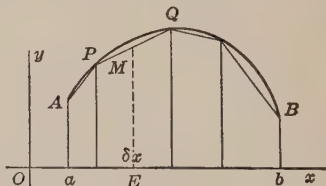


FIG. 97.

$x'$  in  $\delta x$  such that  $EM = f(x')$ . Hence the area generated by the polygon can be expressed in the form

$$\Sigma 2\pi \cdot EM \cdot PQ = 2\pi \Sigma f(x') [1 + f'(x)^2]^{1/2} \delta x$$

By § 145,  $\Sigma f(x') [1 + f'(x)^2]^{1/2} \delta x \rightarrow \int_a^b f(x) [1 + f'(x)^2]^{1/2} dx$ .  
Hence

$$\text{Area generated by arc } AB = 2\pi \int_{x=a}^{x=b} y \, ds \quad (1)$$

**EXAMPLE 1.** Find the area of the surface got by revolving either arc of the parabola  $y^2 = 4x$  between  $x = 0$  and  $x = 8$  about  $Ox$ .

$$S = 2\pi \int_0^8 2x^{1/2} \left[1 + \frac{1}{x}\right]^{1/2} dx = 4\pi \int_0^8 (x+1)^{1/2} dx = \frac{8}{3}\pi(27-1) = \frac{208}{3}\pi.$$

**EXAMPLE 2.** If an arc of the circle  $x^2 + y^2 = a^2$  between  $x = x_1$  and  $x = x_1 + h$  be revolved about  $Ox$ , a spherical zone  $Z$  of altitude  $h$  is generated. Prove that area  $Z = 2\pi ah$ .

$$x^2 + y^2 = a^2 \text{ gives } x + y \frac{dy}{dx} = 0 \quad \therefore 1 + \left(\frac{dy}{dx}\right)^2 = \frac{x^2 + y^2}{y^2} = \frac{a^2}{y^2}.$$

$$\text{Hence area } Z = 2\pi \int_{x_1}^{x_1+h} \frac{a}{y} y \, dx = 2\pi ah.$$

Therefore the area of the sphere is  $2\pi a \cdot 2a = 4\pi a^2$ .

In (1) we may express  $ds$  and  $y$  in terms of any variable. If the arc is circular, it is usually most convenient to make the angle  $\theta$  at the center the independent variable, which gives  $ds = a d\theta$ ,  $a$  being the radius.

**EXAMPLE 3.** Find the area of the torus got by revolving  $x^2 + (y-b)^2 = a^2$ ,  $a < b$ , about  $Ox$ . Since  $y = DP = OC - EC = b - a \cos \theta$ , and  $ds = a d\theta$ , we have  $(1/2)S = 2\pi \int_0^\pi (b - a \cos \theta) a d\theta = 2ab\pi^2$ ,  $S = 4ab\pi^2$ . By the same method we can find the areas of the surfaces generated by the arcs into which  $Ox$  cuts the circle when  $b < a$ .

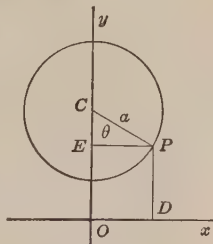


FIG. 98.

**145. Theorem.** The functions  $f(x)$  and  $\phi(x)$  are continuous in  $(a, b)$ . Divide  $(a, b)$  into  $n$  parts  $\delta x$ , all of which  $\rightarrow 0$  when  $n \rightarrow \infty$ , and let  $x$  and  $x'$  denote any two points in the same  $\delta x$ ; then

$$\lim [\Sigma f(x)\phi(x')\delta x]_a^b = \int_a^b f(x)\phi(x)dx \quad (1)$$

For we can express  $\Sigma f(x)\phi(x')\delta x$  in the form

$$\Sigma f(x)\phi(x)\delta x + \Sigma f(x)[\phi(x') - \phi(x)]\delta x$$

The limit of the first term is  $\int_a^b f(x)\phi(x)dx$ ; hence the theorem will be proved if it can be shown that the limit of the second term is 0.

But, since  $\phi(x)$  is continuous, if any positive number  $\epsilon$  be assigned we can find a value  $n'$  of  $n$  such that when  $n > n'$  each of the differences  $\phi(x') - \phi(x)$  is numerically less than  $\epsilon$  (§ 122). Hence, if  $M$  denotes the greatest value of  $|f(x)|$  in  $(a, b)$  we shall have, when  $n > n'$ ,

$$|\Sigma f(x)[\phi(x') - \phi(x)]\delta x| < M\epsilon \Sigma \delta x = M\epsilon(b - a)$$

We can take  $M\epsilon(b - a)$  as small as we please; hence

$$\lim \Sigma f(x)[\phi(x') - \phi(x)]\delta x = 0^1$$

### EXERCISE XXIX

Find the areas got by revolving the following arcs about  $Ox$ .

1.  $y = x^3$ ,  $x = 0$  to  $x = 1$
2.  $y = 2 \sec x$ ,  $x = 0$  to  $x = \pi/4$
3. Loop of  $9y^2 = x(x - 3)^2$
4. Loop of  $8y^2 = x^2 - x^4$
5.  $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$ ,  $x = 0$  to  $x = a$
6.  $y^2 = 1 - x$ ,  $x = 0$  to  $x = 1$
7.  $x^2 + 2y^2 = 2$
8.  $x^2 - y^2 = 1$ ,  $x = 1$  to  $x = 2$
9.  $x = a \cos \theta$ ,  $y = a \sin \theta$ ,  $\theta = \pi/4$  to  $\theta = \pi/3$ .
10. An arch of the cycloid  $x = a(\theta - \sin \theta)$ ,  $y = a(1 - \cos \theta)$
11. The cardioid  $r = a(1 - \cos \theta)$
12.  $y = e^{-x}$ ,  $x = 0$  to  $x = \infty$
13.  $x = 1/(t + 1)$ ,  $y = t/(t + 1)$ ,  $t = 0$  to  $t = 1$
14.  $x = 1/(t^2 + 1)$ ,  $y = t/(t^2 + 1)$ ,  $t = 0$  to  $t = 5$
15.  $6xy = x^4 + 3$ ,  $x = 2$  to  $x = 4$
16.  $x = a \cos^3 \theta$ ,  $y = a \sin^3 \theta$

<sup>1</sup> Similarly, if  $f(x)$ ,  $\phi(x)$ ,  $\psi(x)$  are continuous in  $(a, b)$ , and  $x, x', x''$  denote points in the same  $\delta x$ , we have

$$\lim [\Sigma f(x)\phi(x')\psi(x'')\delta x]_a^b = \int_a^b f(x)\phi(x)\psi(x)dx$$

This follows from § 145 and the fact that

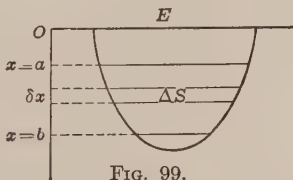
$$f(x)\phi(x')\psi(x'') = f(x)\phi(x)\psi(x'') + f(x)\psi(x'')[\phi(x') - \phi(x)].$$

The extension to the case of any number of factors is obvious.

17. Find the area of a parabolic mirror 2 ft. deep and 6 ft. wide.
18. Show that the area  $S$  got by revolving about  $Oy$  the arc of  $y^2 = 4x$  to the left of the line  $x - 1 = 0$  is given by  $S/2 = 2\pi \int_{x=0}^{x=1} x ds$  and find it.
19. Find the area got by revolving the arc in Ex. 18 about  $x - 1 = 0$ .
20. An arc of length  $\lambda$  is revolved about each of two parallel axes whose distance apart is  $d$ ; show that the difference between the areas generated is  $2\pi d\lambda$ .
21. An arc which subtends the angle  $2\beta$  at the center of a circle of radius  $a$  is revolved about its chord; show that the area generated is  $4\pi a^2(\sin \beta - \beta \cos \beta)$ .
22. Show that the area generated by revolving a quadrant of a circle of radius  $a$  about the tangent at one extremity is  $\pi(\pi - 2)a^2$ .
23. A variable sphere of radius  $r$  has its center on the surface of a fixed sphere of radius  $a$ ; show that the area of the portion of it within the fixed sphere is greatest when  $r = (4/3)a$ .
24. A point  $P$  moves from  $A$  to  $B$  on a curve  $y = f(x)$  in the  $xy$ -plane; and  $Q$  is a point at the distance  $z = \phi(x)$  vertically above  $P$ . Show that area generated by  $PQ$  is  $\int_A^B z ds$ .
25. Show that the area of the portion of the cylindrical surface  $x^2 + y^2 - 2ax = 0$  within the sphere  $x^2 + y^2 + z^2 = 4a^2$  is  $16a^2$ .

## PHYSICAL PROBLEMS

**146. Fluid pressure.** At the distance  $x$  below the surface  $E$  of a fluid the intensity of the pressure in every direction is  $kx$ , where  $k$  denotes the weight of a cubic unit of the fluid. Let there be immersed in the fluid a vertical plane surface  $S$  whose width at the distance  $x$  below  $E$  is  $w = f(x)$ ,  $f(x)$  being continuous. By planes parallel to  $E$  divide the portion  $S_a^b$  of  $S$  between the levels  $x = a$  and  $x = b$  into  $n$  strips  $\Delta S$  of height  $\delta x$  which  $\rightarrow 0$  when  $n \rightarrow \infty$ . The pressure on the strip  $\Delta S$  between  $x$  and  $x + \delta x$  is  $> kx\Delta S$  and  $< k(x + \delta x)\Delta S$  and therefore is  $kx'\Delta S$  where  $x'$  is some



point in  $\delta x$ ; and there is also a point  $x''$  in  $\delta x$  such that  $\Delta S = f(x'')\delta x$  (§ 129 (6)).

When  $n \rightarrow \infty$ , then  $[\Sigma kx'f(x'')\delta x]_a^b \rightarrow k \int_a^b x f(x) dx$  (§ 145).

Therefore the pressure on  $S_a^b$  is

$$P_a^b = k \int_a^b x w dx \quad (1)$$

**EXAMPLE.** A trough whose cross section is a semicircle of radius 3 ft. is full of water. What is the pressure on the end of the trough, the weight of 1 cu. ft. of water being 62 lbs.? Here  $w/2 = (3^2 - x^2)^{1/2}$ . Hence  $P = 62 \int_0^3 2x(9 - x^2)^{1/2} dx = 1116$  lbs.

### EXERCISE XXX

1. A flood gate 10 ft. square has its top at the water surface. What is the pressure on the gate? What are the pressures on the halves into which a diagonal divides it?

2. What is the pressure on the gate closing a water main of 1 ft. radius if the center of the gate be 100 ft. below the surface of the water in the reservoir?

3. Show that the pressure on the part  $S_a^b$  of a plane surface which makes the angle  $\alpha$  with the vertical is  $k \sec \alpha \int_a^b x w dx$ .

4. Show that the pressure on the part  $S_a^b$  of any surface for which the  $x$ -differential of area is of the form  $dS = \phi(x)dx$  is  $P_a^b = k \int_{x=a}^{x=b} x dS$ .

5. Find the pressure on a sphere of 4 ft. radius whose center is 6 ft. below the water surface.

6. A trough 10 ft. long and whose cross section is a triangle 4 ft. deep and 6 ft. across at the top is full of water. Find the pressure on its ends and sides.

7. A cylindrical tank of length 20 ft. and radius 3 ft. is full of oil weighing 50 lbs. per cu. ft. Find the pressure on the tank when set (a) vertically, (b) horizontally.

**147. Work.** When, under the action of a constant force  $F$ , a particle moves the distance  $s$  in the direction of  $F$ , the force is said to do the work  $Fs$ .

Let a particle  $P$  move along  $Ox$  from  $x = a$  to  $x = b$  under the action of a variable force in the direction of the motion, the force being a continuous function  $F(x)$  of the distance  $x$



of  $P$  from  $O$ . Divide  $(a, b)$  into  $n$  parts  $\delta x$  which  $\rightarrow 0$  when  $n \rightarrow \infty$ , and multiply each  $\delta x$  by the value of  $F(x)$  at any point in that  $\delta x$ . The sum of the products thus obtained,  $[\Sigma F(x)\delta x]_a^b$ , is the total work done in the several parts  $\delta x$  by the constant forces represented by the selected values of  $F(x)$ . When  $n \rightarrow \infty$ , then  $[\Sigma F(x)\delta x]_a^b \rightarrow \int_a^b F(x)dx$ .

We therefore define the work done by  $F(x)$  as

$$W = \int_a^b F(x)dx \quad (1)$$

If  $P$  moves in the opposite direction to that of  $F$ , the negative of (1) is called the work done against  $F$ .

### EXERCISE XXXI

1. A particle  $P$  on  $Ox$  is attracted toward  $O$  by a force  $F = kx^2$ . What is the work done when  $P$  moves from the point  $x = 6$  to the point  $x = 3$ ?

2. A positive charge  $m$  of electricity at  $O$  repels a unit positive charge at the distance  $x$  from  $O$  with the force  $m/x^2$ . What is the work done when the unit charge is carried from  $x = 2a$  to  $x = a$ ? from  $x = \infty$  to  $x = a$ ?

3. The work done against gravity in raising a pound weight one foot is called a *foot-pound*. A cistern is bounded laterally by a cylindrical surface of depth 20 ft. and radius 6 ft. and at the bottom by an inverted hemisphere: find in ft. lbs. the work done in pumping it out if at the start it is full of water.

4. The moment about  $O$  of a weight  $m$  at the point  $(x, 0)$  of  $Ox$  is  $mx$ . Show by the reasoning in § 146 that if  $m$  were distributed uniformly along the segment  $ab$  of  $Ox$  the moment would be  $k \int_a^b x dx$ , where  $k = m/(b - a)$ .

5. A light  $A$  of intensity  $k$  is at the distance  $h$  above the center  $O$  of a circular space  $S$  of radius  $a$ . If the intensity of the illumination of  $S$  at any point  $P$  is  $\frac{k \sin OPA}{AP^2}$ , show by the reasoning in § 146 that the total illumination of  $S$  is  $k \int_0^a \frac{2 \pi h r dr}{(h^2 + r^2)^{3/2}}$ .

6. The equation of motion along  $Ox$  of a particle  $P$  of mass  $m$  is  $x = t^3$ . Find the work done by the force causing this motion between  $t = 2$  and  $t = 4$ . ( $F = m d^2x/dt^2$ )

## XVI. DOUBLE INTEGRALS

**148. Continuous functions of two variables.** Let  $S$  denote a closed region, its boundary included, in the  $xy$ -plane, and let  $f(x, y)$  denote a one-valued function of  $x$  and  $y$  in  $S$ , so that to each point  $(x, y)$  of  $S$  corresponds a single real value of  $f(x, y)$ .

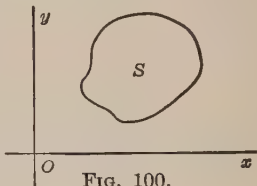


FIG. 100.

1. The function  $f(x, y)$  is said to be *continuous at the point*  $(a, b)$  of  $S$  if  $f(a, b)$  has a definite finite value and if

$$f(x, y) \rightarrow f(a, b) \text{ when } (x, y) \rightarrow (a, b) \text{ in } S$$

2. The function  $f(x, y)$  is said to be *continuous in*  $S$  if it is continuous at every point of  $S$ .

3. It can be proved that among the different values of a function  $f(x, y)$  continuous in  $S$  there is a *greatest value*  $M$  and a *least value*  $m$ , and that  $f(x, y)$  takes in  $S$  every value between  $M$  and  $m$ .

4. The difference between the greatest and least values of  $f(x, y)$  in any part of  $S$  is called the *oscillation* of  $f(x, y)$  in that part.

5. It can be proved that if  $f(x, y)$  is continuous in  $S$  it is *uniformly continuous in*  $S$ : that is, to any assigned positive number  $\epsilon$  there corresponds another positive number  $\delta$  such that in every part of  $S$  which can be inclosed in a circle of radius  $\delta$  the oscillation of  $f(x, y)$  is less than  $\epsilon$ .

**149. The integral  $\int_S f(x, y) dS$ .** The proof of the following theorem is, except for notation, identical with that in § 124.

Let a region  $S$  in which  $f(x, y)$  is continuous be divided into parts each of which can be inclosed in a circle of radius  $\rho$ . Let  $\delta S$  denote any one of these parts and also its area, and let  $(x, y)$  denote any point in  $\delta S$ . When  $\rho \rightarrow 0$ , the sum  $\sum f(x, y)\delta S$  approaches a definite limit, and the value of this limit is independent of the mode of partition of  $S$ .

We represent  $\lim \sum f(x, y)\delta S$  by the symbol  $\int_S f(x, y) dS$ , read "integral  $f(x, y)$  over  $S$ ." Hence, by definition,

$$\int_S f(x, y) dS = \lim \sum f(x, y)\delta S \quad (1)$$

**150. The function  $F(x) = \int_c^d f(x, y) dy$ .** Let  $f(x, y)$  be continuous in the rectangle  $R$  bounded by the lines  $x = a$ ,  $x = b$ ,  $y = c$ ,  $y = d$ . For any assigned value  $x'$  of  $x$  between  $a$  and  $b$  the  $y$ -integral  $\int_c^d f(x', y) dy$  has a definite value; hence  $\int_c^d f(x, y) dy$  is a function of  $x$  in  $(a, b)$ , § 14.

*The function  $F(x) = \int_c^d f(x, y) dy$  is continuous.*

For  $f(x, y)$  is uniformly continuous in  $R$ , § 148, 5. Hence to any assigned positive number  $\epsilon$  there corresponds another positive number  $\delta$  such that everywhere in  $R$

$$|f(x + \Delta x, y) - f(x, y)| < \epsilon \text{ when } |\Delta x| < \delta \quad (1)$$

and when (1) is satisfied, we have, p. 146, Ex. 6,

$$\begin{aligned} |F(x + \Delta x) - F(x)| &\leq \int_c^d |f(x + \Delta x, y) - f(x, y)| dy \\ &< \int_c^d \epsilon dy = \epsilon(d - c) \end{aligned} \quad (2)$$

We can take  $\epsilon(d - c)$  as small as we please; hence

$$\lim_{\Delta x \rightarrow 0} F(x + \Delta x) = F(x).$$

**EXAMPLE.**  $F(x) = \int_1^3 2xy dy = [xy^2]_{y=1}^{y=3} = 8x$ , which is a continuous function of  $x$  in any finite rectangle  $x = a$ ,  $x = b$ ,  $y = 1$ ,  $y = 3$ .

**151. The integral  $\int_a^b \int_0^{\phi(x)} f(x, y) dy dx$ .** Let  $f(x, y)$  be continuous in the space  $S$  bounded by the lines  $x = a$ ,  $x = b$ .  $Ox$ , and a curve  $y = \phi(x)$  which is continuous between  $x = a$  and  $x = b$  (Fig. 101).

In  $S$  the integral  $\int_0^y f(x, y) dy$  is a continuous function of  $x$  (by the reasoning of § 150) and of its upper limit  $y$  (by § 126). Therefore,  $y = \phi(x)$  being continuous, the integral  $\int_0^{\phi(x)} f(x, y) dy$  is a continuous function of  $x$  in the interval  $(a, b)$  and has an  $x$ -integral between the limits  $a$  and  $b$ . This integral

$$\int_a^b [\int_0^{\phi(x)} f(x, y) dy] dx \text{ is written } \int_a^b \int_0^{\phi(x)} f(x, y) dy dx \quad (1)$$

EXAMPLE.  $\int_1^3 \int_0^{\sqrt{x}} 2y dy dx = \int_1^3 [y^2]_0^{\sqrt{x}} dx = \int_1^3 x dx = 4.$

**152. Evaluation of the integral  $\int_S f(x, y) dS$ .** Let  $S$  be the space  $abBA$  bounded by the lines  $x = a$ ,  $x = b$ ,  $Ox$ , and a continuous curve arc  $y = \phi(x)$  along which  $y$  increases only or decreases only as  $x$  increases. And let  $\int_S f(x, y) dS$  denote the integral over this space defined in § 149. We shall prove that

$$\int_S f(x, y) dS = \int_a^b \int_0^{\phi(x)} f(x, y) dy dx$$

Through points  $x_2, x_3, \dots$  of  $Ox$  got by dividing  $(a, b)$  into parts, then subdividing these parts, and so on, draw parallels to  $Oy$ , and so divide  $S$  into strips  $S_1, S_2, \dots, S_m$  whose bases  $h_1, h_2, \dots, h_m \rightarrow 0$  when  $m \rightarrow \infty$ .

Then by parallels to  $Ox$  divide

each strip into a set of rectangles (like  $QR$ ) and a triangle (like  $PVT$ ) whose altitudes  $k_1, k_2, \dots, k_n \rightarrow 0$  when  $n \rightarrow \infty$ .

The parts into which  $S$  is thus divided are of the type  $\delta S$  considered in § 149. But we may ignore the triangles  $PVT$  when finding  $\int_S f(x, y) dS$ ; for their contribution to  $\sum f(x, y) \delta S$  is numerically  $< M \cdot h \cdot CB$ , where  $M$  is the greatest value of  $|f(x, y)|$  in  $S$  and  $h$  is the greatest  $h_i$ ; it therefore  $\rightarrow 0$  when  $m \rightarrow \infty$ . Hence if  $m_i$  and  $M_i$  denote the least and

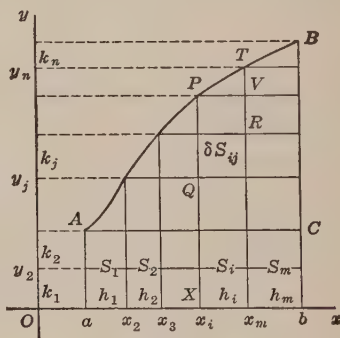


FIG. 101.

greatest values of  $f(x, y)$  in the rectangle  $(QR)$  whose sides are  $h_i$  and  $k_j$ , we have, by § 149,

$$\int_S f(x, y) dS = \lim \Sigma m_{ij} h_i k_j = \lim \Sigma M_{ij} h_i k_j \quad (1)$$

1. The portions of the sums  $\Sigma$  in (1) which belong to the strip  $S_i$  have the factor  $h_i$  in common and may be written  $h_i \Sigma m_{ij} k_j$  and  $h_i \Sigma M_{ij} k_j$ .

Since  $m_{ij} \leq f(x_i, y_j) \leq M_{ij}$ , we have

$$\Sigma m_{ij} k_j \leq \Sigma f(x_i, y_j) k_j \leq \Sigma M_{ij} k_j \quad (2)$$

But in the sum  $\Sigma f(x_i, y_j) k_j$ ,  $x_i$  is a constant; the  $k_j$ 's are a set of parts, which  $\rightarrow 0$ , of the segment  $x_i P$  of the line  $x = x_i$  between  $y = 0$  and  $y = \phi(x_i)$ ; and  $f(x_i, y_j)$  is a value of the  $y$ -function  $f(x_i, y)$  in the part  $k_j$ . Hence when  $n \rightarrow \infty$ ,  $\Sigma f(x_i, y_j) k_j \rightarrow \int_0^{\phi(x_i)} f(x_i, y) dy$ , § 124. Therefore, by (2),

$$\Sigma m_{ij} k_j \leq \int_0^{\phi(x_i)} f(x_i, y) dy \leq \Sigma M_{ij} k_j \quad (3)$$

2. Multiply each member of (3) by  $h_i$ . Then apply the resulting inequalities to each of the strips  $S_1, S_2, \dots S_m$  and add corresponding members of the results. We get

$$\Sigma m_{ij} h_i k_j \leq \Sigma h_i \int_0^{\phi(x_i)} f(x_i, y) dy \leq \Sigma M_{ij} h_i k_j \quad (4)$$

By (1) the limit of the first and last members of (4) is  $\int_S f(x, y) dS$ , and by § 151 the limit of the second member is  $\int_a^b \int_0^{\phi(x)} f(x, y) dy dx$ . Hence

$$\int_S f(x, y) dS = \int_a^b \int_0^{\phi(x)} f(x, y) dy dx \quad (5)$$

This formula also holds good when the curve  $y = \phi(x)$  has any finite number of turning points between  $x = a$  and  $x = b$ ; for parallels to  $Oy$  through these points will divide the corresponding space  $S$  into pieces of the type just considered.

When  $f(x, y)$  is 1, the formula (5) becomes

$$\text{Area } S = \int_a^b \int_0^{\phi(x)} dy dx \quad (6)$$

**153. Computation of volumes.** Let  $K$  denote a solid bounded below by the space  $S$  of § 152, above by a continu-

ous surface  $z = f(x, y)$  which does not cut  $S$ , and laterally by the planes  $x = a$ ,  $x = b$ ,  $y = 0$ , and the cylindrical surface formed by parallels to  $Oz$  through the points of the curve  $y = \phi(x)$ .

The portion of  $K$  which rests on each part  $\delta S$  of  $S$  is included between two cylinders whose volumes are the least and greatest values of  $z\delta S$  in  $\delta S$ . Hence, § 138, the volume  $V$  of  $K$  is  $\int_S z dS$ , and therefore, § 152,

$$V = \int_a^b \int_0^{\phi(x)} z dy dx \quad (1)$$

The first integration gives the area  $A(x)$  of the section of  $K$  by the plane  $x = x$ , the second evaluates  $\int_a^b A(x) dx$ . (Compare § 141.)

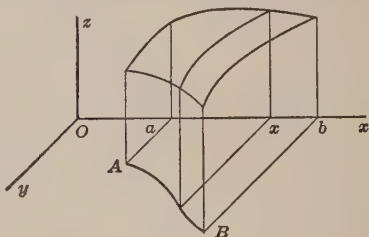


FIG. 102.

EXAMPLE. For the solid bounded by  $x = 2$ ,  $x = 4$ ;  $y = 0$ ,  $y = x$ ;  $z = 0$ ,  $z = x^2 + 3y^2$ , we have

$$V = \int_2^4 \int_0^x (x^2 + 3y^2) dy dx = \int_2^4 (x^2 y + y^3) \Big|_{y=0}^{y=x} dx = 2 \int_2^4 x^3 dx = 120$$

For a solid standing on a space  $S$  bounded by  $x = a$ ,  $x = b$  and two curve arcs  $y = \phi(x)$ ,  $y = \psi(x)$  which do not meet between  $x = a$  and  $x = b$ , we have

$$V = \int_a^b \int_{\phi(x)}^{\psi(x)} z dy dx \quad (2)$$

### EXERCISE XXXII

The student is advised to read §§ 204, 205 and Exs. 2 and 3 in § 213 before doing the following examples.

1. Compute the following and indicate the areas which they represent.

$$1. \int_1^2 \int_{-x^2}^x dy dx \quad 2. \int_2^3 \int_y^{1-y} dx dy \quad 3. \int_0^1 \int_{-2x}^{x+x^2} dy dx$$

2. Compute the following and indicate the volumes which they represent.

$$1. \int_{-1}^2 \int_2^3 (2x + 3y + 6) dy dx \quad 2. \int_2^3 \int_{\sqrt{2}}^x xy dy dx$$

$$3. \int_0^a \int_0^{\sqrt{a^2-x^2}} (x+y) dy dx$$

3. Show that under the conditions of § 150,

$$\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

4. Find the volume of one of the wedges cut from the cylinder  $x^2 + y^2 = a^2$  by the planes  $z = 0$  and  $z = x + y$ .

5. Find the volume of the solid bounded by the planes  $y = x$ ,  $y = 2x$ ,  $x = 1$ ,  $z = 0$  and the surface  $z = xy$ .

6. Find the volume of the solid bounded by the planes  $z = x + y$ ,  $z = 0$ ,  $x = 2$  and the cylindrical surface  $y^2 - 4x = 0$ .

7. Find the volume of the solid bounded by the surfaces  $x^2 + 3y^2 = z$ ,  $x^2 + y^2 = 2x$  and the plane  $z = 0$ .

8. Find the volume of the solid bounded by the surfaces  $y = x^2$ ,  $x = y^2$ , and the planes  $z = x$ ,  $z = 2x$ .

9. Find volume bounded by the cylinders  $x^2 + y^2 = a^2$ ,  $x^2 + z^2 = a^2$ , in the first octant.

10. Find the volume bounded by the planes  $x + 3y + 2z = 6$ ,  $x = 0$ ,  $y = 0$ ,  $z = 0$ .

11. Find the volume bounded by the surface  $x^{1/2} + y^{1/2} + z^{1/2} = 1$ , and the planes  $x = 0$ ,  $y = 0$ ,  $z = 0$ .

12. Set up the integral for the volume of the solid bounded by  $z = x^2 + y^2$  and  $z = 2x$  [Eliminating  $z$  gives  $x^2 + y^2 = 2x$ , which bounds the space in the  $xy$ -plane above which the solid lies.]

13. Let  $S$  be the space  $OAB$  bounded by the curve arc  $r = \phi(\theta)$  and the rays  $\theta = \alpha$  and  $\theta = \beta$ . By rays from  $O$  and circles about  $O$  as center, divide  $S$  into parts  $\delta S$ . Show that  $\delta S = PQTR = r\delta\theta\delta r$ , where  $r = (OP + OR)/2$  and therefore belongs to points in  $\delta S$ . Then show, as in § 152, that if  $f(\theta, r)$  be continuous in  $S$ , we have

$$\int_S f(\theta, r) dS = \int_\alpha^\beta \int_0^{\phi(\theta)} f(\theta, r) r dr d\theta$$

14. Show that the volume of the solid bounded by  $z = 0$ ,  $z = x + 2y + 8$  and the right cylinder through the cardioid  $r = 1 - \cos \theta$  is given by  $\int_0^{2\pi} \int_0^{1 - \cos \theta} [r(\cos \theta + 2 \sin \theta) + 8] r dr d\theta$ .

15. Show that the volume of the solid bounded by the sphere  $x^2 + y^2 + z^2 = a^2$  and the right cylinder through the circle  $r = a \cos \theta$  is  $4 \int_0^{\pi/2} \int_0^{a \cos \theta} (a^2 - r^2)^{1/2} r dr d\theta = \frac{4a^3}{3} \left( \frac{\pi}{2} - \frac{2}{3} \right)$ .

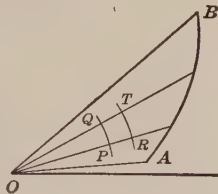


FIG. 103.



## TRIPLE INTEGRALS

**154.** The integral  $\int_S (x, y, z) dS$ . Let  $F(x, y, z)$  denote a one-valued function of the variables  $x, y, z$  in a given closed region  $S$  in space, its boundary included.

The function  $F(x, y, z)$  is said to be *continuous at the point*  $(a, b, c)$  in  $S$ , if  $F(a, b, c)$  has a finite value and if  $F(x, y, z) \rightarrow F(a, b, c)$  when  $(x, y, z) \rightarrow (a, b, c)$  in  $S$ . It is said to be *continuous in  $S$*  if it is continuous at all points of  $S$ .

By a repetition of the argument in §§ 148, 149, we may prove the theorem:

*Let a region  $S$  in which  $F(x, y, z)$  is continuous be divided into parts each of which can be inclosed in a sphere of radius  $\rho$ . Let  $\delta S$  denote any one of these parts and also its volume, and let  $(x, y, z)$  denote any point in  $\delta S$ . When  $\rho \rightarrow 0$ , the sum  $\Sigma F(x, y, z) \delta S$  approaches a definite limit, and the value of this limit is independent of the mode of partition of  $S$ .*

We represent  $\lim \Sigma F(x, y, z) \delta S$  by the symbol  $\int_S F(x, y, z) dS$ , read “integral  $F(x, y, z)$  over  $S$ .” Hence by definition,

$$\int_S F(x, y, z) dS = \lim \Sigma F(x, y, z) \delta S \quad (1)$$

Let  $S$  be the solid  $K$  in Fig. 102. If the upper boundary surface  $z = f(x, y)$  is met by parallels to the  $xy$ -plane in single points only or may be divided into portions of which this is true, it can be proved by an argument like that in § 152, but based on the division of  $S$  into parts by sets of planes parallel to the  $yz$ -,  $zx$ -, and  $xy$ -planes, that for this  $S$

$$\begin{aligned} \int_S F(x, y, z) dS &= \int_a^b \int_0^{\phi(x)} \int_0^{f(x, y)} F(x, y, z) dz dy dx \\ &= \int_a^b \left[ \int_0^{\phi(x)} \left( \int_0^{f(x, y)} F(x, y, z) dz \right) dy \right] dx \quad (2) \end{aligned}$$

In the first integration (with respect to  $z$ ),  $x$  and  $y$  play the rôles of constants, and in the second (with respect to  $y$ ),  $x$  plays this rôle. When  $F(x, y, z) = 1$ , (2) becomes a formula for the volume of the solid  $K$ .

EXAMPLE 1. Find  $\int_S x dS$  when  $S$  is the tetrahedron bounded by the planes  $x = 0$ ,  $y = 0$ ,  $z = 0$ , and  $x + y + z = 4$ .

$$\begin{aligned} \text{We have } \int_S x dS &= \int_0^4 \int_0^{4-x} \int_0^{4-x-y} x dz dy dx \\ &= \int_0^4 \int_0^{4-x} [xz]_{z=0}^{z=4-x-y} dy dx \\ &= \int_0^4 \int_0^{4-x} (4x - x^2 - xy) dy dx \\ &= \int_0^4 \left[ 4xy - x^2y - \frac{xy^2}{2} \right]_{y=0}^{y=4-x} dx \\ &= \frac{1}{2} \int_0^4 (16x - 8x^2 + x^3) dx = 10\frac{2}{3}. \end{aligned}$$

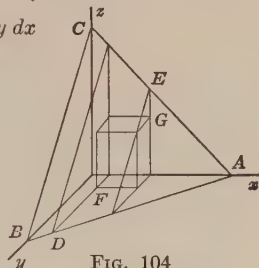


FIG. 104

Divide  $OA$ ,  $OB$ ,  $OC$  into parts  $\delta x$ ,  $\delta y$ ,  $\delta z$ , and through the points of division take planes parallel to the planes  $x = 0$ ,  $y = 0$ ,  $z = 0$ . We thus divide  $OABC$  into parts  $\delta S$  of volume  $\delta x \delta y \delta z$  (except those along  $ABC$  whose sum  $\rightarrow 0$ ). The successive integrations correspond to forming  $\Sigma x \delta S$  for the  $\delta S$ 's in (1) a column ( $FG$ ) parallel to  $Oz$ , (2) a slice ( $DE$ ) parallel to  $Oyz$ , (3) the whole of  $OABC$ .

- EXAMPLE 2. Find (1)  $\int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2) dz dy dx$   
 (2)  $\int_2^3 \int_1^x \int_0^x x dz dy dx$   
 (3)  $\int_0^a \int_0^x \int_0^{x+y} xyz dz dy dx$

EXAMPLE 3. Express  $\int_S F(x, y, z) dS$  as a triple integral when  $S$  is the sphere whose equation is  $x^2 + y^2 + z^2 - 2x = 0$ .

## XVII. MEAN VALUES

**155. Mean values.** 1. Let  $y = f(x)$  be continuous in the  $x$ -interval  $(a, b)$ . Divide  $(a, b)$  into  $n$  equal parts, of length  $h$ , and let  $y_1, y_2, \dots$  be any values taken by  $y$  in the first, second,  $\dots$  of these parts. The arithmetic mean of the numbers  $y_1, y_2, \dots$ , namely

$$(y_1 + y_2 + \dots + y_n)/n \quad (1)$$

will approach a definite limit when  $n \rightarrow \infty$ . For (1) equals

$$(y_1 h + y_2 h + \dots + y_n h)/nh \quad (2)$$

But  $nh = b - a$ , and  $\lim_{n \rightarrow \infty} (y_1 h + y_2 h + \dots + y_n h) = \int_a^b y \, dx$ .

Hence

$$\lim_{n \rightarrow \infty} \frac{\sum y_i}{n} = \frac{\int_a^b y \, dx}{b - a} \quad (3)$$

We call (3) the *mean value of  $y = f(x)$  with respect to  $x$  over the range  $(a, b)$* .

**EXAMPLE 1.** The mean value, with respect to  $x$ , of the ordinates of the parabola  $y = x^2$  between  $x = 0$  and  $x = 2$  is  $[\int_0^2 x^2 \, dx]/2 = 4/3$ .

2. The notion of mean value is readily extended to functions of two or more variables. Thus, let  $S$  denote any closed region in the  $xy$ -plane and also its area; the mean value of  $z = f(x, y)$  with respect to  $x, y$  over the range  $S$  is

$$\bar{z} = [\int_S f(x, y) \, dS]/S \quad (4)$$

We think of  $S$  as divided into equal parts of the type  $\delta S$ , § 149, choose a value taken by  $z = f(x, y)$  in each of these parts and then proceed as in 1. In general,

*The mean value of a function of one or more variables with respect to these variables over a given range is the integral of the function over the range, divided by the measure of the range.*

EXAMPLE 2. Let  $R$  be the rectangle bounded by  $x = 0$ ,  $x = a$ ,  $y = 0$ ,  $y = b$ . The mean value of the squares of the distances of the points of  $R$  from  $O$  is  $[\int_0^a \int_0^b (x^2 + y^2) dy dx] / ab = (a^2 + b^2) / 3$ .

3. Multiplying (4) by  $S$ , we obtain

$$\bar{z}S = \int_S z dS \quad (5)$$

Suppose  $S$  to be divided into  $m$  parts  $\Delta S_1, \Delta S_2, \dots, \Delta S_m$  of any kind, as for example, into strips by parallels to  $Oy$ : then

$$\int_S z dS = \int_{\Delta S_1} z dS + \int_{\Delta S_2} z dS + \dots + \int_{\Delta S_m} z dS \quad (6)$$

Let  $\bar{z}'_1, \bar{z}'_2, \dots$  denote the mean values of  $z$  in  $\Delta S_1, \Delta S_2, \dots$ . Then  $\int_{\Delta S_1} z dS = \bar{z}'_1 \Delta S_1, \int_{\Delta S_2} z dS = \bar{z}'_2 \Delta S_2, \dots$ , by (5). Hence

$$\bar{z}S = \bar{z}'_1 \Delta S_1 + \bar{z}'_2 \Delta S_2 + \dots + \bar{z}'_m \Delta S_m \quad (7)$$

*The product of mean value by measure of range for the whole is the sum of such products for the parts.*

The relation (6) continues to hold good when  $m \rightarrow \infty$  and every  $\Delta S_i \rightarrow 0$ . Hence

$$\bar{z}S = \int_S \bar{z}' dS \quad (8)$$

where the differential  $dS$  represents  $\Delta S$ , and  $\bar{z}'$  the mean value of  $z$  in  $\Delta S$ .

EXAMPLE 3. Find the mean distance of the points of the surface  $S$  of a circle of radius  $a$  from its center  $O$ . Divide  $S$  into circular rings. Let  $\Delta S$  be any one of these rings,  $r$  its inner radius and  $\delta r$  its breadth. The mean distance  $r'$  of the points of  $\Delta S$  from  $O$  is between  $r$  and  $r + \delta r$ ; and  $\Delta S = 2\pi r' \delta r$ , where  $r' = r + \delta r / 2$ , § 139. Hence the required mean distance is  $[\int_0^a r \cdot 2\pi r dr] / a^2\pi = (2/3)a$ , § 145.

### EXERCISE XXXIII

1. Show that the mean value of  $\sin^2 \theta$  with respect to  $\theta$  between  $\theta = 0$  and  $\theta = \pi/2$  is  $1/2$ .

2. The circle of radius  $a$  about  $O$  may be represented by  $x^2 + y^2 = a^2$  or by  $x = a \cos \theta, y = a \sin \theta$ . Hence show that the mean value of the ordinates of the first quadrant is  $a\pi/4$  when based on the division of the radius along  $Ox$  into equal parts, and that it is  $2a/\pi$  when based on the division of the arc into equal parts.

3. The final velocity of a body falling from rest is  $v_1$ . Show that its mean velocity during its motion was  $v_1/2$  with respect to time, but  $(2/3)v_1$  with respect to distance.

4. Show that the mean distance from  $O$  of the points of the surface of the circle  $r = a \cos \theta$  is  $16a/9\pi$ . See p. 173, Ex. 13.

5. Derive the mean value theorem as stated in § 129 (6) from § 155 (3).

6. If  $a$  denote the earth's radius and  $z$  the distance of a point  $P$  of the earth's surface from the plane of the equator, then the latitude of  $P$  is  $\sin^{-1}(z/a)$ . Show that the mean latitude of all points north of the equator is  $\frac{1}{a} \int_0^a \sin^{-1} \frac{z}{a} dz = \frac{\pi}{2} - 1$  radians ( $32.7^\circ$ ).

7. Let  $S_1$  and  $S_2$  be two regions which do not overlap, and let  $M_1$ ,  $M_2$ , and  $M$  be the mean values of  $z$  in  $S_1$ ,  $S_2$ , and  $S_1 + S_2$ . Show that  $M(S_1 + S_2) = M_1S_1 + M_2S_2$ .

8. By Ex. 7, find the mean distance from  $O$  of points between the circles  $r = a$  and  $r = a \cos \theta$ .

9. A number  $c$  is divided at random into two parts; show that the mean value of their product is  $c^2/6$ .

10. Find the mean value of the square of the distance of a point of a square from its center.

11. A number  $c$  is divided at random into three parts; show that the mean value of their product is  $c^3/60$ .

## CENTROIDS AND MASS CENTERS

**156. Centroids.** 1. Let  $S$  denote a given curve arc, surface, or solid; also its numerical measure. The point  $C$  whose coordinates  $\bar{x}$ ,  $\bar{y}$ ,  $\bar{z}$  are the mean values of the coordinates  $x$ ,  $y$ ,  $z$  of the points of  $S$  is called the *centroid*<sup>1</sup> of  $S$ . By § 155 (5),

$$\bar{x}S = \int_S x dS \quad \bar{y}S = \int_S y dS \quad \bar{z}S = \int_S z dS \quad (1)$$

Here  $dS$  stands for any one of a set of equal parts of  $S$  of the type  $\delta S$ , § 149, and  $(x, y, z)$  for any point in  $\delta S$ . When  $S$  is in the  $xy$ -plane,  $\bar{z} = 0$ .

<sup>1</sup> It can be shown that the position of the centroid as thus defined, relative to the points of  $S$ , is independent of the choice of the coordinate axes  $Ox$ ,  $Oy$ ,  $Oz$ .

EXAMPLE 1. Find the centroid of the first quadrant of the circle  $x = a \cos \theta$ ,  $y = a \sin \theta$ .

$$\bar{x} \frac{a\pi}{2} = \int_{\theta=0}^{\theta=\pi/2} x \, ds = \int_0^{\pi/2} a \cos \theta \cdot a \, d\theta = a^2 \quad \therefore \bar{x} = \frac{2a}{\pi}. \quad \text{And } \bar{y} = \bar{x}.$$

EXAMPLE 2. Find the centroid of the surface of the first quadrant of the circle  $x^2 + y^2 = a^2$ .

$$\bar{x} \frac{a^2\pi}{4} = \int_0^a \int_0^{(a^2-x^2)^{1/2}} x \, dy \, dx = \int_0^a x(a^2 - x^2)^{1/2} \, dx = \frac{a^3}{3} \quad \therefore \bar{x} = \frac{4a}{3\pi}.$$

And  $\bar{y} = \bar{x}$ .

EXAMPLE 3. Find the centroid of the solid bounded by the planes  $x = 2$ ,  $y = x$ ,  $y = 2x$ ,  $z = 0$ ,  $z = x$ .

$$\bar{z} \int_0^2 \int_x^{2x} \int_0^x dz \, dy \, dx = \int_0^2 \int_x^{2x} \int_0^x z \, dz \, dy \, dx, \text{ which gives } \bar{z} = 3/4.$$

A similar reckoning gives  $\bar{x} = 3/2$  and  $\bar{y} = 9/4$ .

EXAMPLE 4. Find the centroid of the region bounded by the curves  $2y = 2x - x^2$ ,  $y = x^2 - 2x$ .

EXAMPLE 5. Find the centroid of the solid bounded by the cylindrical surfaces  $y = x^2$ ,  $y^2 = x$ , and the planes  $z = 0$ ,  $z = y$ .

2. If  $S$  is symmetric with respect to a certain plane  $E$ , its centroid is in  $E$ . For refer the points of  $S$  to coordinate axes in which the  $xy$ -plane coincides with  $E$ . Then in the sum  $\sum z \delta S$  every two terms belonging to parts  $\delta S$  which are symmetric with respect to  $E$  will cancel each other; hence  $\int_S z \, dS = 0$  and therefore  $\bar{z} = 0$ .

In like manner, if  $S$  is symmetric with respect to a line  $l$ , its centroid is in  $l$ ; if symmetric with respect to a point  $C$ , its centroid is  $C$ .

Thus the centroid of a square or of an ellipse is its geometric center.

3. Let  $S$  denote the portion of a given solid between the planes  $x = a$  and  $x = b$ , and  $A(x)$  the area of the section  $A_x$  of  $S$  by a plane perpendicular to  $Ox$  at the distance  $x$  from  $O$  [Fig. 92, p. 158]. Then

$$\bar{x}S = \int_a^b xA(x)dx \quad (2)$$

For by (1),  $\bar{x}S = \int_a^b x[\int_{A_x} dz \, dy]dx = \int_a^b xA(x)dx$

Thus for the solid got by revolving about  $Ox$  the space bounded by the curve  $y = f(x)$  and the lines  $y = 0$ ,  $x = a$ ,  $x = b$ , we have

$$\bar{x} \int_a^b y^2 dx = \int_a^b xy^2 dx \quad \bar{y} = 0 \quad z = 0 \quad (3)$$

EXAMPLE 1. Find the centroid of the right circular cone of altitude  $a$  and radius  $b$ . We get the cone by revolving about  $Ox$  the part of the line  $y = (b/a)x$ , between  $x = 0$  and  $x = a$ . Hence

$$\bar{x} \int_0^a \frac{b^2}{a^2} x^2 dx = \int_0^a \frac{b^2}{a^2} x^3 dx, \text{ which gives } \bar{x} = \frac{3}{4} a. \text{ And } \bar{y} = 0, \bar{z} = 0.$$

EXAMPLE 2. Show that the centroid of a solid hemisphere of radius  $a$  is on the central radius at the distance  $(3/8)a$  from the bounding plane.

EXAMPLE 3. Find the centroids of the solids, p. 157, Exs. 4-9, 17-22.

EXAMPLE 4. Find the centroid of the solid bounded by the surface  $4x = y^2 + 2z^2$  and the plane  $x = 4$ .

**157. Theorems of Pappus.** 1. Suppose a curve arc  $AB$  of length  $s$  in the  $xy$ -plane to be revolved about  $Ox$ . By § 144 (1) and § 156 (1), we have

$$\text{Area gen'd by } AB = 2\pi \int y ds = 2\pi \bar{y} \int ds = 2\pi \bar{y} \cdot s \quad (1)$$

*The area generated by the arc equals the length of the arc times the perimeter of the circle described by its centroid.*

2. Suppose a closed region  $S$  in the  $xy$ -plane, which does not cut  $Ox$ , to be revolved about  $Ox$ . Divide  $S$  into parts of the type  $\delta S = \delta x \delta y$ . By § 139 (2) the volume of the solid generated by  $\delta S$  is  $2\pi y \delta S$ , where  $y$  belongs to a point in  $\delta S$ . Hence, § 156 (1),

$$\text{Volume gen'd by } S = 2\pi \int_S y dS = 2\pi \bar{y} \cdot S \quad (2)$$

*The volume generated by  $S$  equals the area of  $S$  times the perimeter of the circle described by the centroid of  $S$ .*

EXAMPLE. For the torus got by revolving the circle  $x^2 + (y-b)^2 = a^2$ ,  $a < b$ , about  $Ox$  we have area  $= 2\pi b \cdot 2\pi a = 4\pi^2 ab$ , and volume  $= 2\pi b \cdot a^2\pi = 2\pi^2 a^2 b$ .

**158. Mass center.** Let  $S$  denote a physical solid and  $m$  its mass. The mass center of  $S$  is the point whose coordinates



$\bar{x}$ ,  $\bar{y}$ ,  $\bar{z}$  are the mean values of the coordinates  $x$ ,  $y$ ,  $z$  of the points of  $S$  corresponding to a division of  $S$  into parts  $\delta S$  of equal mass  $\delta m$ . We therefore have

$$m\bar{x} = \int_S x \, dm \quad m\bar{y} = \int_S y \, dm \quad m\bar{z} = \int_S z \, dm \quad (1)$$

When the ratio  $\delta m/\delta S$  has the same value  $c$  for all parts  $\delta S$ ,  $S$  is said to be *homogeneous and of density  $c$* . In this case  $dm = c \, dS$ ,  $m = cS$ ; hence the formulas (1) reduce to the formulas § 156 (1), that is, the mass center coincides with the centroid of  $S$ .

Suppose  $S$  to be non-homogeneous. Let  $P$  be any given point of  $S$ ,  $\delta S$  a part of  $S$  containing  $P$ , and  $\delta m$  its mass. When  $\delta S \rightarrow 0$ , the ratio  $\delta m/\delta S$  approaches a limit  $\mu$ ; we call  $\mu$  the *density of  $S$  at  $P$* . If  $\mu$  be given for every point  $P$  of  $S$  by some continuous function of the coordinates  $x$ ,  $y$ ,  $z$ , we obtain formulas for  $\bar{x}$ ,  $\bar{y}$ ,  $\bar{z}$  by setting  $dm = \mu \, dS$ ,  $m = \int_S \mu \, dS$  in (1). All that has been said holds good also when a mass  $m$  is supposed to be distributed over a given surface or line  $S$ .

**EXAMPLE.** Find the mass center of the square bounded by  $x = 0$ ,  $x = a$ ,  $y = 0$ ,  $y = a$  when the density at a point varies as the square of the distance from the origin.

Here  $\mu = c(x^2 + y^2)$ , where  $c$  is a constant. Hence

$\bar{x} \int_0^a \int_0^a c(x^2 + y^2) dy \, dx = \int_0^a \int_0^a xc(x^2 + y^2) dy \, dx$  which gives  $\bar{x} = (5/8)a$ . And  $\bar{y} = \bar{x}$ .

#### EXERCISE XXXIV

1. Show that if the origin is the mass center of  $S$ , then  $\int_S x \, dm = 0$ ,  $\int_S y \, dm = 0$ ,  $\int_S z \, dm = 0$ .

2. Show that the centroid of the curved surface of a hemisphere of radius  $a$  is at the distance  $(1/2)a$  from the bounding plane. Find the centroid of the total surface.

3. Find the centroid of the total surface of a right circular cone.

4. Find the mass center of a semicircle when the density varies as (1) the distance from the bounding diameter (2) the distance from the center of the circle.

5. Find the centroid of the solid got by revolving the figure bounded by  $y = x^{1/2}$ ,  $y = -1$ ,  $x = 0$  and  $x = 1$  about  $x = 1$ .

6. Find the centroid of the space bounded by  $Ox$  and an arch of the cycloid  $x = a(\theta - \sin \theta)$ ,  $y = a(1 - \cos \theta)$ . Also of the solid got by revolving this space about  $Ox$ .

7. For the  $S$  bounded by  $r = f(\theta)$ ,  $\theta = 0$ ,  $\theta = \beta$ , show that

$$\bar{x}S = \int_0^\beta \int_0^{f(\theta)} xr \, dr \, d\theta, \quad \bar{y}S = \int_0^\beta \int_0^{f(\theta)} yr \, dr \, d\theta,$$

where  $x = r \cos \theta$ ,  $y = r \sin \theta$ .

8. Find the centroid of the  $S$  between  $Ox$  and the upper half of  $r = a(1 - \cos \theta)$ .

9. Show that the volume of the solid got by revolving the  $S$  of Ex. 7 about  $Ox$  is  $2\pi \int_0^\beta \int_0^{f(\theta)} yr \, dr \, d\theta$ , and that for the centroid of this solid we have  $\bar{x} \int_0^\beta \int_0^{f(\theta)} yr \, dr \, d\theta = \int_0^\beta \int_0^{f(\theta)} xyr \, dr \, d\theta$ .

10. Apply the formulas in Ex. 9 to the solid got by revolving the  $S$  in Ex. 8 about  $Ox$ .

11. Show that the centroid of the tetrahedron bounded by the planes  $x = 0$ ,  $y = 0$ ,  $z = 0$ ,  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$  is  $\left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right)$ .

12. Find the volume and area of the solid got by revolving the rectangle bounded by  $y - x = 0$ ,  $y - x = 2$ ,  $y + x = 0$ ,  $y + x = 6$  about  $Ox$ .

13. Show that the mean distance of points in a sphere of radius  $a$  from a point  $O$  on its surface is  $(6/5)a$ ; also that the mean distance of the points of the surface of the sphere from  $O$  is  $(4/3)a$ .

14. Show that the centroid of a triangle is the point of intersection of its medians.

15. Express the coordinates of the centroid of the perimeter of a triangle in terms of the lengths of the sides and the coordinates of their midpoints.

## XVIII. MOMENTS OF INERTIA

**159. Moment of inertia.** Let  $l$  be a given axis, and  $P$  a particle of mass  $m$  at the distance  $r$  from  $l$ ; the product  $mr^2$  is called the *moment of inertia* of  $P$  about  $l$ .

Suppose that instead of being concentrated at a point  $P$ , the mass  $m$  is distributed through a line, surface or solid  $S$ . Divide  $S$  into parts of the type  $\delta S$ , § 149, multiply the mass  $\delta m$  of each part  $\delta S$  by the square of the distance  $r$  of any point in it from  $l$ , and add the products thus obtained. The resulting sum  $\Sigma r^2 \delta m$  will approach the limit  $\int_S r^2 dm$  when the parts  $\delta S \rightarrow 0$ . This limit is called the *moment of inertia* of  $m$  about  $l$ . Denoting it by  $I_l$ ,

$$I_l = \int_S r^2 dm \quad (1)$$

When the mass in  $S$  is homogeneous and of unit density, (1) reduces to  $I_l = \int_S r^2 dS$ , which is called the moment of inertia of  $S$  itself about  $l$ .

When  $S$  is in a plane  $E$  and  $l$  is the perpendicular to  $E$  through a point  $O$  of  $E$ , we may replace the symbol  $I_l$  by  $I_O$  and call it the moment of inertia of the  $m$  in  $S$  about  $O$ .

**EXAMPLE 1.** To find the moment of inertia of a rod of length  $a$  about an axis perpendicular to the rod and through one of its end points, take the end point as origin and the line of the rod as  $Ox$ ; we then have

$$I_O = \int_0^a x^2 dx = a^3/3 = (a^2/3)m$$

**EXAMPLE 2.** For the space  $S$  between the parabola  $y^2 = 4x$  and the line  $y = x$ , find the moment of inertia about (1)  $Ox$ , (2)  $Oy$ , (3)  $O$ .

$$I_x = \int_S y^2 dS = \int_0^4 \int_x^{2\sqrt{x}} y^2 dy dx = 64/5$$

$$I_y = \int_S x^2 dS = \int_0^4 \int_x^{2\sqrt{x}} x^2 dy dx = 64/7$$

$$I_O = \int_S (x^2 + y^2) dS = \int_S x^2 dS + \int_S y^2 dS = I_y + I_x = 768/35$$

**EXAMPLE 3.** The lengths of the edges  $OA$ ,  $OB$ ,  $OC$  of a rectangular parallelopiped are  $a$ ,  $b$ ,  $c$ ; prove that its moment of inertia about the edge  $OC$  is

$$I_{OC} = \int_0^a \int_0^b \int_0^c (x^2 + y^2) dz dy dx = abc \frac{a^2 + b^2}{3}$$

**EXAMPLE 4.** Show that the moment of inertia of the circumference of a circle of radius  $a$  about its center  $O$  is  $a^2m$  ( $m = 2\pi a$ ). Also, by taking rectangular axes  $Ox$ ,  $Oy$ , and using the relation  $I_O = I_x + I_y$  (Ex. 2), show that its moment of inertia about a diameter is  $(a^2/2)m$ .

**160. Radius of gyration.** Let  $k^2$  denote the ratio  $I_l/m$ ; then

$$I_l = k^2 m \quad (2)$$

Hence  $k$  denotes the distance from  $l$  at which a particle of mass  $m$  has the same moment of inertia that the mass  $m$  in  $S$  has;  $k$  is called the *radius of gyration* of the mass  $m$  in  $S$  about  $l$ .

Thus, in § 159, Ex. 1, we have  $k^2 = a^2/3$ , and in Ex. 3 we have  $k^2 = (a^2 + b^2)/3$ .

From (1) and (2), we have  $k^2 m = \int_S r^2 dm$ . Hence, § 155,  $k^2$  is the mean value of  $r^2$  with respect to  $m$  for the points of  $S$ . But from this it follows that the formula  $I_l = \int_S r^2 dm$  is true when  $dm$  stands for the mass of a part of  $S$  of any type  $\Delta S$ , and  $r$  for the radius of gyration of this  $dm$ , § 155, 3.

**EXAMPLE 1.** Thus, to find the moment of inertia of a circular plate  $S$  of radius  $a$  about its center  $O$ , we may divide  $S$  into circular rings  $\Delta S$ . For the  $\Delta S$  of inner radius  $\rho$  and breadth  $d\rho$ , we may set  $dm = 2\pi\rho d\rho$  and  $r = \rho$  in  $\int_S r^2 dm$ , which gives

$$I_O = \int_0^a 2\pi\rho^3 d\rho = \frac{\pi a^4}{2} = \frac{a^2}{2} \pi a^2 = \frac{a^2}{2} m$$

The moment of inertia of the plate about a diameter is  $(a^2/4)m$ .

**EXAMPLE 2.** For the surface  $S = abBA$  in Fig. 105, show that

$$1. I_x = \int_a^b \frac{y^2}{3} \cdot y dx \quad 2. I_y = \int_a^b x^2 \cdot y dx$$

For the solid  $V_x$  got by revolving  $S$  about  $Ox$ , that

$$3. I_x = \int_a^b \frac{y^2}{2} \pi y^2 dx$$

For the solid  $V_y$  got by revolving  $S$  about  $Oy$ , that

$$4. I_y = \int_a^b x^2 \cdot 2\pi xy dx$$

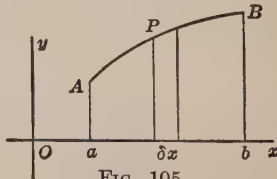


FIG. 105.

**EXAMPLE 3.** Apply the formulas 1.-4. of Ex. 2. to the  $S$  bounded by  $Ox$ ,  $y = x$ , and  $x = 2$ .

**EXAMPLE 4.** Show that the moment of inertia of a sphere of radius  $a$  about a diameter is  $(2/5)a^2m$ ,  $m = (4/3)\pi a^3$ .

**EXAMPLE 5.** For the surface of the ellipse  $x^2/a^2 + y^2/b^2 = 1$  ( $m = ab\pi$ ) show that

$$I_x = \frac{4}{3} \int_0^a y^3 dx = \frac{b^2}{4} m \quad I_y = \frac{4}{3} \int_0^b x^3 dy = \frac{a^2}{4} m \quad I_O = \frac{a^2 + b^2}{4} m$$

**EXAMPLE 6.** Referring to § 141, Ex. 1, show that the moment of inertia of the ellipsoid  $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$  about the axis  $Oz$  is

$$I_z = 2 \int_0^c \frac{a^2 + b^2}{4} ab\pi \left(1 - \frac{z^2}{c^2}\right)^2 dz = \frac{a^2 + b^2}{5} m, \quad \left(m = \frac{4}{3} \pi abc\right)$$

**161. Theorem.** Let  $m$  be a mass distributed through a space  $S$ ,  $l$  any given axis,  $l'$  the axis parallel to  $l$  through the mass center of  $S$ , and  $a$  the distance between  $l$  and  $l'$ ; then

$$I_l = I_{l'} + a^2 m \quad (\because k_l^2 = k_{l'}^2 + a^2) \quad (3)$$

For take the mass center of  $S$  as origin,  $l'$  as  $Oz$ , and the perpendicular to  $l$  through  $O$  as  $Ox$ . Let  $P(x, y, z)$  be any point of  $S$ , and let  $r$  and  $r'$  denote the distances  $PD$  and  $PE$  of  $P$  from  $l$  and  $l'$ .

$$PD^2 = (x - a)^2 + y^2$$

$$PE^2 = x^2 + y^2$$

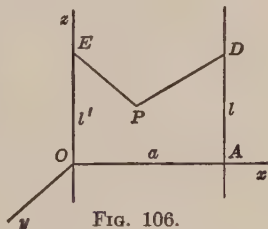
$$\therefore r^2 = (x^2 + y^2) + a^2 - 2ax = r'^2 + a^2 - 2ax$$

$$\therefore \int_S r^2 dm = \int_S r'^2 dm + a^2 m - 2a \int_S x dm$$

This is the formula (3); for,  $O$  being the mass center,  $\int_S x dm = 0$ , § 158 (1).

**EXAMPLE 1.** Find the moment of inertia of a circle of radius  $a$  about a tangent  $l$ . Let  $l'$  be the diameter parallel to  $l$ ; then  $I_l = I_{l'} + a^2 m$   
 $= \frac{a^2}{4} m + a^2 m = \frac{5}{4} a^2 m$ .

**EXAMPLE 2.** Find the moment of inertia of a cylinder of radius  $a$  and length  $2b$  with respect to a line through the center and perpendicular to the axis of the cylinder.



Let  $O$  be the center and  $Ox$  the axis of the cylinder. The moment of inertia about  $Oz$  of the section of the cylinder by a plane perpendicular to  $Ox$  at the distance  $x$  from  $O$  is  $\left(\frac{a^2}{4} + x^2\right)a^2\pi$ . Therefore

$$I_z = 2 \int_0^b \left(\frac{a^2}{4} + x^2\right) a^2 \pi dx = 2 a^2 \pi b \left(\frac{b^2}{3} + \frac{a^2}{4}\right)$$

EXAMPLE 3. Show that the moment of inertia of a cone of radius  $a$  and altitude  $b$  about a line through its vertex and perpendicular to its axis is

$$I_z = \int_0^b \left[ \frac{a^2}{4b^2} x^2 + x^2 \right] \frac{a^2}{b^2} x^2 \pi dx = \frac{3}{20} m(4b^2 + a^2)$$

**162. Kinetic energy of rotation.** Let  $P$  be a particle of mass  $m$  at the distance  $r$  from an axis  $l$  and rotating about  $l$  with the angular velocity  $\omega$ . Its speed is  $v = r\omega$ , § 88; hence its kinetic energy, namely  $E = mv^2/2$ , is

$$E = \frac{1}{2} mr^2 \omega^2 = \frac{1}{2} I_l \omega^2$$

Suppose that the mass  $m$  is distributed through a body  $S$  which is rotating about  $l$  with the angular velocity  $\omega$ . As in § 159, divide  $S$  into parts  $\delta S$  of mass  $\delta m$ . The kinetic energy of  $m$  is the sum of the kinetic energies of the parts  $\delta m$ , and the kinetic energy of the  $\delta m$  in each  $\delta S$  is between the least and greatest values of  $r^2 \delta m \cdot \omega^2/2$  for the points  $P$  of that  $\delta S$ . Hence the kinetic energy of the mass  $m$  in  $S$  is

$$E = \frac{1}{2} I_l \omega^2, \text{ where } I_l = \int_S r^2 dm \quad (4)$$

EXAMPLE 1. For a thin homogeneous rod of length  $a$  and mass  $m$  which is rotating about an end point with the angular velocity  $\omega$ , we have  $E = \frac{1}{2} \frac{a^2}{3} m \omega^2$ .

When the mass of the body  $S$  is expressed in pounds, and  $r$  and  $\omega$  in feet and radians per second, the formula (4) gives  $E$  in "foot-pounds." If the kinetic energy be required in "foot-pounds," this result must be divided by  $g = 32$ , the acceleration due to gravity.

EXAMPLE 2. For a homogeneous solid wheel of 4 ft. radius, weighing 1000 lbs., and making 30 revolutions per minute, we have

$$E = \frac{1}{2} \frac{4^2}{2} \frac{1000}{32} \left(\frac{30}{60} 2\pi\right)^2 = 1233 \text{ ft. lbs.}$$

## EXERCISE XXXV

1. Find  $I_x$ ,  $I_y$ , and  $I_O$  for the space  $S$  bounded by the parabolas  $y = 2x^2$ ,  $y^2 = 4x$ .
2. Find  $I_z$  for the  $S$  bounded by the paraboloid  $z = x^2 + 2y^2$  and the planes  $z = 0$ ,  $x = 0$ ,  $x = 2$ ,  $y = 0$ ,  $y = 1$ .
3. Find  $I_x$ ,  $I_y$ ,  $I_O$  for the triangle  $y = x$ ,  $y = 2x$ ,  $x + y - 6 = 0$ .
4. For the solid got by revolving the space bounded  $y^2 = 4x$  and  $x = 3$  about  $Ox$  find  $I_x$ ,  $I_y$ .
5. Find the moment of inertia of a cylinder of radius  $a$  and altitude  $b$  about its axis.
6. Show that the moment of inertia of a cone of radius  $a$  and altitude  $b$  about its axis is  $(3/10)ma^2$ .
7. Find the moment of inertia of the perimeter of a square of side  $a$  about (1) a side, (2) the center, (3) a diagonal.
8. Show that the moment of inertia of a rectangle, of sides  $a$ ,  $b$ , about a corner is  $m \frac{a^2 + b^2}{3}$ ; about its center,  $m \frac{a^2 + b^2}{12}$ .
9. From Ex. 8 deduce that the moment of inertia of a rectangular parallelepiped of edges  $a$ ,  $b$ ,  $c$  about the edge  $c$  is  $m \frac{a^2 + b^2}{3}$ .
10. Show that the moment of inertia of a parallelogram of sides  $a$ ,  $b$  and angle  $\theta$  about the side  $a$  is  $(mb^2 \sin^2 \theta)/3$ .
11. Show that the moment of inertia of a triangle of base  $b$  and altitude  $h$  about the base is  $mh^2/6$ ; about the line through the mass center parallel to the base,  $mh^2/18$ .
12. A right pyramid has a square base of side  $a$ , and its altitude  $AD = h$ . Show that its moment of inertia about  $AD$  is  $ma^2/10$ .
13. For the surface got by revolving the arc of  $y = f(x)$  between  $x = a$  and  $x = b$  about  $Ox$  show that  $I_x = 2\pi \int_{x=a}^{x=b} y^3 ds$ .
14. Show that the moment of inertia of a spherical surface of radius  $a$  about a diameter is  $\frac{2}{3}ma^2$ .
15. Find the moment of inertia of a solid sphere of radius  $a$  about a diameter by dividing it into spherical shells.
16. For the torus got by revolving the circle  $x^2 + (y - b)^2 = a^2$ ,  $b > a$ , about  $Ox$ , Fig. 98, p. 163, show that  $I_x = m(b^2 + \frac{3}{4}a^2)$ .
17. For the surface of the torus in Ex. 16 show that  $I_x = m(b^2 + \frac{3}{2}a^2)$ .



18. For the area bounded by the curve  $\rho = f(\theta)$  and the rays  $\theta = \alpha$ ,  $\theta = \beta$ , show that  $I_O = \iint_S \rho^2 d\theta d\rho = \int_\alpha^\beta \frac{\rho^4}{4} d\theta$ .

19. Find  $I_O$  for the area of the cardioid  $\rho = a(1 - \cos \theta)$ .

20. Find the moment of inertia of a rod of length  $a$  about an end point when the density varies as the distance from this end point.

21. Find the moment of inertia of a sphere of radius  $a$  about a diameter when,  $\rho$  denoting the distance of a point from the center, the density varies as  $a - \rho$ .

22. The rim of a wheel weighs 500 pounds. It has 8 spokes, each 3 feet long and weighing 100 pounds. The wheel is making 40 revolutions per minute. Find its kinetic energy in foot-pounds.

23. Show that the moment of inertia of a hollow sphere of outer radius  $a$  and inner radius  $b$  with respect to a diameter is

$$\frac{2}{5} m(a^5 - b^5)/(a^3 - b^3).$$

## XIX. ATTRACTION OF GRAVITATION

**163. Attraction of gravitation.** By the law of gravitation, a particle  $P$  of mass  $m$  attracts a particle  $P'$  of mass  $m'$  at the distance  $r$  from  $P$  with a force of intensity

$$F = k \frac{mm'}{r^2} \quad (1)$$

where  $k$  denotes a constant which can be found by experiment.

*A particle of mass  $m'$  at the origin  $O$  is attracted by a mass  $m$  distributed through a space  $S$  not containing  $O$  with a force  $F$  whose  $x$ -,  $y$ -,  $z$ -components are*

$$F_x = km' \int_S \frac{x}{r^3} dm \quad F_y = km' \int_S \frac{y}{r^3} dm \quad F_z = km' \int_S \frac{z}{r^3} dm \quad (2)$$

For divide  $S$  into parts, as in § 154. Let  $\delta S$  denote any one of these parts,  $\delta m$  its mass,  $P$  any point in  $\delta S$ , and  $r$  the length of  $OP$ . A particle of mass  $\delta m$  at  $P$  would attract the mass  $m'$  at  $O$  with the force  $km'\delta m/r^2$  in the direction  $OP$ ; and the  $x$ -component of this force, got by multiplying it by  $\cos(OP, x)$ , or  $x/r$ , is  $km'(x/r^3)\delta m$ . By the axioms of mechanics, the  $x$ -component of the attraction due to the  $\delta m$  distributed through  $\delta S$  is between the least and greatest values of this product  $km'(x/r^3)\delta m$  for the points  $P$  of  $\delta S$ ; and  $F_x$  is the sum of these  $x$ -components for all the parts  $\delta S$ . Hence  $F_x = \lim \Sigma km'(x/r^3)\delta m = km' \int_S (x/r^3) dm$ . Similarly for  $F_y$  and  $F_z$ .

The intensity of  $F$  is  $[F_x^2 + F_y^2 + F_z^2]^{1/2}$ ; its direction is from  $O$  to  $S$  on the line  $\frac{x}{F_x} = \frac{y}{F_y} = \frac{z}{F_z}$ .

EXAMPLE 1. With what force is a mass  $m'$  at  $O$  attracted by a homogeneous rod of density  $\mu$  which extends from  $x = a$  to  $x = b$  on  $Ox$ ?

$$F = F_x = km' \int_a^b \frac{\mu dx}{x^2} = km' \mu \left( \frac{1}{a} - \frac{1}{b} \right) = km' \frac{\mu(b-a)}{ab} = k \frac{m'm}{ab}.$$

EXAMPLE 2. Find the  $x$ - and  $y$ -components of the force with which a particle of mass  $m'$  at  $O$  is attracted by a homogeneous rod of length  $l$  and density  $\mu$  which stands on  $Ox$  at  $x = c$ . Also the direction of this force when  $l = 3$  and  $c = 4$ .

$$F_x = km' \mu \int_0^l \frac{c dy}{(y^2 + c^2)^{3/2}} = k \frac{m' \mu}{c} \left[ \frac{y}{(y^2 + c^2)^{1/2}} \right]_0^l = k \frac{m' \mu l}{c(l^2 + c^2)^{1/2}}$$

$$F_y = km' \mu \int_0^l \frac{y dy}{(y^2 + c^2)^{3/2}} = km' \mu \left[ \frac{1}{c} - \frac{1}{(l^2 + c^2)^{1/2}} \right]$$

When  $l = 3$  and  $c = 4$ , we have  $(l^2 + c^2)^{1/2} = 5 \quad \therefore F_x = km' \mu (3/20)$ ,  
 $F_y = km' \mu (1/20) \quad \therefore F_y/F_x = 1/3 \quad \therefore$  the direction of  $F$  makes the angle  $\tan^{-1} (1/3)$  with  $Ox$ , and  $F = km' \mu \sqrt{10}/20$ .

EXAMPLE 3. Show that the attraction of a homogeneous spherical surface  $S$  on a particle at a point outside of  $S$  is the same that it would be were the mass of  $S$  concentrated at its center. Show also that this theorem holds good for a homogeneous solid sphere.

1. Let the particle, of mass  $m'$ , be at  $O$ , let  $\mu$  be the density of  $S$ , and let  $OC = c$  and  $AC = a$ . Since  $S$  is symmetric to  $Ox$ , we have  $F_y = 0$ ,  $F_z = 0$ .

By planes perpendicular to  $Ox$  divide  $S$  into zones  $\Delta S$  of altitude  $\delta x$ ;  $\Delta S = 2\pi a \delta x$  (p. 163, Ex. 2).

We have  $OP = OQ$ , and the equation of the circle  $AQDB$  being  $(x-c)^2 + z^2 = a^2$ , we have  $OQ^2 = x^2 + z^2 = 2cx - (c^2 - a^2)$ .

Hence show that

$$F = F_x = km' 2\pi a \mu \int_{c-a}^{c+a} \frac{x dx}{[2cx - (c^2 - a^2)]^{3/2}} = km' \mu \frac{4a^2\pi}{c^2} = k \frac{m'm}{c^2}$$

2. Dividing the solid sphere of density  $\mu$  bounded by  $S$  into shells of thickness  $\delta \rho$ , we find

$$F = F_x = km' \mu \int_0^a \frac{4\pi \rho^2}{c^2} d\rho = \frac{4}{3} \frac{\pi a^3}{c^2} km' \mu = k \frac{m'm}{c^2}$$

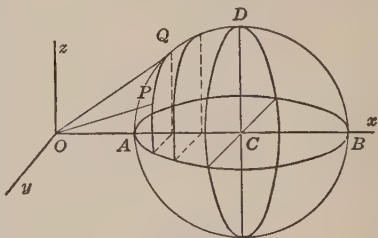


FIG. 107.

EXAMPLE 4. Show that the attraction of a homogeneous spherical surface on a point inside the surface is 0.

EXAMPLE 5. Show that each of two homogeneous solid spheres attracts the other with the force  $k mm'/r^2$  where  $m$ ,  $m'$  are their masses and  $r$  the distance between their centers.

EXAMPLE 6. Find the attraction of a homogeneous hemispherical shell on a particle situated at the center of the sphere.

EXAMPLE 7. Find the attraction of a homogeneous circular plate on a particle situated on the line through its center and perpendicular to its plane.

EXAMPLE 8. Find the attraction of a homogeneous cylinder on a particle situated on its axis produced.

EXAMPLE 9. Let  $A$  be one of the points where a circle with center  $O$  and radius  $c$  cuts  $Ox$ , let  $B$  and  $C$  be any two points on the tangent to the circle at  $A$ , and let  $B'$  and  $C'$  be the points where the circle is cut by the lines  $OB$  and  $OC$ . Regarding the line segment  $BC$  and the arc  $B'C'$  as homogeneous masses of unit density, show that their attractions on a particle of mass  $m'$  at  $O$  are the same in both magnitude and direction and therefore that the direction is along the bisector of the angle  $BOC$ . Derive the results in Ex. 2 from this theorem.

EXAMPLE 10. Given any homogeneous rod  $AB$  and any point  $C$  not on the line of  $AB$ , find the direction of the force  $F$  with which  $AB$  attracts a particle at  $C$ .

## XX. INFINITE SERIES

**164. Infinite series.** An expression of the form

$$u_1 + u_2 + \cdots + u_n + \cdots \quad (1)$$

where  $u_1, u_2, \cdots, u_n, \cdots$  is any given never-ending sequence of numbers, is called an *infinite series*. Thus

$$1/2 + 1/2^2 + 1/2^3 + \cdots.$$

When we say that  $u_1, u_2, \dots$  is *given*, we mean that for every assigned value of  $n$  the value of  $u_n$  can be found. In the series that we shall meet, a formula for  $u_n$  in terms of  $n$  is given or indicated by the first few terms. Thus the series for which  $u_n = n/(n^2 + 1)$  is  $1/2 + 2/5 + \cdots$ ; and for the series  $1/(2 \cdot 3) + 2/(3 \cdot 4) + \cdots$  we have  $u_n = n/(n + 1)(n + 2)$ .

The series (1) is often represented by the symbol  $\Sigma u_n$ .

**165. Convergence and divergence.** Let  $S_n$  denote the sum of the first  $n$  terms of (1), so that  $S_1 = u_1, S_2 = u_1 + u_2, \cdots, S_n = u_1 + u_2 + \cdots + u_n$ . If, when  $n \rightarrow \infty$ ,  $S_n$  approaches a finite limit, (1) is said to be *convergent*, and  $\lim S_n$  is called its *sum*; but if  $S_n$  does not approach a finite limit, (1) is said to be *divergent*.

Thus, for the series  $1/2 + 1/4 + 1/8 + \cdots$  the successive values of  $S_n$  are  $1/2, 3/4, 7/8, \cdots$ , and  $S_n \rightarrow 1$ ; hence this series is convergent and its sum is 1.

On the other hand, the series  $1 + 1 + 1 + \cdots$  and  $1 - 1 + 1 - \cdots$  are divergent.

If the series (1) is convergent and  $S$  denotes its sum, we may write

$$S = S_n + R_n \quad (2)$$

where  $R_n$  denotes the sum of the part of (1) which follows the term  $u_n$ . Since  $\lim S_n = S$ , we have  $\lim R_n = 0$ .

Evidently  $S_n$  cannot approach a finite limit unless  $u_n \rightarrow 0$ . But  $\lim u_n = 0$  does not ensure convergence. Thus in

$1 + 1/2 + 1/3 + \dots$ , we have  $u_n = 1/n \rightarrow 0$ ; but this series is divergent, § 171.

If  $u_1 + u_2 + \dots$  (1) is convergent and has the sum  $S$ , then  $cu_1 + cu_2 + \dots$  (2) is convergent and has the sum  $cS$ ; for  $cu_1 + \dots + cu_n = cS_n \rightarrow cS$ . If (1) is divergent, so is (2).

**166. Positive series.** We call  $u_1 + u_2 + \dots$  a *positive series* when all its terms are positive. For such series we have the following theorems:

**167. Theorem 1.** *A positive series  $u_1 + u_2 + \dots$  is convergent if, as  $n$  increases,  $S_n$  remains always less than some finite number  $c$ .*

For since the series is positive,  $S_n$  continually increases as  $n$  increases. But it remains less than  $c$ . Hence it approaches a limit ( $\leq c$ ), § 5.

**168. Theorem 2.** *A positive series  $u_1 + u_2 + \dots$  is convergent if its terms are less than the corresponding terms of a positive series  $a_1 + a_2 + \dots$  known to be convergent.*

For if the sum of  $a_1 + a_2 + \dots$  be  $A$ , we have  $S_n = (u_1 + u_2 + \dots + u_n) < A$ . Hence  $u_1 + u_2 + \dots$  is convergent, by Theorem 1. Similarly

**169. Theorem 3.** *A positive series  $u_1 + u_2 + \dots$  is divergent if its terms are greater than the corresponding terms of a positive series  $b_1 + b_2 + \dots$  known to be divergent.*

In the next two sections we consider the *test series* most frequently used in applying Theorems 2. and 3.

**170. The geometric series.** *The infinite geometric series  $a + ar + ar^2 + \dots$  is convergent when  $|r| < 1$ .*

For  $S_n = a \frac{1 - r^n}{1 - r}$ , and since  $|r| < 1$ ,  $\lim r^n = 0$ .

Hence 
$$\lim S_n = \frac{a}{1 - r}$$

Evidently  $a + ar + ar^2 + \dots$  is divergent when  $|r| \geq 1$ .

**171. The series  $\Sigma 1/n^p$ .** The series  $1 + 1/2^p + 1/3^p + \dots$  is convergent when  $p > 1$ , divergent when  $p \leq 1$ .

1.  $p > 1$ . Combining the two terms beginning at  $1/2^p$ , the four beginning at  $1/4^p$ , and so on, we obtain

$$1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \dots \quad (1)$$

whose terms, after the first, are less than those of

$$\begin{aligned} &1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \dots \\ &= 1 + \frac{2}{2^p} + \frac{4}{4^p} + \dots = 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \dots \end{aligned} \quad (2)$$

But since  $p > 1$  and therefore  $1/2^{p-1} < 1$ , the geometric series (2) is convergent. Hence (1) is convergent by Theorem 2.

2.  $p = 1$ . Combining the two terms ending at  $1/4$ , the four ending at  $1/8$ , and so on, we obtain

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots \quad (3)$$

whose terms after the second are greater than those of

$$1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots = 1 + \frac{1}{2} + \frac{1}{2} + \dots \quad (4)$$

But (4) is divergent. Hence (3) is divergent by Theorem 3.

3.  $p < 1$ . In this case  $1 + 1/2^p + 1/3^p + \dots$  is divergent since its terms are greater than those of the divergent series  $1 + 1/2 + 1/3 + \dots$ .

Thus the series  $1 + 1/2^2 + 1/3^2 + \dots$  is convergent, but the series  $1 + 1/\sqrt{2} + 1/\sqrt{3} + \dots$  is divergent.

**172. Applications of these test series.** Consider the following examples:

**EXAMPLE 1.** The series  $x^2/(1+x^2) + x^4/(1+x^4) + \dots$  converges when  $|x| < 1$  since its terms are less than those of the geometric series  $x^2 + x^4 + x^6 + \dots$ , which converges when  $|x| < 1$ .

**EXAMPLE 2.** The series  $1/(2 \cdot 3) + 1/(3 \cdot 4) + \dots$  is convergent since its terms are less than those of the convergent series  $1/2^2 + 1/3^2 + \dots$ .

**EXAMPLE 3.** Is the series for which  $u_n = (3n+2)/(n^3+1)$  convergent or divergent?

$$\text{Here } u_n = \frac{3n+2}{n^3+1} = \frac{1}{n^2} \frac{3+2/n}{1+1/n^3}; \text{ and } \frac{3+2/n}{1+1/n^3} \rightarrow 3 \text{ when } n \rightarrow \infty.$$

Hence after a certain term all terms of  $\Sigma u_n$  are less than the corresponding terms of the series  $\Sigma k/n^2$  where  $k$  is any assigned number  $> 3$ . Hence  $\Sigma u_n$  is convergent.



By the reasoning in Ex. 3 it can be shown that if  $u_n = f(n)/\phi(n)$ , where  $f(n)$  and  $\phi(n)$  are integral expressions in  $n$ , the series  $\Sigma u_n$  is convergent when the degree of  $\phi(n)$  exceeds that of  $f(n)$  by more than 1, divergent when this is not the case.

**EXAMPLE 4.** In the following find  $u_n$  and test for convergence or divergence:

$$1. \frac{1}{1} + \frac{1}{\sqrt{2^3}} + \frac{1}{\sqrt{3^3}} + \dots \quad 2. \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots$$

$$3. \frac{2}{1 \cdot 3} + \frac{4}{3 \cdot 5} + \frac{6}{5 \cdot 7} + \dots \quad 4. \frac{2}{1 \cdot 2 \cdot 3} + \frac{4}{2 \cdot 3 \cdot 4} + \frac{6}{3 \cdot 4 \cdot 5} + \dots$$

$$5. \frac{1}{a} + \frac{1}{2a+b} + \frac{1}{3a+b} + \dots$$

$$6. \frac{2}{1+2\sqrt{2}} + \frac{3}{1+3\sqrt{3}} + \frac{4}{1+4\sqrt{4}} + \dots$$

**EXAMPLE 5.** Is the series for which  $u_n = (n^2 + n)^{1/2}/(n^4 + 1)^{1/3}$  convergent or divergent?

**EXAMPLE 6.** The curve in Fig. 108 is the graph (for  $x > 0$ ) of a positive decreasing function  $y = f(x)$  which  $\rightarrow 0$  when  $x \rightarrow \infty$ . The series  $f(2) + f(3) + f(4) + \dots$  is represented by the sum of the rectangles  $2A_1, 3A_2, \dots$  and therefore converges if  $\int_1^x f(x)dx \rightarrow$  a finite limit  $l$  when  $x \rightarrow \infty$ . The series  $f(1) + f(2) + f(3) + \dots$  is represented by the sum of rectangles  $1B_2, 2B_3, \dots$  and therefore diverges if  $\int_1^x f(x)dx \rightarrow \infty$  when  $x \rightarrow \infty$ . Fill in the details of this proof and show that the theorem of § 171 follows from it.

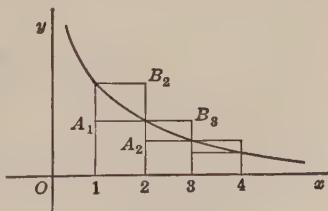


FIG. 108.

**173. Theorem 4.** A positive series  $u_1 + u_2 + \dots$  is convergent if the ratio of each of its terms to the immediately preceding term is less than some number  $r$  which itself is less than 1.

By hypothesis,  $\frac{u_2}{u_1} < r$ ,  $\frac{u_3}{u_2} < r$ ,  $\frac{u_4}{u_3} < r$ ,  $\dots$ ,  $\frac{u_{n+1}}{u_n} < r$ ,  $\dots$

Multiplying the first of these inequalities by the second, the result by the third, and so on, and then clearing of fractions, we get

$$u_2 < u_1 r, u_3 < u_1 r^2, u_4 < u_1 r^3, \dots u_{n+1} < u_1 r^n, \dots$$

Hence the terms of  $u_1 + u_2 + \dots$  after the first are less than the corresponding terms of the convergent series  $u_1 + u_1 r + u_1 r^2 + \dots$ . It is therefore convergent.

If  $u_{n+1}/u_n \geq 1$ , the series  $u_1 + u_2 + \dots$  is divergent.

**174. Corollary.** *If, as  $n \rightarrow \infty$ , the ratio  $u_{n+1}/u_n$  approaches a limit  $l$ , the series  $u_1 + u_2 + \dots$  is convergent when  $l < 1$ , divergent when  $l > 1$ .*

For if  $l < 1$ , and  $r$  denote any number between  $l$  and 1, then after  $n$  reaches a certain value  $n'$  the ratio  $u_{n+1}/u_n$  will remain between  $l$  and  $r$ , that is, less than  $r$ ; hence the series is convergent, § 173.

If  $l > 1$ , then after a certain value  $n'$  of  $n$  the ratio  $u_{n+1}/u_n$  will remain  $> 1$ ; hence the series is divergent.

**EXAMPLE 1.** Prove that  $\frac{3}{4} + \frac{3 \cdot 5}{4 \cdot 7} + \dots + \frac{3 \cdot 5 \dots (2n+1)}{4 \cdot 7 \dots (3n+1)} + \dots$  is convergent.

$$\text{Here } \lim \frac{u_{n+1}}{u_n} = \lim \frac{2n+1}{3n+1} = \lim \frac{2 + 1/n}{3 + 1/n} = \frac{2}{3}, \text{ which is } < 1.$$

A series in which  $\lim u_{n+1}/u_n = 1$  may be convergent or divergent. It can be proved of such series that if  $u_{n+1}/u_n$  is reducible to the form

$$u_{n+1}/u_n = (n^p + an^{p-1} + \dots)/(n^p + a'n^{p-1} + \dots) \quad (1)$$

the series is convergent when  $a' - a > 1$ , divergent when  $a' - a \leq 1$ .

$$\text{Thus in } \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \dots, \quad \frac{u_{n+1}}{u_n} = \frac{2n+1}{2n+2} = \frac{n+1/2}{n+1}$$

Here  $a' - a = 1 - 1/2 = 1/2$ ; hence the series is divergent.

**EXAMPLE 2.** Test the following series for convergence or divergence.

1.  $\frac{1 \cdot 2}{1 \cdot 3} + \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 5} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 3 \cdot 5 \cdot 7} + \dots$
2.  $\frac{3}{4} + 2(\frac{3}{4})^2 + 3(\frac{3}{4})^3 + \dots$
3.  $\frac{4}{3} + \frac{4 \cdot 8}{3 \cdot 8} + \frac{4 \cdot 8 \cdot 12}{3 \cdot 8 \cdot 13} + \dots$
4.  $\frac{3}{2} + \frac{3 \cdot 7}{2 \cdot 9} + \frac{3 \cdot 7 \cdot 11}{2 \cdot 9 \cdot 16} + \dots$
5.  $\frac{2}{3} + \frac{2 \cdot 3}{3 \cdot 4} + \frac{2 \cdot 3 \cdot 4}{3 \cdot 4 \cdot 5} + \dots$
6.  $\frac{2}{5} + \frac{2 \cdot 4}{5 \cdot 8} + \frac{2 \cdot 4 \cdot 6}{5 \cdot 8 \cdot 11} + \dots$

**EXAMPLE 3.** Show that  $\frac{a}{1} + \frac{a(a+1)}{1 \cdot 2} + \frac{a(a+1)(a+2)}{1 \cdot 2 \cdot 3} + \dots$  is divergent if  $a > 0$ .

**EXAMPLE 4.** Show by § 174 (1) that  $\sum 1/n$  is divergent and  $\sum 1/n^2$  convergent.

**EXAMPLE 5.** Show that  $u_1 + u_2 + \dots$  is convergent if  $\sqrt[n]{u_n} < r < 1$ .

**175. Series with positive and negative terms.** The most important theorem respecting the convergence of such series is the following:

**176. Theorem 5.** *A series which has both positive and negative terms is convergent if the corresponding positive series is convergent.*

This is evident when the number of positive or of negative terms is finite; we therefore suppose that there are infinitely many terms of both kinds.

Let  $u_1 + u_2 + \dots$  (1) be the given series, and let  $u'_1 + u'_2 + \dots$  (2) denote (1) with the signs of all its negative terms changed. Also let  $S_n$  and  $S'_n$  be the sums of the first  $n$  terms in (1) and (2).

If there are  $p$  positive and  $q$  negative terms in  $S_n$ , and if  $P_p$  and  $-N_q$  denote the sums of these terms, we have

$$S_n = P_p - N_q \qquad S'_n = P_p + N_q$$

By hypothesis,  $S'_n$  approaches the finite limit  $S'$  when  $n \rightarrow \infty$ . Hence  $P_p$  and  $N_q$ , both of which increase with  $n$  but remain less than  $S'$ , also approach limits  $P$  and  $N$ , § 5, and therefore  $S_n = P_p - N_q$  approaches the limit  $P - N$ . Hence  $u_1 + u_2 + \dots$  is convergent.

**177. Absolute and conditional convergence.** The converse of Theorem 5 is not true. For  $S_n = P_p - N_q$  may also approach a finite limit when both  $P_p$  and  $N_q$  become infinite; then  $\Sigma u_n$  is convergent, but  $\Sigma u'_n$  divergent.

A convergent series is said to be *absolutely convergent* or *conditionally convergent* according as it remains convergent or becomes divergent when the signs of its negative terms, if any, are changed.

Thus in the next section it is shown that the series  $1 - 1/2 + 1/3 - 1/4 + \dots$  is convergent. But it is only conditionally convergent since the series  $1 + 1/2 + 1/3 + \dots$  is divergent, § 171.

**178. Theorem 6.** *An alternating series, that is, one whose terms are alternately positive and negative, is convergent if each term is numerically less than the one which precedes it and if the limit of the  $n$ th term is 0.*

Let  $a_1 - a_2 + a_3 - \dots$  be the series,  $a_1, a_2, a_3, \dots$ , being positive. When  $n$  is even, we can group the terms in  $S_n$  in the two ways:

$$S_n = (a_1 - a_2) + (a_3 - a_4) + \dots + (a_{n-1} - a_n) \quad (1)$$

$$= a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (a_{n-2} - a_{n-1}) - a_n \quad (2)$$

Since  $a_1 > a_2 > a_3 > \dots$ , the bracketed expressions in (1) and (2) are all positive. Hence when  $n$  increases through even values,  $S_n$  increases, by (1), but remains less than  $a_1$ , by (2); it therefore approaches a limit  $l$  (§ 5). But  $S_{n+1}$  also approaches this limit  $l$  since  $S_{n+1} = S_n + a_{n+1}$  and  $a_{n+1} \rightarrow 0$ . Hence  $S_n \rightarrow l$  when  $n \rightarrow \infty$  through the sequence 1, 2, 3,  $\dots$ , and the series is convergent.

The expressions (1) and (2) also show that the sums  $S_1 = a_1$ ,  $S_2 = a_1 - a_2$ ,  $S_3 = a_1 - a_2 + a_3$ , and so on, are alternately greater and less than the sum  $S$  of the series, and therefore that  $S_n$  differs from  $S$  by less than  $a_{n+1}$ .

Thus the sum of the geometric series  $1 - (.1)^2 + (.1)^4 - (.1)^6 + \dots$  is 100/101, § 170; and  $S_3$  exceeds 100/101 by less than  $(.1)^6$  or .000001.

## EXERCISE XXXVI

1. Which of the following series are divergent, conditionally convergent, absolutely convergent?

$$1. \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \dots$$

$$3. \frac{2}{3} - (\frac{2}{3})^2 + (\frac{2}{3})^3 - \dots$$

$$2. \frac{1}{\sqrt{2}} - \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[4]{2}} - \dots$$

$$4. \frac{1}{1-c} + \frac{1}{1-2c} + \frac{1}{1-3c} + \dots$$

2. Show that the sum  $S$  of  $1 - 1/3! + 1/5! - 1/7! + \dots$  lies between 1 and  $5/6$ . To what decimal figure will  $S_3$  represent  $S$ ?

3. Add enough terms of  $\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \dots$  to find  $S$  to the third decimal figure.

4. Show that the remainder after  $n$  terms in  $1 + x + x^2 + \dots$  is  $R_n = x^n/(1 - x)$ . How great must  $n$  be to make  $R_n < .1$  when  $x = .9$ ? when  $x = -.9$ ?

## POWER SERIES

**179. Power series in  $x$ .** This name is given to a series of the form

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \quad (1)$$

where  $x$  is a variable and  $a_0, a_1, \dots$  are constants.

**Theorem.** *If in the series  $a_0 + a_1x + \dots$  the ratio  $|a_n/a_{n+1}|$  approaches a limit  $l$ , the series is absolutely convergent when  $|x| < l$ , divergent when  $|x| > l$ .*

For, by § 174, the positive series  $|a_0| + |a_1x| + \dots$  converges when

$$\lim \left| \frac{a_{n+1}x^{n+1}}{a_nx^n} \right| < 1, \quad \text{that is, when } |x| < \lim \left| \frac{a_n}{a_{n+1}} \right|$$

Similarly,  $a_0 + a_1x + \dots$  diverges when  $|x| > \lim |a_n/a_{n+1}|$ .

We call  $l$  the *limit of convergence* of the series, and  $(-l, l)$  its *interval of convergence*. The series may be convergent or divergent when  $x = l$  or  $-l$ .<sup>1</sup>

<sup>1</sup> A series in which  $a_n/a_{n+1}$  does not approach a limit may also have a limit of convergence. Thus it is easily shown that the series  $1 + 2x + x^2 + 2x^3 + \dots$  converges when  $|x| < 1$ , diverges when  $|x| \geq 1$ .

The following important series illustrate this theorem.

1. *The exponential series.* The series

$$1 + \frac{x}{1} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \quad (2)$$

is convergent for all finite values of  $x$ .

For  $\lim \left| \frac{a_n}{a_{n+1}} \right| = \lim (n+1) = \infty$ , that is,  $l = \infty$ .

2. *The logarithmic series.* The series

$$\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-)^{n-1} \frac{x^n}{n} + \cdots \quad (3)$$

is convergent when  $|x| < 1$ , divergent when  $|x| > 1$ .

For  $\lim \left| \frac{a_n}{a_{n+1}} \right| = \lim \frac{n+1}{n} = \lim \left( 1 + \frac{1}{n} \right) = 1$ , that is,  $l = 1$ .

The series is convergent when  $x = 1$ , divergent when  $x = -1$ .

3. *The binomial series.* When  $m$  is not a positive integer,

$$1 + \frac{m}{1}x + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)\cdots(m-n+1)}{n!}x^n + \cdots \quad (4)$$

is an infinite series. It converges when  $|x| < 1$ , diverges when  $|x| > 1$ . For

$$\frac{a_n}{a_{n+1}} = \frac{m(m-1)\cdots(m-n+1)}{n!} \div \frac{m(m-1)\cdots(m-n)}{(n+1)!} = \frac{n+1}{m-n}$$

Hence  $\lim \left| \frac{a_n}{a_{n+1}} \right| = \lim \left| \frac{n+1}{m-n} \right| = \lim \left| \frac{1+1/n}{1-m/n} \right| = 1$ , or  $l = 1$ .

Similarly,  $a_0 + a_1(x-a) + \cdots$  is called a power series in  $x-a$ , and  $a_0 + a_1(1/x) + \cdots$  a power series in  $1/x$ . Evidently if  $\sum a_n x^n$  converges when  $|x| < l$ , then  $\sum a_n (x-a)^n$  converges when  $|x-a| < l$ , and  $\sum a_n (1/x^n)$  converges when  $|x| > l$ .

EXAMPLE 1. Show that the following series converge for the values of  $x$  indicated.

1.  $3x + \frac{(3x)^2}{2} + \frac{(3x)^3}{3} + \cdots$   $|x| < 1/3$
2.  $(x-2) + 2^2(x-2)^2 + 3^2(x-2)^3 + \cdots$   $1 < x < 3$
3.  $\frac{1}{3}x + \frac{1 \cdot 3}{3 \cdot 6}x^2 + \frac{1 \cdot 3 \cdot 5}{3 \cdot 6 \cdot 9}x^3 + \cdots$   $|x| < 3/2$
4.  $mx + \frac{m(m+3)}{2!}x^2 + \frac{m(m+3)(m+6)}{3!}x^3 + \cdots$   $|x| < 1/3$
5.  $\frac{1}{x} + \frac{2}{x^2} + \cdots + \frac{n}{x^n} + \cdots$   $|x| > 1$

$$6. \dots \left(\frac{x}{2}\right)^{-3} + \left(\frac{x}{2}\right)^{-2} + \left(\frac{x}{2}\right)^{-1} + 1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 + \dots, \quad 2 < |x| < 3$$

EXAMPLE 2. Show that  $1 + x + 2!x^2 + 3!x^3 \dots$  converges when  $x = 0$  only.

**180. Functions defined by power series.** For each value of  $x$  within its interval of convergence  $(-l, l)$ , the series  $a_0 + a_1x + \dots$  (1) has a definite sum; it therefore represents a function of  $x$  within  $(-l, l)$ , § 14. This function  $f(x)$  we call the sum of the series (1) and write

$$f(x) = a_0 + a_1x + a_2x^2 + \dots \quad |x| < l$$

Conversely, it is often advantageous to represent a *given* function of  $x$  by a power series in  $x$  or  $x - a$ ; but such a representation is valid within  $(-l, l)$  only.

Thus  $1/(1-x)$  may be represented by  $1 + x + x^2 + \dots$  when  $|x| < 1$ , but then only.

**181. Continuity.** The function  $f(x) = a_0 + a_1x + a_2x^2 + \dots$ ,  $|x| < l$ , is continuous at every point  $x = x_1$  such that  $|x_1| < l$ .

If  $S_n(x)$  be the sum of the first  $n$  terms of  $a_0 + a_1x + \dots$ , and  $R_n(x)$  the remainder after the  $n$ th term,

$$f(x) - f(x_1) = [S_n(x) - S_n(x_1)] + R_n(x) - R_n(x_1)$$

It is to be proved that  $\lim_{x \rightarrow x_1} [f(x) - f(x_1)] = 0$ .

Let  $c$  be any number between  $|x_1|$  and  $l$ , and  $R_n$  the remainder after  $n$  terms in the series  $|a_0| + |a_1c| + |a_2c^2| + \dots$ . This series being convergent, if any positive number  $\epsilon$  be assigned, we can find a value  $n'$  of  $n$  such that when  $n \geq n'$ , then  $R_n < \epsilon$ . But  $|R_n(x_1)| < R_n$ ; and if  $|x| < c$ , then also  $|R_n(x)| < R_n$ ; hence when  $n \geq n'$  and  $|x| < c$ , we shall have  $|R_n(x_1)|, |R_n(x)| < \epsilon$ .

Again, since  $S_n'(x)$  is a polynomial in  $x$  and is therefore continuous, we can find a positive number  $\delta$  such that, when  $|x - x_1| < \delta$ , then  $|S_n'(x) - S_n'(x_1)| < \epsilon$ .

We thus make  $|f(x) - f(x_1)| < 3\epsilon$ . We can take  $3\epsilon$  as small as we please. Hence  $\lim_{x \rightarrow x_1} [f(x) - f(x_1)] = 0$ .



**182. Integration.** If  $f(x) = a_0 + a_1x + a_2x^2 + \dots$ ,  $|x| < l$ , and  $x_1$  be such that  $|x_1| < l$ , we can find  $\int_0^{x_1} f(x)dx$  by integrating the series  $a_0 + a_1x + a_2x^2 + \dots$  term by term.

For, expressing  $f(x)$  in the form  $a_0 + a_1x + \dots + a_{n-1}x^{n-1} + R_n(x)$ , we have

$$\int_0^{x_1} f(x)dx = a_0x_1 + a_1\frac{x_1^2}{2} + \dots + a_{n-1}\frac{x_1^n}{n} + \int_0^{x_1} R_n(x)dx$$

By § 181, 1., if any positive number  $\epsilon$  be assigned we can find  $n'$  such that, when  $n \geq n'$ , then for every value of  $x$  in the interval  $(0, x_1)$  we shall have  $|R_n(x)| < \epsilon$ , and this makes  $|\int_0^{x_1} R_n(x)dx| < \epsilon|x_1|$ . Hence  $\lim_{n \rightarrow \infty} \int_0^{x_1} R_n(x)dx = 0$ , and

$$\int_0^{x_1} f(x)dx = a_0x_1 + a_1\frac{x_1^2}{2} + \dots + a_{n-1}\frac{x_1^n}{n} + \dots \quad (1)$$

**183. Differentiation.** Given  $f(x) = a_0 + a_1x + a_2x^2 + \dots$ ,  $|x| < l$ . If  $|x| < l$ , we can find  $f'(x)$  by differentiating  $a_0 + a_1x + a_2x^2 + \dots$  term by term.

For the series got by differentiating  $a_0 + a_1x + a_2x^2 + \dots$  term by term, namely  $a_1 + 2a_2x + \dots$ , also converges<sup>1</sup> when  $|x| < l$ . Hence, by § 182 (1), when  $|x| < l$ ,

$$\begin{aligned} \int_0^x [a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots] dx \\ = a_1x + a_2x^2 + \dots + a_nx^n + \dots \end{aligned} \quad (2)$$

But equating the derivative of the second member of (2) to that of the first gives

$$\begin{aligned} \frac{d}{dx} [a_1x + a_2x^2 + \dots + a_nx^n + \dots] \\ = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots \end{aligned} \quad (3)$$

Hence, for  $|x| < l$ , the derivative of  $a_1x + a_2x^2 + \dots$ , and therefore that of  $f(x) = a_0 + a_1x + a_2x^2 + \dots$ , can be got by differentiating term by term.

<sup>1</sup> See p. 220, Ex. 12. In case  $|a_n/a_{n+1}| \rightarrow l$ , the following proof suffices:

$$\text{If } \left| \frac{a_n}{a_{n+1}} \right| \rightarrow l, \text{ then } \left| \frac{na_n}{(n+1)a_{n+1}} \right| \rightarrow l \text{ since } \frac{n}{n+1} \rightarrow 1.$$

**184. The logarithmic series.** We have

$$\log(1+x) = \int_0^x \frac{dx}{1+x}$$

$$\begin{aligned} \text{Also } \int_0^x \frac{dx}{1+x} &= \int_0^x (1 - x + x^2 - \dots) dx \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots, \quad |x| < 1 \end{aligned}$$

Hence for  $|x| < 1$

$$\log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + \dots \quad (1)$$

This series is not very serviceable<sup>1</sup> for computing logarithms; but

$$\begin{aligned} \log \frac{1+x}{1-x} &= \log(1+x) - \log(1-x) \\ &= 2 \left[ \frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \dots \right] \end{aligned} \quad (2)$$

and from (2) by setting  $\frac{1+x}{1-x} = \frac{n+1}{n}$ ,  $\therefore x = \frac{1}{2n+1}$ , we get

$$\log(n+1) = \log n + 2 \left[ \frac{1}{2n+1} + \frac{1}{3} \frac{1}{(2n+1)^3} + \dots \right] \quad (3)$$

Hence, since  $\log 1 = 0$ , we have

$$\log 2 = 2 \left[ \frac{1}{3} + \frac{1}{3^3} + \frac{1}{5} \frac{1}{3^5} + \dots \right] = .6931 \dots;$$

$$\log 3 = \log 2 + 2 \left[ \frac{1}{5} + \frac{1}{3} \frac{1}{5^3} + \dots \right] = 1.0986; \text{ and so on.}$$

**EXAMPLE.** Using  $\tan^{-1} x = \int_0^x \frac{dx}{1+x^2}$ , prove that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots, \quad |x| < 1$$

**185. The binomial series.** Let  $f(x)$  denote the sum of the binomial series, § 179, 3, so that, for  $|x| < 1$ ,

$$f(x) = 1 + \frac{m}{1} x + \frac{m(m-1)}{2!} x^2 + \dots \quad (1)$$

We are to prove that  $f(x) = (1+x)^m$ .

<sup>1</sup> It is available only when  $|x| < 1$ , and even then it converges slowly.

Differentiating both members of (1), we obtain

$$f'(x) = m \left[ 1 + \frac{m-1}{1} x + \frac{(m-1)(m-2)}{2!} x^2 + \dots \right] \quad (2)$$

But if to the series in brackets we add  $x$  times this series, combining terms in like powers of  $x$ , we obtain the series (1). Hence

$$f'(x)(1+x) = mf(x)$$

We therefore have successively,

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{m}{1+x}, & \int_0^x \frac{f'(x)}{f(x)} dx &= m \int_0^x \frac{dx}{1+x}, \\ \log f(x) &= \log (1+x)^m, & f(x) &= (1+x)^m \end{aligned}$$

Hence, for any real value of  $m$ ,

$$\begin{aligned} (1+x)^m &= 1 + \frac{m}{1} x + \frac{m(m-1)}{2!} x^2 + \dots \\ &+ \frac{m(m-1) \dots (m-n+1)}{n!} x^n + \dots \quad |x| < 1 \quad (3) \end{aligned}$$

### EXERCISE XXXVII

By aid of (3), show that

$$1. \quad \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{2 \cdot 4}x^2 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \dots \quad |x| < 1$$

$$2. \quad \frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \dots \quad |x| < 1$$

$$3. \quad \frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \dots \quad |x| < 1$$

$$4. \quad \text{Using } \sin^{-1} x = \int_0^x \frac{dx}{\sqrt{1-x^2}} \text{ and the series in Ex. 3, prove that}$$

$$\sin^{-1} x = \frac{x}{1} + \frac{1}{2} \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{x^7}{7} + \dots \quad |x| < 1$$

$$5. \quad \text{Using } \sin^{-1} (1/2) = \pi/6, \text{ show that } \pi = 3.14 \dots$$

$$6. \quad \text{Show that replacing } x \text{ by } x/a \text{ in (3) gives, for } |x| < |a|$$

$$(a+x)^m = a^m + \frac{m}{1} a^{m-1}x + \dots + \frac{m(m-1) \dots (m-n+1)}{n!} a^{m-n}x^n + \dots$$

$$7. \quad \text{Compute } \sqrt{17} = (16+1)^{1/2} = 4(1+1/16)^{1/2} \text{ to the fourth decimal figure.}$$

8. By aid of (3), show that  $\int_0^1 (4 + x^3)^{1/2} dx = 2.0604$ .

9. For  $|x| > 1$ , show that

$$\sqrt{x+1} = \sqrt{x} \left[ 1 + \frac{1}{2} \cdot \frac{1}{x} - \frac{1}{2 \cdot 4} \frac{1}{x^2} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} \frac{1}{x^3} - \dots \right]$$

10. Show that the difference between corresponding ordinates of the hyperbola  $x^2 - y^2 = 1$  and its asymptote  $y = x$  is approximately  $-1/2 x$ .

11. Find to the third decimal figure the length of an arc of the parabola  $y^2 = 4x$  between  $x = 10$  and  $x = 100$ .

12. Show that when  $0 < x < 1$ , then  $\sqrt{1+x} = 1 + \frac{x}{2}$  with an error which is  $< \frac{x^2}{8}$ , and  $\sqrt{1-x} = 1 - \frac{x}{2}$  with an error which is  $< \frac{x^2}{8(1-x)}$ .

13. If the coefficients of  $f(x) = a_0 + a_1x + \dots$  from  $a_n$  on are numerically less than some positive number  $c$ , show that when  $|x| < 1$ , then  $f(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$  with an error which is numerically less than  $cx^n/(1-x)$ .

14. Compute  $\log 4$ ,  $\log 5$ ,  $\log 6$ , each to the fourth decimal figure.

15. Express the coefficients of  $x^n$  in Exs. 1. and 2. in terms of factorials, and powers of 2.

16. If  $f(x) = 1 + \frac{x}{1} + \dots + \frac{x^n}{n!} + \dots$ , show successively that

$$f(x) = f'(x), \quad \log f(x) = x, \quad f(x) = e^x$$

**186. Taylor's and Maclaurin's series.** Under what conditions may a given function  $f(x)$  be represented by a power series, and what is the general form of this series? The second question is readily answered, as follows.

Let  $f(x)$  denote a given function of  $x$ , and  $a$  any assigned value of  $x$ . Assume that a power series of the form  $A_0 + A_1(x-a) + A_2(x-a)^2 + \dots$  exists which in its interval of convergence  $(-l, l)$  represents  $f(x)$ , so that, for  $|x-a| < l$ ,

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + \dots + A_n(x-a)^n + \dots$$

Then, § 183, we also have, for  $|x-a| < l$ ,

$$f'(x) = A_1 + 2 A_2(x-a) + 3 A_3(x-a)^2 + \dots$$

$$f''(x) = 2 \cdot 1 \cdot A_2 + 3 \cdot 2 \cdot A_3(x-a) + \dots$$

and so on; and by setting  $x = a$  in these equations we find  $A_0 = f(a)$ ,  $A_1 = f'(a)$ ,  $A_2 = f''(a)/2!$ , ...,  $A_n = f^{(n)}(a)/n! \dots$

Hence, if  $f(x)$  admits of being expressed as a power series in  $x - a$ , the expression is

$$f(x) = f(a) + f'(a) \frac{x - a}{1} + f''(a) \frac{(x - a)^2}{2!} + \dots + f^{(n)}(a) \frac{(x - a)^n}{n!} + \dots \quad (1)$$

The series (1) is called *Taylor's series*. When  $a$  is 0, it becomes

$$f(x) = f(0) + f'(0) \frac{x}{1} + f''(0) \frac{x^2}{2!} + \dots + f^{(n)}(0) \frac{x^n}{n!} + \dots \quad (2)$$

which is called *Maclaurin's series*.

EXAMPLE 1. If  $\sin x$  can be represented by a power series in  $x$ , what is the series?

$$\begin{array}{llll} f(x) = \sin x & f'(x) = \cos x & f''(x) = -\sin x & f'''(x) = -\cos x, \dots \\ \therefore f(0) = 0 & f'(0) = 1 & f''(0) = 0 & f'''(0) = -1, \dots \end{array}$$

$$\text{Hence} \quad \sin x = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

EXAMPLE 2. Assuming that the following functions have Maclaurin series, show that

$$1. \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$2. e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$3. \tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots$$

$$4. \sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \dots$$

EXAMPLE 3. Show by (1) that

$$x^3 = 1 + 3(x - 1) + 3(x - 1)^2 + (x - 1)^3.$$

EXAMPLE 4. Verify the fact that the series already got for  $\log(1 + x)$  and  $(1 + x)^m$  are Maclaurin series.

EXAMPLE 5. Show that  $f(a + x) = f(a) + f'(a) \frac{x}{1} + f''(a) \frac{x^2}{2!} + \dots$

**187. An extension of the mean value theorem.** To find under what conditions a given function  $f(x)$  is expressible by a Taylor series, we extend the mean value theorem (§ 97) as follows:

Suppose that  $f(x)$  has a finite  $n$ th derivative throughout the interval  $(a, b)$ , and therefore that  $f^{(n-1)}(x), \dots, f'(x), f(x)$  are continuous in  $(a, b)$ .

Let  $K$  denote the constant determined by the equation:

$$\begin{aligned} f(b) - f(a) - f'(a) \frac{b-a}{1} - f''(a) \frac{(b-a)^2}{2!} - \dots \\ - f^{(n-1)}(a) \frac{(b-a)^{n-1}}{(n-1)!} - K \frac{(b-a)^n}{n!} = 0 \end{aligned} \quad (1)$$

If we replace  $a$  by  $x$  in the first member of (1), we get a function

$$\begin{aligned} F(x) = f(b) - f(x) - f'(x) \frac{b-x}{1} - \dots \\ - f^{(n-1)}(x) \frac{(b-x)^{n-1}}{(n-1)!} - K \frac{(b-x)^n}{n!} \end{aligned} \quad (2)$$

which satisfies the conditions of Rolle's theorem, § 96; for  $F(a) = 0$  by (1),  $F(b) = 0$  identically, and  $F'(x)$ , which reduces to

$$F'(x) = -f^{(n)}(x) \frac{(b-x)^{n-1}}{(n-1)!} + K \frac{(b-x)^{n-1}}{(n-1)!} \quad (3)$$

is finite in  $(a, b)$ .

Hence there exists a value  $x_1$  of  $x$  between and distinct from  $a$  and  $b$ , such that  $F'(x_1) = 0$ , and therefore, by (3), such that

$$-f^{(n)}(x_1) \frac{(b-x_1)^{n-1}}{(n-1)!} + K \frac{(b-x_1)^{n-1}}{(n-1)!} = 0, \text{ or } K = f^{(n)}(x_1) \quad (4)$$

Substituting this value of  $K$  in (1) and transposing terms, we obtain

$$\begin{aligned} f(b) = f(a) + f'(a) \frac{b-a}{1} + \dots + f^{(n-1)}(a) \frac{(b-a)^{n-1}}{(n-1)!} \\ + f^{(n)}(x_1) \frac{(b-a)^n}{n!} \end{aligned} \quad (5)$$

where  $x_1$  is a definite number between  $a$  and  $b$  and may be written  $x_1 = a + \theta(b - a)$  where  $0 < \theta < 1$  (6)

**188. Existence of a Taylor series.** Let  $x$  denote a variable in  $(a, b)$ . In (5) replace  $b$  by  $x$ . We get

$$f(x) = f(a) + f'(a) \frac{x-a}{1} + \dots + f^{(n-1)}(a) \frac{(x-a)^{n-1}}{(n-1)!} + R_n \quad (7)$$

where 
$$R_n = f^{(n)}[a + \theta(x-a)] \frac{(x-a)^n}{n!}$$

We call (7) *Taylor's series with remainder term*.

Suppose that the condition that  $f^{(n)}(x)$  is finite in  $(a, b)$  holds good for all finite values of  $n$ , and that  $R_n \rightarrow 0$  when  $n \rightarrow \infty$ . We then have

$$f(x) = f(a) + f'(a) \frac{x-a}{1} + \dots + f^{(n)}(a) \frac{(x-a)^n}{n!} + \dots \quad (8)$$

*The Taylor series (8) converges and represents  $f(x)$  in  $(a, b)$  if all the derivatives of  $f(x)$  are finite in  $(a, b)$  and if  $\lim_{n \rightarrow \infty} R_n = 0$ .*

In the case of Maclaurin's series,  $a$  is 0 and (7) becomes

$$f(x) = f(0) + f'(0) \frac{x}{1} + \dots + f^{(n-1)}(0) \frac{x^{n-1}}{(n-1)!} + R_n$$

where 
$$R_n = f^{(n)}(\theta x) \frac{x^n}{n!} \quad (9)$$

*If in  $(0, b)$  every  $f^{(n)}(x)$  is finite and  $\lim_{n \rightarrow \infty} R_n = 0$ , then in  $(0, b)$*

$$f(x) = f(0) + f'(0) \frac{x}{1} + \dots + f^{(n)}(0) \frac{x^n}{n!} + \dots \quad (10)$$

**189. The function  $e^x$ .** Here  $f^{(n)}(x) = e^x$  is finite for every finite  $n$  and  $x$ .

$$f^{(n)}(\theta x) \frac{x^n}{n!} = e^{\theta x} \frac{x^n}{n!}$$

which  $\rightarrow 0$  when  $n \rightarrow \infty$ ; for  $e^{\theta x}$  is finite, and since  $x^n/n! \rightarrow 0$  is the  $n$ th term of a convergent series, § 179,  $\lim_{n \rightarrow \infty} x^n/n! = 0$ . Hence, by (10),

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \quad |x| < \infty \quad (11)$$



**190.** The functions  $\sin x$ ,  $\cos x$ . If  $f(x) = \sin x$ , then  $f'(x) = \cos x = \sin(x + \pi/2)$ , and by repetitions of this process  $f^{(n)}(x) = \sin(x + n\frac{\pi}{2})$ . Hence

$$f^{(n)}(\theta x) \frac{x^n}{n!} = \sin\left(\theta x + n\frac{\pi}{2}\right) \frac{x^n}{n!}$$

which  $\rightarrow 0$  when  $n \rightarrow \infty$ , if  $|x| < \infty$ . Similarly for  $\cos x$ . Hence by (10)

$$\sin x = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad |x| < \infty \quad (12)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad |x| < \infty \quad (13)$$

Here  $x$  represents the circular measure of the angle:  $1^\circ = \pi/180$  or .017453 radians.<sup>1</sup>

**EXAMPLE 1.** Compute  $\sin 10^\circ$  to the fourth decimal figure.

Since  $10^\circ = .17453$  radians, and (12) is an alternating series, we have, by the remark at the end of § 178,

$$\sin 10^\circ = .17453 - \frac{(.17453)^3}{6}, \text{ with error } < \frac{(.17453)^5}{120}, \therefore < (.1)^5.$$

$$\text{Hence} \quad \sin 10^\circ = .17453 - .00089 = .1736 \dots$$

**EXAMPLE 2.** For how large an angle may we set  $\sin x = x$  with an error  $< .001$ ?

**EXAMPLE 3.** For an arc which subtends the angle  $\theta$  at the center of a circle of radius  $r$  show that the difference between the arc and its chord is  $r[\theta - 2 \sin(\theta/2)]$ .

**EXAMPLE 4.** Calling the radius of the earth 4000 m., show of two points whose distance along the earth's surface is 100 m. that their distance in a straight line is about 13.75 ft. less than this.

**EXAMPLE 5.** Compute  $\cos 20^\circ$  to the fourth decimal figure.

<sup>1</sup> In the case of  $\log(1+x)$  and of  $(1+x)^m$ , and  $0 < x < 1$ , it will be found that we cannot prove that  $R_n \rightarrow 0$  if we express  $R_n$  in the form  $f^{(n)}(\theta x)x^n/n!$  (Lagrange's form). But if we replace the last term in § 187 (1) by  $K(b-a)$ , the reasoning in §§ 187, 188 will lead to the following expression for  $R_n$  due to Cauchy:

$$R_n = f^{(n)}(\theta x)(1-\theta)^{n-1} \frac{x^n}{(n-1)!} \quad 0 < \theta < 1$$

and for  $\log(1+x)$  and  $(1+x)^m$ , and  $|x| < 1$ , this  $R_n \rightarrow 0$ .

EXAMPLE 6. For a circle of 4000 m. radius show that an arc 1 m. long differs from its chord less than .0002 in. and recedes from the chord less than 2 in.

EXAMPLE 7. Show that  $\int_0^1 \frac{\sin x}{x} dx = \frac{17}{18}$ , with an error  $< \frac{1}{600}$ .

EXAMPLE 8. Show by (11) that  $e = 2.71828 \dots$

EXAMPLE 9. From (11), (12), (13) derive the expansions of the following in powers of  $x$ :

1.  $\sqrt{e^x} = e^{x/2}$     2.  $1/e^x$     3.  $\cos^2 x = (1 + \cos 2x)/2$     4.  $\sin x \cos x$

**191. Operations with power series.** The series for  $e^x$ ,  $\sin x$ ,  $\cos x$ ,  $\log(1+x)$  and  $(1+x)^m$  are very important and should be memorized. For other functions  $f(x)$  than these the direct derivation of the series  $f(0) + f'(0)x + \dots$  is, generally speaking, difficult, and the proof of its equivalence to  $f(x)$  — by showing that  $\lim R_n = 0$  — impracticable. But for a function which is expressible in terms of  $e^x$ ,  $\sin x$ ,  $\dots$  this may often be accomplished indirectly by aid of the following theorems. For, by § 186, a power series in  $x$ , however obtained, which in its interval of convergence equals  $f(x)$ , is the Maclaurin series of  $f(x)$ .

1. The rules for adding and multiplying polynomials in  $x$  may be extended as follows to functions defined by power series in  $x$ .

*If when  $|x| < l$  we have*

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$$

$$\phi(x) = b_0 + b_1x + b_2x^2 + \dots + b_nx^n + \dots$$

*then also, for  $|x| < l$ ,*

$$f(x) + \phi(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots$$

$$f(x)\phi(x) = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \dots$$

For if  $S_n$  and  $S'_n$  be the sums of the first  $n$  terms of the  $f(x)$  and  $\phi(x)$  series, we have

$$\lim S_n + \lim S'_n = \lim (S_n + S'_n)$$

which proves the theorem for  $f(x) + \phi(x)$ . The proof for  $f(x)\phi(x)$  will be given later.

EXAMPLE 1. Find to four terms the Maclaurin series for the product  $(1+x)^{-3}(1+2x)^{-1}$ .

By § 185 (3),  $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \dots$   $|x| < 1$   
 $\frac{(1+2x)^{-1} = 1 - 2x + 4x^2 - 8x^3 + \dots}{(1+x)^{-3}(1+2x)^{-1} = 1 - 3x + 6x^2 - 10x^3 + \dots}$   $|x| < 1/2$   
 $\begin{array}{r} -2 \mid +6 \mid -12 \\ \phantom{-2 \mid} +4 \mid -12 \\ \phantom{-2 \mid} \phantom{+4 \mid} -8 \end{array}$   
 $= 1 - 5x + 16x^2 - 42x^3 + \dots$   $|x| < 1/2$

This is the Maclaurin series of  $f(x) = (1+x)^{-3}(1+2x)^{-1}$  to four terms, that is,  $f(0) = 1$ ,  $f'(0) = -5$ ,  $f''(0) = 32$ ,  $f'''(0) = -252$ . Observe also that the reckoning not only gives the series but also proves its equivalence to  $f(x)$  for  $|x| < 1/2$ .

2. By the process of long division, or the method of undetermined coefficients, a rational fraction

$$\frac{a_0 + a_1x + \dots + a_nx^n}{b_0 + b_1x + \dots + b_mx^m}, \quad b_0 \neq 0,$$

may be transformed into an infinite series which converges and represents the fraction when  $|x|$  is less than the numerical value of the numerically least root, real or imaginary, of  $b_0 + b_1x + \dots + b_mx^m = 0$ . The like is true of the fraction  $f(x)/\phi(x)$  when  $f(x)$  and  $\phi(x)$  are infinite series, the resulting series then being convergent and representing  $f(x)/\phi(x)$  when  $|x|$  is less than the limit of convergence of  $f(x)$  and also less than the numerical value of the numerically least root of  $\phi(x) = 0$ . This and the theorem in 3 will be proved later.

**EXAMPLE 1.** Express  $(2-3x)/(1-3x+2x^2)$  by a Maclaurin series.

Let  $\frac{2-3x}{1-3x+2x^2} = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots$

### Clearing of fractions,

$$2 - 3x = C_0 + C_1|x + C_2|x^2 + C_3|x^3 + \dots$$

Hence  $C_0 = 2, C_1 - 3C_0 = -3, C_2 - 3C_1 + 2C_0 = 0, \dots$

Therefore  $C_0 = 2, C_1 = 3, C_2 = 5, C_3 = 9, \dots$

and the required series, to four terms, is  $2 + 3x + 5x^2 + 9x^3 + \dots$ . The roots of  $1 - 3x + 2x^2 = 0$  are  $1/2$  and  $1$ ; hence the series converges when  $|x| < 1/2$ . This may be proved by resolving the given fraction into its partial fractions:

$$\frac{2 - 3x}{1 - 3x + 2x^2} = \frac{1}{1 - x} + \frac{1}{1 - 2x} = (1 + x + x^2 + \dots) + (1 + 2x + 4x^2 + \dots) = 2 + 3x + 5x^2 + \dots, \quad |x| < 1/2$$

**EXAMPLE 2.** Find to three terms the Maclaurin series for  $\tan x$ .

$$\begin{array}{r|l} \sin x = x - x^3/6 + x^5/120 - \dots & 1 - x^2/2 + x^4/24 + \dots = \cos x \\ x - x^3/2 + x^5/24 - \dots & x + x^3/3 + 2x^5/15 + \dots = \tan x \\ \hline x^3/3 - x^5/30 + \dots & \\ x^3/3 - x^5/6 + \dots & \\ \hline 2x^5/15 - \dots & \end{array}$$

The series converges when  $|x| < \pi/2$ , one of the numerically smallest roots of  $\cos x = 0$ .

3. If in the series  $z = a_0 + a_1y + a_2y^2 + \dots$   $|y| < l$  we substitute  $y = b_0 + b_1x + b_2x^2 + \dots$   $|b_0| < l$  and then compute  $y^2, y^3, \dots$  by 1, and finally collect terms which involve like powers of  $x$ , we obtain a power series in  $x$  which converges and represents  $z$ , at least when  $|b_0| + |b_1x| + \dots < l$ .

**EXAMPLE.** Find the Maclaurin series for  $(\cos x)^{1/3}$  to the term in  $x^4$ .

$$\begin{aligned} (\cos x)^{1/3} &= \left[ 1 + \left( -\frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) \right]^{1/3} \\ &= 1 + \frac{1}{3} \left( -\frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) - \frac{1}{9} \left( -\frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right)^2 \\ &= 1 - \frac{x^2}{6} - \frac{x^4}{72} - \dots \end{aligned}$$

4. All that has been said in this section applies also to the representation of a function in powers of  $x - a$ . In the case of a rational fraction  $F(x)$  in particular, often the Taylor series in  $x - a$  is most easily got by setting  $x = a + (x - a)$  in  $F(x)$  and then applying the theorem in 2. The expression of this  $F(x)$  in process of  $1/x$  may also be got by first expressing  $F(x)$  in terms of  $1/x$  and then applying 2.

Thus the expansions of  $1/(x+1)$  in powers of  $x-1$  and of  $1/x$  may be got as follows:

$$\begin{aligned}\frac{1}{x+1} &= \frac{1}{2+(x-1)} = \frac{1}{2} \frac{1}{1+(x-1)/2} \\ &= \frac{1}{2} \left[ 1 - \frac{x-1}{2} + \frac{(x-1)^2}{4} - \dots \right] \quad |x-1| < 2 \\ \frac{1}{x+1} &= \frac{1}{x} \left[ \frac{1}{1+1/x} \right] = \frac{1}{x} \left[ 1 - \frac{1}{x} + \frac{1}{x^2} - \dots \right] = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \dots \quad |x| > 1\end{aligned}$$

**192. Indeterminate forms.** Infinite series may often be used to advantage in evaluating indeterminate forms.

**EXAMPLE 1.**  $F(x) = (\sin x - x)/x^2 \sin x$  becomes  $0/0$  when  $x = 0$ . Find  $\lim_{x \rightarrow 0} F(x)$ .

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin x - x}{x^2 \sin x} &= \lim_{x \rightarrow 0} \frac{-x^3/3! + x^5/5! - \dots}{x^3 - x^5/3! + \dots} \\ &= \lim_{x \rightarrow 0} \frac{-1/3! + x^2/5! \dots}{1 - x^2/3! + \dots} = -\frac{1}{6}\end{aligned}$$

**EXAMPLE 2.** Prove that  $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$

$$\log \left(1 + \frac{x}{n}\right)^n = n \log \left(1 + \frac{x}{n}\right) = n \left( \frac{x}{n} - \frac{1}{2} \frac{x^2}{n^2} + \dots \right) \rightarrow x$$

Hence 
$$\left(1 + \frac{x}{n}\right)^n \rightarrow e^x$$

### EXERCISE XXXVIII

1. Find the following limiting values:

$$1. \lim_{x \rightarrow 0} \frac{2x \cos x - \sin 2x}{x^2 \log(1+x)}$$

$$2. \lim_{x \rightarrow 0} \frac{(1+x) - (1+2x)^{1/2}}{(1+3x)^{1/3} - (1+4x)^{1/4}}$$

$$3. \lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{\tan^2 x}$$

$$4. \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^{bx}$$

2. Using  $\sec x = 1/\cos x$ , prove that

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots, |x| < \frac{\pi}{2}.$$

3. In leveling, the curvature of the earth is ordinarily neglected. Show that the consequent error for points 1 m. apart is about 8 in.

4. Show that the difference between the perimeter of the circumscribed and inscribed regular polygons of  $n$  sides to a circle of radius  $r$  is  $r \left( \frac{\pi^3}{n^2} + \frac{\pi^5}{4n^4} + \dots \right)$ .

5. Referring to § 73, show that for  $|x| < \infty$

$$\sinh x = \frac{x}{1} + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \quad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

6. Show that  $\log(x+3)^{1/2} = \frac{1}{2} \left( \log 3 + \frac{x}{3} - \frac{1}{2} \frac{x^2}{9} + \dots \right) \quad |x| < 3$

7. Find the Maclaurin series to the term in  $x^4$  and its limit of convergence for

1.  $\frac{2-x}{(2+x)(1+x^2)}$

2.  $(1-2x-x^2)^{1/2}$

3.  $e^{\sin x}$

8. Show that  $\frac{x^2}{x^2-3x+2} = 1 + \frac{3}{x} + \frac{7}{x^2} + \frac{15}{x^3} + \dots$  for  $|x| > 2$ .

### 193. Taylor series with remainder term. Approximations.

1. Suppose that  $f(x)$  has a finite  $n$ th derivative in the interval  $(a, a+h)$ . Then by setting  $x = a+h$  in the formula § 188 (7) we have

$$\begin{aligned} f(a+h) &= f(a) + f'(a) \frac{h}{1} + \dots \\ &\quad + f^{(n-1)}(a) \frac{h^{n-1}}{(n-1)!} + f^n(a+\theta h) \frac{h^n}{n!} \end{aligned} \quad (1)$$

If  $M_n$  denote the greatest value of  $|f^{(n)}(x)|$  in  $(a, a+h)$ , the last term in (1) is numerically  $\leq M_n h^n / n!$  and can therefore, by taking  $h$  sufficiently small, be made as small as we please. Hence, for small values of  $h$ ,

$$f(a+h) = f(a) + f'(a) \frac{h}{1} + \dots + f^{(n-1)}(a) \frac{h^{n-1}}{(n-1)!} \quad (2)$$

approximately, with an error numerically  $\leq M_n h^n / n!$

We have  $a+h = x$ ; hence when  $a$  is 0, then  $h$  is  $x$  and (1) is the Maclaurin expansion, with remainder term, of  $f(x)$ .

EXAMPLE 1. The Maclaurin expansion of  $(1-x)^{1/2}$  to three terms is  $1 - \frac{x}{2} - \frac{x^2}{8}$ . How great is the error involved in setting

$$(1-x)^{1/2} = 1 - \frac{x}{2} - \frac{x^2}{8} \quad \text{when } 0 < x \leq 1/4?$$

$$f'''(x) = -\frac{3}{8}(1-x)^{-5/2} \quad \therefore M_3 = \frac{3}{8} \left(\frac{4}{3}\right)^{5/2} = .7698.$$

$$\therefore \text{error is } < \frac{.7698}{6 \cdot 4^3} = .002005$$

2. In (1) transpose the term  $f(a)$ , then replace  $a$  by  $x$  and  $h$  by  $\Delta x$ ; we get

$$\Delta f(x) = f'(x)\Delta x + f''(x)\frac{(\Delta x)^2}{2!} + \dots + f^{(n-1)}(x)\frac{(\Delta x)^{n-1}}{(n-1)!}$$

with an error  $E$  numerically less than  $M_n(\Delta x)^n/n!$ ,  $M_n$  denoting the greatest value of  $|f^{(n)}(x)|$  in  $(x, x + \Delta x)$ . Thus

$$\Delta f(x) = f'(x)\Delta x, \quad |E| \leq M_2 \frac{(\Delta x)^2}{2!} \quad (3)$$

$$\Delta f(x) = f'(x)\Delta x + f''(x)\frac{(\Delta x)^2}{2!}, \quad |E| \leq M_3 \frac{(\Delta x)^3}{3!} \quad (4)$$

and so on.

EXAMPLE 2. If  $f(x) = 1/x$ , and  $x$  and  $\Delta x$  are positive, the greatest value of  $f''(x) = 2/x^3$  in  $(x, x + \Delta x)$ , is  $2/x^3$ , and therefore, by (3),

$$\Delta\left(\frac{1}{x}\right) = -\frac{1}{x^2}\Delta x, \text{ with error } < \frac{(\Delta x)^2}{x^3}$$

For what values of  $x$  will this formula give  $\Delta(1/x)$  correctly to the fifth decimal figure when  $\Delta x = .1$ ?

$$\text{We must have } \frac{(.1)^2}{x^3} < .00001 \quad \therefore x^3 > 1000 \quad \therefore x > 10$$

Hence for  $x > 10$ , and  $0 < \Delta x \leq .1$ , we have

$$\frac{1}{x + \Delta x} = \frac{1}{x} - \frac{\Delta x}{x^2}, \text{ with error } < .00001.$$

Show that for  $\Delta x = .1$  the formula (4) gives

$$\Delta\left(\frac{1}{x}\right) \text{ with an error } < .00001 \text{ when } x > 5.63.$$

**194. Rule of proportional parts.** Suppose that, as in the case of  $\log_{10} x$  and  $\sin x$ , the function  $f(x)$  is tabulated for a sequence of values of  $x$  having some common difference  $h$ . It is customary to compute  $f(x)$  for intermediate values of  $x$  by a rule which assumes that for small changes in  $x$  the change in  $f(x)$  is proportional to that in  $x$ : so that if  $a$  be a number in the table,  $a + h$  the next number, and  $a + th$  ( $0 < t < 1$ ) a given intermediate number, then

$$\frac{f(a + th) - f(a)}{f(a + h) - f(a)} = \frac{th}{h} = \frac{t}{1} \quad (1)$$



By how much may the value of  $f(a + th)$  given by (1) be in error? By (1) and § 193 (1),

$$\begin{aligned} f(a + th) &= f(a) + t[f(a + h) - f(a)] \\ &= f(a) + th f'(a) + \frac{th^2}{2} f''(x_1) \quad a < x_1 < a + h \quad (2) \end{aligned}$$

And, by § 193 (1), the actual value of  $f(a + th)$  may be expressed by

$$f(a + th) = f(a) + th f'(a) + \frac{t^2 h^2}{2} f''(x_2) \quad a < x_2 < a + th \quad (3)$$

Hence the error in question is the difference between (2) and (3), or  $[f''(x_1)t - f''(x_2)t^2] h^2/2$ , which is obviously less than  $M_2 h^2$ , and can be proved to be less than  $M_2 h^2/8$ ,  $M_2$  denoting the greatest value of  $|f''(x)|$  in  $(a, a + h)$ .

EXAMPLE 1. In a seven-place logarithmic table the logarithms of all numbers from 10000 to 100000 are given at unit intervals. Here  $h = 1$  and

$$f(x) = \log_{10} x \quad \therefore f''(x) = -\frac{\mu}{x^2} \quad \text{where } \mu = \log_{10} e = .43429 \dots$$

Hence the greatest value of  $|f''(x)|$  for  $x \leq 10000$  is  $.43429/(10000)^2$ , and the error resulting from the use of the method of proportional parts at any point in the table is less than  $.43429/8(10000)^2 = .00000000543$  or far less than a unit in the seventh place.

EXAMPLE 2. Find the corresponding error in the case of a four-place table.

EXAMPLE 3. Let  $A$  and  $B$  be the points of the graph of  $y = f(x)$  corresponding to  $x = a$  and  $x = a + h$ , and  $P$  and  $Q$  the points of the arc  $AB$  and the chord  $AB$  corresponding to  $x = a + th$ . Show that the length of  $PQ$  cannot exceed  $M_2 h^2/8$ .

**195. Area under a parabolic arc. The prismoid formula.**

1. Let  $y_a, y_c, y_b$  be any positive ordinates erected at points  $a, c, b$  of  $Ox$  such that  $c - a = b - c = h$ , and  $C_a, C_c, C_b$  their end points. There is a single parabola with axis parallel to  $Oy$  through the

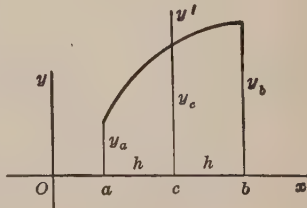


FIG. 109.

points  $C_a, C_c, C_b$ , and the area  $T$  of the space bounded by this parabola and  $Ox$ ,  $x = a$ ,  $x = b$  is

$$T = \frac{h}{3}(y_a + 4y_c + y_b) \quad (1)$$

For let the equation of the parabola when  $c$  is taken as origin be  $y = A + Bx' + Cx'^2$ . Then

$$T = \int_{-h}^h (A + Bx' + Cx'^2) dx' = 2Ah + \frac{2}{3}Ch^3 \quad (2)$$

$$y_a = A - Bh + Ch^2 \quad y_c = A \quad y_b = A + Bh + Ch^2 \quad (3)$$

If we solve the equations (3) for  $A$  and  $C$  in terms of  $y_a, y_c, y_b$ , and  $h$ , and substitute the results in (2), we get (1).

2. Let  $y = f(x)$  be any function whose graph passes through  $C_a, C_c, C_b$  and which has a finite fourth derivative in  $(a, b)$ . The area  $S = \int_a^b f(x) dx$  of the space bounded by  $y = f(x)$  and  $Ox$ ,  $x = a$ ,  $x = b$  differs numerically from  $T$  by less than  $M_4 h^5 / 90$ , where  $M_4$  denotes the greatest value of  $|f^{iv}(x)|$  in  $(a, b)$ .

For,<sup>1</sup> setting  $a = c - h$ ,  $b = c + h$  in (1), we find

$$T = \frac{h}{3} [f(c - h) + 4f(c) + f(c + h)]$$

and if  $F(x)$  denote an integral of  $f(x)$ , then

$$S = F(c + h) - F(c - h)$$

Let  $\phi(h) = T - S$ . It will be found that  $\phi(h)$ ,  $\phi'(h)$ ,  $\phi''(h)$  vanish when  $h = 0$  and that

$$\phi'''(h) = \frac{h}{3} [f'''(c + h) - f'''(c - h)]$$

which by the mean value theorem, § 97, can be reduced to the form

$$\phi'''(h) = \frac{2h^2}{3} f^{iv}(x_1) \quad c - h < x_1 < c + h$$

Since  $\phi''(0) = 0$  and  $|f^{iv}(x_1)| < M_4$ , we have, by § 129, Ex. 7,

$$|\phi''(h)| = \left| \int_0^h \phi'''(x) dx \right| < M_4 \int_0^h \frac{2x^2}{3} dx = \frac{2}{3 \cdot 3} M_4 h^3$$

and repetitions of this process give

$$|\phi(h)| < \frac{2}{3 \cdot 3 \cdot 4 \cdot 5} M_4 h^5 = \frac{M_4 h^5}{90}$$

<sup>1</sup> Ch.-J. de la Vallée Poussin, Cours d'Analyse Infinitésimale.

3. When  $f(x)$  is a polynomial in  $x$  of degree less than four, we have  $f^{iv}(x) \equiv 0$ , hence  $S = T$ , and therefore, by (1),

$$\int_a^b f(x)dx = \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \frac{b-a}{6} \quad (8)$$

By § 141, if the area of a variable section of a solid by a plane perpendicular to  $Ox$  be  $f(x)$ , the volume  $V$  of the solid between the planes  $x = a$  and  $x = b$  is  $\int_a^b f(x)dx$ . Hence, when  $f(x)$  is a polynomial in  $x$  of degree less than four,

$$V = \frac{d}{6}(B_1 + 4B_2 + B_3) \quad (9)$$

where  $d$  is the distance between the end sections,  $B_1$ ,  $B_3$  the areas of these sections, and  $B_2$  that of the midsection. This is called the *prismoid formula*.<sup>1</sup> All the volume formulas of elementary geometry follow from it.

EXAMPLE. For the sphere got by revolving the circle  $x^2 + y^2 = a^2$  about  $Ox$ , we have  $f(x) = (a^2 - x^2)\pi$ ; also

$$B_1 = 0, B_3 = 0, B_2 = \pi a^2 \quad \therefore \text{ by (9), } V = \frac{2a}{6} (4\pi a^2) = \frac{4}{3}\pi a^3.$$

**196. Simpson's rule.** A number of methods have been devised for finding approximate values of a definite integral  $\int_a^b f(x)dx$  when  $\int f(x)dx$  cannot be got by the formulas of integration and when the Taylor series for  $f(x)$  in powers of  $x - a$  converges too slowly to be serviceable. One of the most useful is the following, deduced from the formula § 195 (1) and called *Simpson's rule*.

To find an approximate value of  $S = \int_a^b f(x)dx$  when  $f(x)$  is positive and has a finite fourth derivative in  $(a, b)$ , divide  $(a, b)$  into  $2n$  equal parts of length  $h = (b - a)/2n$ . Let  $x_1, x_2, \dots, x_{2n-1}$  denote the points of division. Compute the corre-

<sup>1</sup> A prismoid is a solid whose bases  $B_1, B_3$  are polygons in parallel planes and whose lateral surface can be divided into triangles whose bases and vertices are sides and angular points of  $B_1$  and  $B_3$ . One can show for any prismoid that if  $Ox$  be taken perpendicular to  $B_1, B_3$  the area  $f(x)$  of a variable section parallel to  $B_1, B_3$  is of the 2nd, 1st, or 0 degree in  $x$ , therefore that (9) holds good.

sponding ordinates  $y_1, y_2, \dots y_{2n-1}$ , also  $y_a$  and  $y_b$ , and substitute in the formula:

$$T = \frac{h}{3} \{y_a + y_b + 4(y_1 + y_3 + \dots) + 2(y_2 + y_4 + \dots)\} \quad (1)$$

The numerical difference between  $S$  and  $T$  is less than  $\frac{M(b-a)}{180} h^4$ ,  $M$  being the greatest value of  $|f^{iv}(x)|$  in  $(a, b)$ .

For let  $C_a, C_1, C_2, \dots C_b$  be the end points of the ordinates  $y_a, y_1, y_2, \dots y_b$ . The right member of (1) is the sum of the areas under the parabolic arcs  $C_a C_1 C_2, C_2 C_3 C_4$ , and so on, as given by § 195 (1). There are  $n$  of these areas and each differs from the corresponding part of  $S$  by less than  $Mh^5/90$ . Therefore, since  $b - a = 2nh$ ,  $T$  differs from  $S$  by less than  $M(b-a)h^4/180$ .

EXAMPLE 1. Find the value of  $\pi$  from the formula  $\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}$ .

If we divide  $(0, 1)$  into four equal parts ( $h = 1/4$ ) at  $x_1 = 1/4, x_2 = 1/2, x_3 = 3/4$ , we find

$$y_a = 1, y_1 = .941, y_2 = .8, y_3 = .64, y_b = .5$$

Hence, by (1),

$$\frac{\pi}{4} = \frac{1}{4 \cdot 3} [1.5 + 4(.941 + .64) + 2(.8)] = \frac{1}{4 \cdot 3} (9.424) = \frac{1}{4} (3.14133) \dots$$

which gives  $\pi = 3.14133$ , while the actual value is  $3.14159 \dots$ . If we divide  $(0, 1)$  into ten equal parts ( $h = .1$ ) we get  $\pi = 3.141593$ , which is correct to the last figure.

EXAMPLE 2. Find  $\log 2$  from the formula  $\log 2 = \int_0^1 \frac{dx}{1+x}$  by Simpson's rule, dividing  $(0, 1)$  into four equal parts. Using the test number  $\frac{M(b-a)}{180} h^4$ , show that the result should be correct to the third decimal figure.

### EXERCISE XXXIX

1. Evaluate  $\int_0^1 (1+x^3)^{1/2} dx$  by Simpson's rule ( $h = 1/4$ ).
2. Show by integration of series that  $\int_0^1 e^{-x^2} dx = .7467$ .
3. By the prismoid formula, show that the volume of a spherical segment of height  $h$  is  $\frac{2}{3}\pi(a - h/3)h^2$ ,  $a$  being the radius of the sphere.

4. For what values of  $\Delta x$  will the formula  $\Delta \sin x = \cos x \Delta x$  give results which are correct to the second decimal figure when  $x = \pi/3$ ?

5. Two curves  $y = f(x)$  and  $y = \phi(x)$  are said to have contact of order  $n$  at  $x = a$  if  $f(a) = \phi(a)$ ,  $f'(a) = \phi'(a)$ , ...,  $f^{(n)}(a) = \phi^{(n)}(a)$ , but  $f^{(n+1)}(a) \neq \phi^{(n+1)}(a)$ . Show that the curves also cross at  $x = a$  when  $n$  is even, but not when  $n$  is odd.

6. Find the curve  $y = A + Bx + Cx^2 + Dx^3$  which has contact of the third order with the ellipse  $x^2 - xy + y^2 = 1$  at the point  $(1, 1)$ .

7. Prove Huygen's formula: the length of a circular arc is approximately  $2B + \frac{1}{3}(2B - A)$  where  $A$  is the chord and  $B$  the chord of half the arc.

8. Show that the greatest distance from the shore at which a mast head 70 ft. above the water level would be visible is about 10.3 m.

9. Verify the formula  $4 \tan^{-1}(1/5) - \tan^{-1}(1/239) = \pi/4$ , and by its aid and the series  $\tan^{-1} x = x - x^3/3 + \dots$ , compute  $\pi$  to the sixth decimal figure.

10. By Example 2, p. 85, the differential of arc of the ellipse  $x = a \cos \phi$ ,  $y = b \sin \phi$  is  $ds = a(1 - e^2 \cos^2 \phi)^{1/2} d\phi$  where  $e = (a^2 - b^2)^{1/2}/a$  is less than 1. Show that the length of a quadrant of the ellipse is

$$\frac{\pi}{2} a \left[ 1 - \left(\frac{1}{2}\right)^2 e^2 - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{e^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{e^6}{5} - \dots \right]$$

11. Verify the fact that the formula § 195 (8) gives the correct value of the area under the curve  $y = x^3 + 2x^2$  between  $x = 1$  and  $x = 3$ .

12. Supply the details of the following general proof of the theorem (assumed in § 183) that, if  $l$  be the limit of convergence of  $\Sigma a_n x^n$ , then  $\Sigma n a_n x^{n-1}$  also converges when  $|x| < l$ .

Suppose  $|x| < l$ , and let  $c$  be any number such that  $|x| < c < l$ . Since  $\Sigma |a_n c^n|$  is convergent, every  $|a_n c^n|$  is less than some finite number  $M$  and therefore

$$|n a_n x^{n-1}| = n \frac{|a_n c^n|}{c} \left(\frac{|x|}{c}\right)^{n-1} < n \frac{M}{c} \left(\frac{|x|}{c}\right)^{n-1}$$

The series  $\Sigma n \frac{M}{c} \left(\frac{|x|}{c}\right)^{n-1}$  converges since  $\frac{u_{n+1}}{u_n} \rightarrow \frac{|x|}{c}$ , which is  $< 1$ .

Hence  $\Sigma |n a_n x^{n-1}|$  converges when  $|x| < c \therefore$  when  $|x| < l$ .

## XXI. DETERMINANTS

**197. On permutations.** When considering the permutations of a set of objects taken all at a time, we may choose some particular order of the objects as the normal order. Any given permutation is then said to have as many *inversions* as it presents instances in which an object precedes one which in the normal order it follows.

Thus, if the objects in normal order are 1, 2, 3, 4, the permutation 3241 has the four inversions 32, 31, 21, 41.

*If two of the objects in a permutation are interchanged, the number of inversions is changed by an odd number.*

For obviously when the objects are adjacent, there is a change of one in the number of inversions; and an interchange of two separated objects can be effected by an odd number of interchanges of adjacent objects.

**198. Determinants.** Take any *square array* of numbers, as

$$\begin{array}{ccccccc}
 a_1 & & a_1 & a_2 & & a_1 & a_2 & a_3 & & \text{and so on,} \\
 & & b_1 & b_2 & & b_1 & b_2 & b_3 & & \\
 & & & & & c_1 & c_2 & c_3 & & 
 \end{array}$$

where the letter indicates the row and the subscript the column in which each number or *element* of the array occurs.

With the elements of any such array, form all the products that can be formed by taking one factor and but one from each row and column. In each product arrange the factors so that the row marks (the letters) are in normal order, and then count the inversions of the column marks (the subscripts). When their number is even (or 0), give the product

the plus sign; when odd, the minus sign. The algebraic sum of all these signed products is called the *determinant* of the array and is represented by the array itself with a bar placed at either side of it.

The number of rows or columns in the array is called the *order* of the determinant, and the products just described are called its *terms*. The determinant may also be represented by the more compact symbol  $|a_1 \ b_2|$ , or  $|a_1 \ b_2 \ c_3|$ , and so on.

The determinants of the second and third orders are

$$1. \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$2. \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 \\ - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1$$

Observe that in 2. the products with + signs are got by following the direction of the "leading diagonal"  $a_1 b_2 c_3$ ; those with - signs, that of the other diagonal. This is not true of determinants of higher order.

**EXAMPLE 1.** Prove the truth of the following:

$$1. \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 5 & 2 \end{vmatrix} = 9$$

$$2. \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$3. \begin{vmatrix} a & b & c \\ 0 & 0 & 0 \\ d & e & f \end{vmatrix} = 0$$

$$4. \begin{vmatrix} ka & a & d \\ kb & b & e \\ kc & c & f \end{vmatrix} = k \begin{vmatrix} a & a & d \\ b & b & e \\ c & c & f \end{vmatrix} = 0$$

**EXAMPLE 2.** Show that the determinant of the fourth order  $|a_1 b_2 c_3 d_4|$  has  $4! = 24$  terms, of which half are + and half -. What are the signs of the following terms?

$$a_4 b_2 c_3 d_1$$

$$a_3 b_4 c_1 d_2$$

$$a_2 b_1 c_4 d_3$$

**EXAMPLE 3.** Show that the signs of the terms of a determinant may also be found by arranging the factors in the normal order of the column marks and then counting the inversions of the row marks.

**199. Properties of determinants.** The following properties of determinants are immediate consequences of the definition in § 198.



1. *If the first, second, ... rows of a determinant are written as its first, second, ... columns, the value of the determinant is not changed.*

2. *If all the elements of a row or column are 0, the determinant has the value 0.*

3. *If two of the rows or two of the columns are interchanged, the value of the determinant merely changes sign.*

Thus,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} (1) = - \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} (2)$$

For suppose the factors in the terms of each determinant to be arranged in the order of the rows of that determinant. Then each term of (1) appears in (2) with its first and third factors interchanged, therefore with the number of inversions of its subscripts changed by an odd number, § 197, therefore with its sign changed. Thus for  $a_2b_3c_1$  in (1), we have  $-c_1b_3a_2$  in (2).

4. *If two of the rows, or two of the columns, be identically the same, the determinant has the value zero.*

For if the value were not 0, it would change sign when the identical rows or columns are interchanged; but the determinant remains unchanged.

5. *If all the elements of a row or a column have a common factor, that factor may be separated and written before the determinant.*

Thus,

$$|ka_1 \ b_2 \ c_3| = k |a_1 \ b_2 \ c_3|$$

6. *If the corresponding elements of two rows or two columns are proportional, the determinant has the value zero.*

7. *Since  $(a_1 + a'_1)b_2c_3 = a_1b_2c_3 + a'_1b_2c_3$ , and so on, we have*

$$\begin{vmatrix} a_1 + a'_1 & a_2 & a_3 \\ b_1 + b'_1 & b_2 & b_3 \\ c_1 + c'_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a'_1 & a_2 & a_3 \\ b'_1 & b_2 & b_3 \\ c'_1 & c_2 & c_3 \end{vmatrix}$$

Any of the numbers  $a'_1, b'_1, c'_1$  may be 0.

8. *The value of a determinant will not be changed if to the elements of any column (or row) there be added the corresponding*

elements of any other column (or row) all multiplied by the same number  $k$ .

This follows from the theorems 6. and 7. Thus

$$\begin{vmatrix} a_1 + ka_3 & a_2 & a_3 \\ b_1 + kb_3 & b_2 & b_3 \\ c_1 + kc_3 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} ka_3 & a_2 & a_3 \\ kb_3 & b_2 & b_3 \\ kc_3 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

EXAMPLE. Prove that the following determinants have the values indicated, shortening the reckoning as much as possible by aid of 6., 7., 8.

$$1. \begin{vmatrix} 10 & 8 & 2 \\ 15 & 12 & 3 \\ 20 & 32 & 13 \end{vmatrix} = 0 \quad 2. \begin{vmatrix} 2 & 3 & 4 \\ 25 & 38 & 50 \\ 15 & 20 & 27 \end{vmatrix} = -3 \quad 3. \begin{vmatrix} 9 & 27 & 18 \\ 15 & 42 & 12 \\ 18 & 51 & 14 \end{vmatrix} = 108$$

**200. Minors.** In any determinant,  $\Delta$ , cancel the row and column in which some particular element, call it  $e$ , lies and then form the determinant of the remaining elements without disturbing their relative positions. The determinant thus got is called the *complementary minor* of  $e$  and may be represented by  $\Delta_e$ .

1. In the expansion of  $\Delta$ , the sum of all the terms that involve the leading element  $a_1$  is  $a_1\Delta_{a_1}$ .

$$\text{Thus, in } \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}, \text{ this sum is } a_1 \begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix}$$

For, apart from sign, each term of  $\Delta$  that involves  $a_1$  is got by multiplying  $a_1$  by elements taken from the remaining rows and columns of  $\Delta$ , one and but one from each; in other words, by multiplying  $a_1$  by a term of  $\Delta_{a_1}$ . Moreover the sign of the  $\Delta$  term is the same as that of the  $\Delta_{a_1}$  term since it depends solely on the inversions of the subscripts 2, 3, 4. Conversely, the product of  $a_1$  by any term of  $\Delta_{a_1}$  is a term of  $\Delta$ .

2. If  $e$  denote the element in the  $i$ th row and  $k$ th column of  $\Delta$ , the sum of all the terms of  $\Delta$  that involve  $e$  is  $(-1)^{i+k} e\Delta_e$ .

For we can bring  $e$  into the position of leading element, without disturbing the relative positions of the elements outside of the row and column in which  $e$  stands, by first interchanging  $e$ 's row with each preceding row in turn, and then interchanging its column with each preceding column. In carrying out these successive interchanges,

we merely change the sign of the determinant  $(i-1)+(k-1)$  times. Hence, if  $\Delta'$  denote the determinant in its final form, we have

$$\Delta' = (-1)^{i+k-2}\Delta = (-1)^{i+k}\Delta$$

The minor of  $e$  in  $\Delta'$  is the same as in  $\Delta$ . Hence in  $\Delta'$  the sum of all terms that involve  $e$  is  $e\Delta_e$ , by 1., and therefore in  $\Delta$  this sum is  $(-1)^{i+k}e\Delta_e$ .

We call  $(-1)^{i+k}\Delta_e$  the *cofactor* of  $e$  in the determinant  $\Delta$ .

3. *A determinant may be expressed as the sum of the products of the elements of one of its rows (or columns) and their complementary minors, with signs which are alternately plus and minus, or minus and plus.*

$$\text{Thus, } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

and if  $\Delta$  be the determinant  $|a_1 \ b_2 \ c_3 \ d_4|$ , we have

$$\Delta = a_1\Delta_{a_1} - a_2\Delta_{a_2} + a_3\Delta_{a_3} - a_4\Delta_{a_4} \quad (1)$$

$$= a_1\Delta_{a_1} - b_1\Delta_{b_1} + c_1\Delta_{c_1} - d_1\Delta_{d_1} \quad (2)$$

with similar developments with respect to the other rows and columns.

4. Hence the value of a determinant of the fourth order may always be found by computing the values of four determinants of the third order. But a simpler method is illustrated in the following example.

$$\begin{vmatrix} 1 & 1 & -1 & 2 \\ 2 & 3 & 1 & 1 \\ -1 & 2 & 2 & 0 \\ 3 & 1 & 1 & 2 \end{vmatrix} \begin{matrix} \\ (1) \end{matrix} = \begin{vmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 3 & -3 \\ 0 & 3 & 1 & 2 \\ 0 & -2 & 4 & -4 \end{vmatrix} \begin{matrix} \\ \\ (2) \end{matrix} = \begin{vmatrix} 1 & 3 & -3 \\ 3 & 1 & 2 \\ -2 & 4 & -4 \end{vmatrix} \begin{matrix} \\ \\ (3) \end{matrix} = -30$$

We get (2) by subtracting from the second, third, and fourth rows of (1) the first row multiplied by 2, -1, and 3; and (2) equals (3) by the theorem in 3. This method is directly applicable to any determinant which has an element 1. And a determinant which has no such element can be transformed into another which has one by aid of the theorems of § 199.

**201. Cofactors.** 1. The cofactors of  $a_1, a_2, \dots$  in the determinant  $\Delta = |a_1 \ b_2 \ c_3 \ d_4|$ , namely  $\Delta_{a_2}, -\Delta_{a_1}, \dots$ , are represented by  $A_1, A_2, \dots$ . Hence, by § 200, (1), (2),

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 \quad (1)$$

$$= a_1A_1 + b_1B_1 + c_1C_1 + d_1D_1 \quad (2)$$

*The sum of the products of the elements of any row or column of a determinant  $\Delta$  and their cofactors is equal to  $\Delta$ .*

2. Since  $A_1, B_1, C_1, D_1$  do not involve  $a_1, b_1, c_1, d_1$ , the effect of replacing  $a_1, b_1, c_1, d_1$  by  $a_2, b_2, c_2, d_2$  in (2) is the same as that of replacing the first column of  $\Delta$  by  $a_2, b_2, c_2, d_2$ , thus making two columns equal; hence, § 199, 4.,

$$a_2A_1 + b_2B_1 + c_2C_1 + d_2D_1 = 0 \quad (3)$$

*The sum of the products of the elements of a column of a determinant  $\Delta$  and the cofactors of the corresponding elements of another column is 0. Similarly for the rows.*

**202. Linear equations.** 1. It is required to solve the following set of simultaneous equations for  $x, y, z$ .

$$\begin{aligned} a_1x + a_2y + a_3z &= k \\ b_1x + b_2y + b_3z &= l \\ c_1x + c_2y + c_3z &= m \end{aligned} \quad (1)$$

Let  $\Delta$  denote the determinant of the coefficients of  $x, y, z$ , and let  $A_1, B_1, \dots$  denote the cofactors of  $a_1, b_1, \dots$  in  $\Delta$ . To eliminate  $y$  and  $z$ , multiply the first equation by  $A_1$ , the second by  $B_1$ , the third by  $C_1$ , and then add. In the result, the coefficient of  $x$  is  $a_1A_1 + b_1B_1 + c_1C_1$  or  $\Delta$ ; the coefficients of  $y$  and  $z$  are 0, § 201, 2.; and the second member is  $kA_1 + lB_1 + mC_1$ , which represents the determinant  $|k \ b_2 \ c_3|$  got by replacing the first column of  $\Delta$  by  $k, l, m$ . We therefore have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} x = \begin{vmatrix} k & a_2 & a_3 \\ l & b_2 & b_3 \\ m & c_2 & c_3 \end{vmatrix} \quad (2)$$

In like manner, we find

$$\Delta \cdot y = |a_1 \ l \ c_3| \quad (3) \quad \Delta \cdot z = |a_1 \ b_2 \ m| \quad (4)$$

Therefore, if  $\Delta \neq 0$ , we have, on dividing (2), (3), (4) by  $\Delta$ ,

$$x = \frac{|k \ b_2 \ c_3|}{\Delta} \quad y = \frac{|a_1 \ l \ c_3|}{\Delta} \quad z = \frac{|a_1 \ b_2 \ m|}{\Delta} \quad (5)$$

We have thus proved that if the equations (1) have any solution, it is (5). That (5) actually is a solution may be shown by substituting (5) in (1).

Thus, if we substitute in  $a_1x + a_2y + a_3z - k = 0$  and clear of fractions, we find that the first member becomes the expansion of a determinant of the fourth order whose first two rows are  $a_1, a_2, a_3, k$ , and which is therefore 0.

This argument may be extended to a set of  $n$  linear equations in  $n$  unknown letters and establishes the theorem:

*A set of  $n$  linear equations in  $n$  unknown letters has one and but one solution if the determinant of the coefficients of the unknown letters is not 0.*

EXAMPLE 1. Solve the following set of equations by this method:

$$2x - y - 3z = 3 \quad 4x + 5y + 7z = 7 \quad 3x + 2y - 8z = 15$$

2. When  $\Delta = 0$ , it follows from (2), (3), (4) that the set of equations (1) has no solution unless  $|k \ b_2 \ c_3|$ ,  $|a_1 \ l \ c_3|$ ,  $|a_1 \ b_2 \ m|$  are also 0; but if these three determinants are 0, the set (1) may have infinitely many solutions.

Consider the *matrices*, or rectangular arrays of numbers:

$$A \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad B \begin{vmatrix} a_1 & a_2 & a_3 & k \\ b_1 & b_2 & b_3 & l \\ c_1 & c_2 & c_3 & m \end{vmatrix}$$

From any matrix we may obtain a number of square arrays by crossing out columns, rows, or both. The determinants of all these arrays (including that of the matrix itself when square) are called the *determinants of the matrix*.

The order of the determinant (or group of determinants) of highest order that does not vanish is called the *rank* of the matrix.

Since every determinant of  $A$  is a determinant of  $B$ , the rank of  $A$  cannot exceed that of  $B$ . It can be proved that

*If the ranks of  $A$  and  $B$  are the same, the set of equations (1) has one solution, or infinitely many solutions.*

*If the rank of  $A$  is less than that of  $B$ , the set of equations (1) has no solution.*

These theorems may be extended<sup>1</sup> to any set of  $m$  linear equations in  $n$  unknown letters, not only when  $m = n$  but also when  $m < n$ .

<sup>1</sup> See Bôcher: Introduction to Higher Algebra, § 16.

**EXAMPLE 2.** Show that the set of equations  $x + z = k$ ,  $y + z = l$ ,  $2x + y + 3z = m$  has infinitely many solutions, or no solution, according as  $2k + l$  is, or is not, equal to  $m$ .

3. When  $k$ ,  $l$ , and  $m$  are 0, the equations (1) become

$$\begin{aligned} a_1x + a_2y + a_3z &= 0 \\ b_1x + b_2y + b_3z &= 0 \\ c_1x + c_2y + c_3z &= 0 \end{aligned} \quad (6)$$

and are said to be *homogeneous*. Equations (2), (3), (4) become  $\Delta \cdot x = 0$   $\Delta \cdot y = 0$   $\Delta \cdot z = 0$  (7) and (7) implies that either  $\Delta = 0$  or  $x$ ,  $y$ , and  $z = 0$ .

In this case the ranks of the matrices  $A$  and  $B$  are the same. If  $\Delta \neq 0$ , the rank is 3, and (6) has the single solution  $(0, 0, 0)$ . If  $\Delta = 0$ , the rank is 2 or 1, and (6) has infinitely many other solutions. For suppose that the rank is 2 and that  $|a_1 \ b_2|$  is one of the non-vanishing determinants of the second order. We can then solve the first two equations for  $x$ ,  $y$  in terms of  $z$ . The result may be written

$$x : y : z = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} : - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} : \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \quad (8)$$

$$\text{or} \quad x = rC_1 \quad y = rC_2 \quad z = rC_3 \quad (9)$$

where  $r$  is an arbitrary constant. And these values (9) satisfy the third equation of (6); for, substituting them in its first member, we obtain  $r(c_1C_1 + c_2C_2 + c_3C_3) = r\Delta = 0$ .

If the rank is 1, every solution of one of the equations will satisfy the other two.

**EXAMPLE 3.** Show that the set of equations

$$2x + 3y - 5z = 0, \quad x + y + 6z = 0, \quad 4x + 5y + 7z = 0$$

has the solutions

$$x : y : z = 23 : -17 : -1, \text{ or } x = 23r, y = -17r, z = -r$$

**203. Multiplication of determinants.** It can be shown that the product of two determinants  $\Delta$  and  $\Delta'$  of the same order  $n$  may be expressed in the form of a third determinant  $\Delta''$  of order  $n$ , obtained as follows:

Multiply the elements of the  $i$ th row of  $\Delta$  by the corresponding elements of the  $k$ th column of  $\Delta'$ . The sum of the products thus obtained is the element in the  $i$ th row and  $k$ th column of  $\Delta''$ .

$$\text{Thus, } \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = \begin{vmatrix} a_1p_1 + a_2q_1 & a_1p_2 + a_2q_2 \\ b_1p_1 + b_2q_1 & b_1p_2 + b_2q_2 \end{vmatrix}$$

For, by § 199, 7., the third determinant is the sum of

$$\begin{vmatrix} a_1p_1 & a_1p_2 \\ b_1p_1 & b_1p_2 \end{vmatrix} (1) \quad \begin{vmatrix} a_1p_1 & a_2q_2 \\ b_1p_1 & b_2q_2 \end{vmatrix} (2) \quad \begin{vmatrix} a_2q_1 & a_1p_2 \\ b_2q_1 & b_1p_2 \end{vmatrix} (3) \quad \begin{vmatrix} a_2q_1 & a_2q_2 \\ b_2q_1 & b_2q_2 \end{vmatrix} (4)$$

But (1) and (4) are 0, by § 199, 6., and the sum of (2) and (3) is

$$p_1q_2 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + p_2q_1 \begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}$$

### EXERCISE XI

1. If the elements of  $\begin{vmatrix} a_1 & b_2 & c_3 \end{vmatrix}$  are functions of  $x$ , show that

$$\frac{d}{dx} \begin{vmatrix} a_1 & b_2 & c_3 \end{vmatrix} = \begin{vmatrix} \frac{da_1}{dx} & b_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & \frac{db_2}{dx} & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_2 & \frac{dc_3}{dx} \end{vmatrix}$$

2. Express  $\begin{vmatrix} a_1b_2c_3 \\ p_1q_2r_3 \end{vmatrix}$  as a determinant of the third order.

3. Show, by eliminating  $x, y$  as in § 202, that the set of equations

$a_1x + a_2y + a_3 = 0 \quad b_1x + b_2y + b_3 = 0 \quad c_1x + c_2y + c_3 = 0$   
has no solution unless  $\Delta = \begin{vmatrix} a_1b_2c_3 \end{vmatrix} = 0$ . If  $\Delta = 0$ , when does the set have one solution; no solution; infinitely many solutions?

4. If  $ax + by + cz = 0$ ,  $a'x + b'y + c'z = 0$ , and  $a : b : c \neq a' : b' : c'$ , show that

$$x : y : z = \begin{vmatrix} b & c \\ b' & c' \end{vmatrix} : \begin{vmatrix} c & a \\ c' & a' \end{vmatrix} : \begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$$

5. The coefficients  $A, B, C$  of the equation of the straight line determined by two points  $(x_1, y_1), (x_2, y_2)$  satisfy the equations

$$Ax + By + C = 0 \quad Ax_1 + By_1 + C = 0 \quad Ax_2 + By_2 + C = 0$$

Show that solving the second and third of these equations for  $A : B : C$  and substituting the result in the first gives

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \quad (1)$$

Show also by § 199, 4. that (1) is the equation of the line.

6. By the reasoning in § 202, 3., show that the set of equations § 202, (1) has infinitely many solutions when the matrices  $A$  and  $B$  are both of rank 2, but no solution when  $A$  is of rank 2, and  $B$  of rank 3.



## XXII. SOLID GEOMETRY

**204. Space coordinates.** Through the point  $O$  chosen as origin, take three mutually perpendicular lines as *axes of coordinates in space*,  $Ox$  and  $Oz$  in the plane of the paper, and  $Oy$  perpendicular to that plane; also take as the positive directions along  $Ox$ ,  $Oy$ ,  $Oz$  those indicated by the arrow heads in the figure.

The axes  $Ox$ ,  $Oy$ ,  $Oz$  determine three mutually perpendicular planes, the  $xy$ -,  $yz$ -, and  $xz$ -planes. They are called the *coordinate planes*.

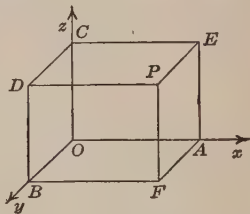


FIG. 110.

The distances of any point  $P$  from the  $yz$ -,  $xz$ -, and  $xy$ -planes are denoted by  $x$ ,  $y$ , and  $z$  respectively, and are called the *coordinates* of  $P$ .

By taking planes through  $P$  parallel to the coordinate planes, construct the parallelepiped in Fig. 110.

The points  $D$ ,  $E$ ,  $F$  are the projections of  $P$  on the  $yz$ -,  $xz$ -,  $xy$ -planes.

The points  $A$ ,  $B$ ,  $C$  are the projections of  $P$  on the  $x$ -,  $y$ -,  $z$ -axes.

$$x = DP = OA \qquad y = EP = OB \qquad z = FP = OC$$

In Fig. 110 the directions of  $DP$ ,  $EP$ ,  $FP$  are positive, and the signs of  $x$ ,  $y$ ,  $z$  therefore plus. But when  $P$  is to the left of the  $yz$ -plane, or behind the  $xz$ -plane, or below the  $xy$ -plane, the sign of  $x$ , or  $y$ , or  $z$  is minus.

Conversely, let  $a$ ,  $b$ ,  $c$  denote any given set of values of  $x$ ,  $y$ ,  $z$ . The point  $P(a, b, c)$  of which  $a$ ,  $b$ ,  $c$  are the  $x$ -,  $y$ -,  $z$ -coordinates can be got by taking  $OA = a$ , then  $AF = b$ , then  $FP = c$ .

**205. Equations in  $x$ ,  $y$ ,  $z$  and their loci.** For points in the  $xy$ -plane, and such points only, we have  $z = 0$ . Hence  $z = 0$  is called the equation of the  $xy$ -plane, and the  $xy$ -plane is called the locus of the equation  $z = 0$ . Similarly  $z = 2$  is the equation of the plane through  $(0, 0, 2)$  parallel to the

$xy$ -plane. The like is true of equations of the forms  $x = a$  and  $y = b$ .

For points on the line  $Oz$ , and such points only, we have  $x = 0, y = 0$ ; hence  $x = 0, y = 0$  are called the equations of  $Oz$ , and  $Oz$  is called the locus of  $x = 0, y = 0$ . Similarly for  $x = a, y = b$ ; and so on.

*An equation in  $x, y, z$  which is true for every point of a given set of points, and for such points only, is called the equation of the set; and the set is called the locus of the equation.*

*Similarly for a pair of simultaneous equations in  $x, y, z$ .*

Generally speaking, the locus of an algebraic equation,  $f = 0$ , in  $x, y, z$  is a surface — a plane when the equation is of the first degree, § 213; and the locus of a pair of simultaneous equations,  $f = 0, \phi = 0$ , is the curve or curves, if any, in which the surfaces  $f = 0, \phi = 0$  intersect.

The shape of a surface  $f = 0$  may be learned from its sections by planes parallel to the coordinate planes, as is illustrated in Ex. 1 on p. 158.

**EXAMPLE 1.** Let the curve  $C$  in Fig. 111 be the locus of  $f(x, y) = 0$  in the  $xy$ -plane. If through any point  $Q$  of  $C$  a line  $l$  be taken parallel to  $Oz$ , the  $x, y$  coordinates of every point  $P$  of  $l$  will be the same as those of  $Q$  and will therefore satisfy  $f(x, y) = 0$ . Hence the locus of  $f(x, y) = 0$  in space is the *cylindrical surface* through  $C$  whose elements are parallel to  $Oz$ . In space, the curve  $C$  is represented by the pair of equations  $f(x, y) = 0, z = 0$ .

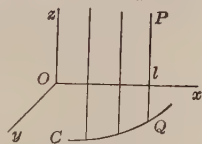


FIG. 111.

Similarly for equations of the forms  $f(y, z) = 0$  and  $f(z, x) = 0$ .

**EXAMPLE 2.** What are the loci in space of the following equations?

1.  $x - y = 0$     2.  $y^2 = x$     3.  $y + 3z = 4$     4.  $x^2 + z^2 = a^2$

**EXAMPLE 3.** The locus of  $x^2/a^2 - y^2/b^2 = z$  is a saddle shaped surface called the *hyperbolic paraboloid*, Fig. 112. Its sections by planes  $x = k$  are parabolas  $z = -y^2/b + k^2/a^2$ ,  $x = k$  which open downwards. Its sections by planes  $y = k$  are parabolas which open upwards. Its

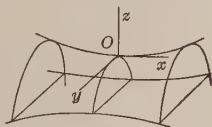


FIG. 112.

section by the plane  $z = 0$  is the pair of lines  $x/a + y/b = 0, z = 0$  (1) and  $x/a - y/b = 0, z = 0$  (2). Its sections by planes  $z = k (k \neq 0)$  are hyperbolas with asymptotes parallel to the lines (1) and (2).

EXAMPLE 4. The locus of the equation  $x^2/a^2 + y^2/b^2 = z$  is an *elliptic paraboloid*. Sketch this surface.

**206. Length and direction cosines of  $OP$ .** 1. Let  $P$  be the point  $(x, y, z)$ .

We have  $OP^2 = OF^2 + FP^2$ ,  $OF^2 = OA^2 + AF^2$

Hence  $OP^2 = x^2 + y^2 + z^2$  (1)

2. The cosines of the angles  $POx$ ,  $POy$ ,  $POz$ , which the direction  $OP$ , from  $O$  to  $P$ , makes with the positive directions on the  $x$ -,  $y$ -,  $z$ -axes, are called the *direction cosines* of  $OP$ . Let  $\alpha$ ,  $\beta$ ,  $\gamma$  denote the angles, and  $l$ ,  $m$ ,  $n$  their cosines.

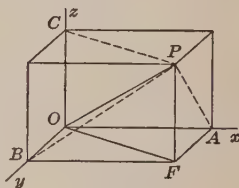


FIG. 113.

Since  $OAP$ ,  $OBP$ ,  $OCP$  are right angles, we have

$$\cos \alpha = x/OP \quad \cos \beta = y/OP \quad \cos \gamma = z/OP \quad (2)$$

Hence, by (1),

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = (x^2 + y^2 + z^2)/OP^2 = 1$$

The direction cosines  $l$ ,  $m$ ,  $n$ , of any direction  $OP$  are connected by the relation

$$l^2 + m^2 + n^2 = 1 \quad (3)$$

The direction angles of the direction  $PO$  are  $\pi - \alpha$ ,  $\pi - \beta$ ,  $\pi - \gamma$ ; hence its direction cosines are  $-l$ ,  $-m$ ,  $-n$ .

EXAMPLE 1. Find  $l$ ,  $m$ ,  $n$  for the direction from  $O$  to  $P$   $(2, 3, -6)$ .

$$OP = (4 + 9 + 36)^{1/2} = 7 \quad \therefore l = 2/7, m = 3/7, n = -6/7$$

EXAMPLE 2. Find  $l$ ,  $m$ ,  $n$  for the positive and negative directions on  $Ox$ ,  $Oy$ , and  $Oz$ .

**207. Direction cosines of a line.** Let  $L$  denote any given line. On the parallel to  $L$  through  $O$ , take  $OP$  (Fig. 113) in either of the directions along  $L$ . The  $l$ ,  $m$ ,  $n$  of this  $OP$  (or those of  $PO$ ) are called the *direction cosines* of  $L$ .

The direction cosines  $l, m, n$  of  $L$  are determined, apart from sign, when their ratios are given; hence these ratios may be called the *direction ratios* of  $L$ .

For if  $a : b : c$  are the given ratios, then  $l = ka, m = kb, n = kc$ , where  $k$  is some constant and, by § 206 (3),

$$k^2(a^2 + b^2 + c^2) = l^2 + m^2 + n^2 = 1$$

Hence  $k = 1/\pm(a^2 + b^2 + c^2)^{1/2}$ , and therefore, for one of the directions on  $L$ ,

$$l, m, n = a, b, c, \text{ each divided by } (a^2 + b^2 + c^2)^{1/2} \quad (1)$$

Thus, if  $l : m : n = 1 : 2 : 2$ , then  $l = 1/3, m = 2/3, n = 2/3$ .

**208. Projections of directed line segments.** Let  $L$  be a given line, and  $AB$  a directed segment of another line  $L'$ . Let planes taken through  $A$  and  $B$  perpendicular to  $L$  cut  $L$  at  $A_0$  and  $B_0$ . The segment  $A_0B_0$  is called the *projection of  $AB$  on  $L$* .

If positive directions be chosen on  $L$  and  $L'$ , and these directions make the angle  $\theta$ , then

$$\begin{aligned} A_0B_0 &= AB \cdot \cos \theta \\ B_0A_0 &= BA \cdot \cos \theta \end{aligned} \quad (1)$$

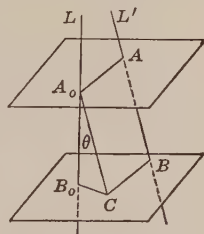


FIG. 114.

**209. Length and direction cosines of  $P'P''$ .** Let  $P'(x', y', z')$  and  $P''(x'', y'', z'')$  be two given points; the length  $s$  and the direction cosines  $l, m, n$  of the directed line segment  $P'P''$  may be found as follows:

The projection of  $P'P''$  on  $Ox$  is  $A'A'' = x'' - x'$ . Similarly, the projections of  $P'P''$  on  $Oy$  and  $Oz$  are  $y'' - y'$  and  $z'' - z'$ . Hence, by § 208 (1),

$$P'P'' \cdot l = x'' - x' \quad P'P'' \cdot m = y'' - y' \quad P'P'' \cdot n = z'' - z' \quad (1)$$

Squaring, adding, and applying § 206 (3), we get

$$P'P''^2 = (x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2$$

Hence

$$s = [(x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2]^{1/2} \quad (2)$$

$$l = (x'' - x')/s \quad m = (y'' - y')/s \quad n = (z'' - z')/s \quad (3)$$

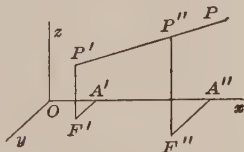


FIG. 115.

**210. Equations of a line.** By § 209 (1), if  $L$  denote a given line,  $P'$  and  $P''$  any two of its points, and  $l, m, n$  its direction cosines, then

$$(x'' - x') : (y'' - y') : (z'' - z') = l : m : n \quad (1)$$

Hence the equations of a line  $L$  through a given point  $P'$  and having given direction ratios  $a : b : c$  are

$$\frac{x - x'}{a} = \frac{y - y'}{b} = \frac{z - z'}{c} \quad (2)$$

For (2) is true for every point  $P(x, y, z)$  of  $L$  and for such points only.

**EXAMPLE 1.** Show that the equations of the line through  $P'$  and  $P''$  are

$$\frac{x - x'}{x' - x''} = \frac{y - y'}{y' - y''} = \frac{z - z'}{z' - z''} \quad (3)$$

and find these equations when  $P'$  and  $P''$  are  $(2, 3, 5)$  and  $(4, -1, 6)$ .

**EXAMPLE 2.** Representing  $P'P$  by  $r$ , show by § 209 (1) that the line through  $P'$  and having the direction cosines  $l, m, n$ , has the parametric equations

$$x = x' + lr \quad y = y' + mr \quad z = z' + nr \quad (4)$$

**211. Angle between two given directions.** Let the direction cosines of  $OP$  and  $OP'$  be  $l, m, n$  and  $l', m', n'$ ; and let  $\theta$  be the angle between  $OP$  and  $OP'$ . Make the construction in Fig. 116. Since  $O$  is joined to  $P$  by  $OP$ , and also by the broken line  $OAFP$ , the projection  $OD$  of  $OP$  on  $OP'$  equals the algebraic sum of the projections of  $OA$ ,  $AF$ ,  $FP$  on  $OP'$ . Hence, § 208 (1),

$$\begin{aligned} OP \cos \theta &= OA \cdot l' + AF \cdot m' + FP \cdot n' \\ &= OP \cdot ll' + OP \cdot mm' + OP \cdot nn' \end{aligned}$$

Therefore  $\cos \theta = ll' + mm' + nn'$  (1)

**212. Condition of perpendicularity.** Since  $\cos \pi/2 = 0$ , it follows from § 211 (1) and § 207 (1) that the condition

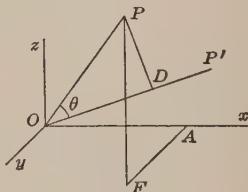


FIG. 116.

that two lines whose direction ratios are  $a : b : c$  and  $a' : b' : c'$  shall be perpendicular is that

$$aa' + bb' + cc' = 0 \quad (1)$$

Thus the lines joining  $O$  to  $P$  (2, 3, -2) and  $P'$  (3, 4, 9) are perpendicular; for  $2 \cdot 3 + 3 \cdot 4 - 2 \cdot 9 = 0$ .

**213. Equation of a plane.** 1. Let  $P'(x', y', z')$  be a known point of a given plane  $E$ , and let  $a : b : c$  be the direction ratios of the normals to  $E$ .

Take any representative point  $P(x, y, z)$  of  $E$ . By § 210 (1), the direction ratios of the line  $PP'$  are  $(x - x') : (y - y') : (z - z')$ . Therefore, since the normals to  $E$  are perpendicular to  $PP'$ , we have, § 212 (1),

$$a(x - x') + b(y - y') + c(z - z') = 0 \quad (1)$$

Since (1) is true for every point  $P(x, y, z)$  of the plane, and for such points only, it is the equation of the plane.

2. Every equation of the first degree in  $x, y, z$ ,

$$Ax + By + Cz + D = 0 \quad (2)$$

represents a plane. For let  $x', y', z'$  be any solution of (2) so that

$$Ax' + By' + Cz' + D = 0 \quad (3)$$

Subtracting (3) from (2), we get

$$A(x - x') + B(y - y') + C(z - z') = 0$$

which is of the form (1). Hence the equation (2) represents a plane, and  $A : B : C$  are the direction ratios of the normals to this plane.

**EXAMPLE 1.** Show that the plane through the point (1, 0, -2) and parallel to the plane  $3x - 8y + 2z = 0$  has the equation  $3(x - 1) - 8y + 2(z + 2) = 0$ .

**EXAMPLE 2.** Plot the plane  $12x + 10y + 15z - 60 = 0$ .

Setting  $y = 0, z = 0$ , we get  $x = 5$ ; hence the  $x$ -intercept is 5. Similarly, the  $y$ -intercept is 6, and the  $z$ -intercept 4. Hence the plane is that marked  $ABC$  in Fig. 117.

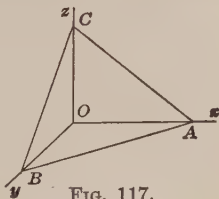


FIG. 117.

EXAMPLE 3. Plot the plane  $2x - 2y - z = 0$  by aid of its traces on the  $xy$ - and  $xz$ -planes.

EXAMPLE 4. Find the direction ratios of the line of intersection of the planes  $2x - y + z = 0$ ,  $x + y + z = 0$ .

The line is perpendicular to the normals to both planes. Hence, its  $l$ ,  $m$ ,  $n$  satisfy the equations  $2l - m + n = 0$ ,  $l + m + n = 0$ , § 212 (1), and therefore, p. 229, Ex. 4,

$$l : m : n = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} : \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} : \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = -2 : -1 : 3$$

### EXERCISE XLI

1. Show that the equation of a sphere with center  $(a, b, c)$  and radius  $r$  is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

2. Find the center and radius of the sphere

$$x^2 + y^2 + z^2 - 4x + 5z = 0.$$

3. Find the length and direction cosines of the vector from the origin to the point  $(3, -4, 12)$ ; also the lengths of its projections on the  $yz$ -,  $xz$ -, and  $xy$ -planes.

4. Find the cosine of the angle between two lines whose direction ratios are  $1 : 2 : 2$  and  $3 : 0 : 4$ .

5. Find the  $l$ ,  $m$ ,  $n$  of a line perpendicular to two lines whose direction ratios are  $2 : -1 : 1$  and  $1 : 2 : 3$ . If the line passes through  $(0, 1, -1)$ , what are its equations.

6. Find the length of the projection of the vector from  $(0, 4, 1)$  to  $(1, 2, -1)$  upon a line whose direction ratios are  $2 : 3 : -6$ ; upon the plane  $7x - 2y + 4z = 5$ .

7. Find the equation of the plane through  $(1, 0, -2)$  and perpendicular to the line of intersection of the planes  $3x + 4y + 7z = 0$ , and  $x - y + 2z = 0$ .

8. What determinant equated to 0 represents the plane determined by three given points  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$ ,  $(x_3, y_3, z_3)$ ?

9. Eliminating  $z$  between  $x^2 + y^2 + z^2 = 4$  (1) and  $z = y$  (2) gives  $x^2 + 2y^2 = 4$  (3). Show that (3) and  $z = 0$  are the equations of the projection on the  $xy$ -plane of the circle in which the plane (2) cuts the sphere (1).

10. Show that the equation of the conical surface got by revolving the line  $y = 2x$ ,  $z = 0$  about  $Ox$  may be found by replacing  $y$  by  $\sqrt{y^2 + z^2}$  in  $y = 2x$  and then rationalizing, which gives

$$y^2 + z^2 - 4x^2 = 0.$$



**11.** Show that if  $x_1, y_1, z_1$  is any real solution of a homogeneous equation in  $x, y, z$ , as  $x^2 - xy + 3z^2 = 0$  (1), so also is  $kx_1, ky_1, kz_1$ ; hence that the line through  $O$  and the point  $(x_1, y_1, z_1)$  lies wholly in the surface (1), which is therefore a conical surface.

**12.** Through  $O'$  ( $a, b, c$ ), and parallel to  $Ox, Oy, Oz$  (1), take  $O'x', O'y', O'z'$  (2). Show that if the coordinates of any point  $P$  referred to the axes (1) and (2) are  $x, y, z$  and  $x', y', z'$ , then

$$x = x' + a \qquad y = y' + b \qquad z = z' + c \qquad (1)$$

**13.** Let  $Ox, Oy, Oz$  (1) and  $Ox', Oy', Oz'$  (2) be two sets of rectangular axes through  $O$ , and let  $(x, x')$  denote the angle between  $Ox$  and  $Ox'$ , and so on. By the method of projection used in § 211, show that if the coordinates of any point  $P$  referred to (1) and (2) are  $x, y, z$  and  $x', y', z'$ , then

$$\begin{aligned} x' &= x \cos(x, x') + y \cos(y, x') + z \cos(z, x') \\ y' &= x \cos(x, y') + y \cos(y, y') + z \cos(z, y') \\ z' &= x \cos(x, z') + y \cos(y, z') + z \cos(z, z') \end{aligned} \qquad (2)$$

What are the corresponding expressions for  $x, y, z$  in terms of  $x', y', z'$ ?

**14.** By aid of the formulas in Exs. 12, 13, justify the footnote respecting centroids on p. 178.

**15.** By § 203, prove that the square of the determinant of the coefficients of  $x, y, z$  in the equations (2) of Ex. 13 equals 1.

### XXIII. PARTIAL DIFFERENTIATION

#### 214. Continuous functions of two or more variables.

1. In the case of any number of independent variables  $x, y, \dots$ , as when the number is one, two, or three, we call any particular set of values of the variables, as  $(a, b, \dots)$ , a *point*.

2. The set of all points  $(x, y, \dots)$  got by giving  $x$  any value in a certain number interval  $(\alpha_1, \beta_1)$ ,  $y$  any value in a second interval  $(\alpha_2, \beta_2)$ , and so on, may be called a *region*  $R$ .

3. Let  $f(x, y, \dots)$  denote a *one-valued function* of  $x, y, \dots$  in  $R$ , so that to each point of  $R$  corresponds a single real value of  $f(x, y, \dots)$ .

4. We say that  $f(x, y, \dots)$  is *continuous*<sup>1</sup> at the point  $(a, b, \dots)$  of  $R$ , if  $f(a, b, \dots)$  is finite, and if

$$f(x, y, \dots) \rightarrow f(a, b, \dots) \text{ when } (x, y, \dots) \rightarrow (a, b, \dots) \text{ in } R$$

5. We say that  $f(x, y, \dots)$  is *continuous in*  $R$  when it is continuous at all points of  $R$ , its boundary included.

6. We say that  $f(x, y, \dots)$  is *continuous in the neighborhood*<sup>2</sup> of the point  $(a, b, \dots)$  if it is continuous in a region  $S$  consisting of all points  $(x, y, \dots)$  for which  $|x - a|, |y - b|, \dots$  are each less than some positive number  $d$ .

<sup>1</sup> Obviously a continuous function of two or more independent variables is a continuous function of the variables taken separately. But the converse is not always true. Thus, the function  $f(x, y)$ , defined as equal to  $xy/(x^2 + y^2)$  when  $x, y$  are not both 0, and as 0 when  $x, y$  are both 0, is a continuous function of either variable when the other has any constant value, 0 included. But  $f(x, y)$  is not a continuous function of  $x, y$  at 0. If, for example,  $(x, y) \rightarrow 0$  on the line  $y = 2x$ , then  $f(x, y) \rightarrow 2/5$ , and not 0.

<sup>2</sup> A function of one or more variables may be continuous at a point but not continuous in the neighborhood of the point. Thus the function  $f(x)$  which is defined as equal to  $x$  when  $x$  is irrational, but as equal to 0 when  $x$  is rational, is continuous at 0, but not in its neighborhood.

**215. Partial derivatives.** The result of differentiating  $u = f(x, y)$  with respect to  $x$ , treating  $y$  as a constant, is called the *partial derivative of  $u$  with respect to  $x$*  and is denoted by one of the symbols  $\partial u / \partial x$ ,  $\partial f / \partial x$ , or  $f_x(x, y)$ . Hence, by definition,

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

and  $\partial f / \partial x$  exists at any point  $(x, y)$  when and only when this limit exists at that point.

The partial derivative of  $f(x, y)$  with respect to  $y$  is similarly defined; so also are the partial derivatives of  $f(x, y, z)$  with respect to  $x, y, z$ ; and so on.

**216. Total increments and differentials.** 1. Let  $f(x, y)$ ,  $f_x$ , and  $f_y$  be continuous at the point  $(x, y)$  and in its neighborhood, that is, in a region  $S$  about  $(x, y)$  of the kind described in § 214, 6. Give  $x, y$  any increments  $\Delta x, \Delta y$  such that the point  $(x + \Delta x, y + \Delta y)$  also is in  $S$ . Then  $u = f(x, y)$  receives the increment

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y) \quad (1)$$

Subtract and add  $f(x, y + \Delta y)$ ; we obtain

$$\begin{aligned} \Delta u &= [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)] \\ &\quad + [f(x, y + \Delta y) - f(x, y)] \end{aligned} \quad (2)$$

The first bracketed expression is the increment which  $f(x, y + \Delta y)$  receives when  $x$  is given the increment  $\Delta x$ . Hence, by the mean value theorem, § 97 (6),

$$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) = f_x(x + \theta_1 \Delta x, y + \Delta y) \Delta x \quad (3)$$

where  $\theta_1$  is some number between 0 and 1. Similarly

$$f(x, y + \Delta y) - f(x, y) = f_y(x, y + \theta_2 \Delta y) \Delta y \quad 0 < \theta_2 < 1 \quad (4)$$

Substituting (3) and (4) in (2), we obtain

$$\Delta u = f_x(x + \theta_1 \Delta x, y + \Delta y) \Delta x + f_y(x, y + \theta_2 \Delta y) \Delta y \quad (5)$$

Since  $f_x$  and  $f_y$  are continuous in  $S$ , the coefficients of  $\Delta x$  and  $\Delta y$  in (5)  $\rightarrow f_x(x, y)$  and  $f_y(x, y)$  when  $\Delta x, \Delta y \rightarrow 0$ , and

therefore differ from  $f_x(x, y)$  and  $f_y(x, y)$  by infinitesimals  $\epsilon_1, \epsilon_2$ , which  $\rightarrow 0$  when  $\Delta x, \Delta y \rightarrow 0$ . Hence

$$\Delta u = \left[ \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y \right] + [\epsilon_1 \Delta x + \epsilon_2 \Delta y] \quad (6)$$

Ordinarily  $[\epsilon_1 \Delta x + \epsilon_2 \Delta y]/[f_x \Delta x + f_y \Delta y] \rightarrow 0$  when  $\Delta x, \Delta y \rightarrow 0$ , that is,  $f_x \Delta x + f_y \Delta y$  is the *principal part* of  $\Delta u$ , and represents  $\Delta u$  approximately for small values of  $\Delta x, \Delta y$ . It is called the *total differential* of  $u$  and is denoted by  $du$ ; so that, by definition,

$$du = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y \quad (7)$$

When  $u = x$ , (7) becomes  $dx = \Delta x$ ; and when  $u = y$ , it becomes  $dy = \Delta y$ . Hence (7) implies that the total differentials of the independent variables  $x, y$  are the same as their increments, and it may also be written

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad (8)$$

2. We can deal in a similar manner with the increment of a function of more than two variables. Thus if the function  $u = f(x, y, z)$ , and  $f_x, f_y, f_z$  are continuous at and in the neighborhood of the point  $(x, y, z)$ , we can reduce  $\Delta u$  to a form, like (6), whose principal part, the differential of  $u$ , is given by the formula

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \quad (9)$$

**EXAMPLE 1.** If  $u = x^2 y^3 z$ , then  $du = 2xy^3z dx + 3x^2y^2z dy + x^2y^3 dz$ .

**EXAMPLE 2.** A rectangle whose sides have the lengths  $x, y$  has the area  $u = xy$ . Hence, by (8), the change in the area due to small changes  $dx, dy$ , in the lengths of the sides is approximately  $du = y dx + x dy$ . Or, if  $x, y$  denote the measurements made of the sides, and  $dx, dy$  the errors in the measuring,  $du$  is the approximate error in the area. Thus, if the measurements are  $x, y = 5, 6$  with possible errors  $dx = dy = \pm .01$ , the approximate maximum error is  $du = 6(.01) + 5(.01) = .11$ .

**EXAMPLE 3.** Find the total differentials of  $y/x, xy/z$ , and  $\sin xyz$ .

**EXAMPLE 4.** What approximately is the change in  $(x^2 + y^2)^{1/2}$  when  $x, y$  change from 3, 4, to 3.01, 3.99?

**EXAMPLE 5.** A steamship is supposed to have made 40 miles in 2 hours. If the estimates of the distance and time may have been in error by 100 ft. and 2 min., how much may the actual average speed have differed from 20 m./h.?

**EXAMPLE 6.** If small changes be made in the lengths  $x, y$  of two sides of a triangle and in the included angle  $\theta$ , what approximately is the corresponding change in the area?

**217. Derivatives of composite functions.** Let the variables  $x, y$  in  $u = f(x, y)$  be differentiable functions of another variable  $t$ , and let the increments  $\Delta x, \Delta y$  in § 216 be those which correspond to the increment  $\Delta t$  of  $t$ . Dividing both members of § 216 (5) by  $\Delta t$ , we have

$$\frac{\Delta u}{\Delta t} = f_x(x + \theta_1 \Delta x, y + \Delta y) \frac{\Delta x}{\Delta t} + f_y(x, y + \theta_2 \Delta y) \frac{\Delta y}{\Delta t}$$

and therefore, making  $\Delta t \rightarrow 0$ ,

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} \quad (1)$$

Similarly, if  $x, y$  are functions of  $t$  and other variables,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} \quad (2)$$

In like manner if  $u = f(x, y, z)$ , we have

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt} \quad (3)$$

and the corresponding formula for  $\partial u / \partial t$ ; and so on.

**EXAMPLE.** If  $u = ax + by$ , and  $x = a_1x' + b_1y'$ ,  $y = a_2x' + b_2y'$ , then  $\frac{\partial u}{\partial x'} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial x'} = aa_1 + ba_2$ . Similarly,  $\frac{\partial u}{\partial y'} = ab_1 + bb_2$ .

### EXERCISE XLII

1. If  $u = x(y - z)$ , and  $x = r + 2s$ ,  $y = s - t$ ,  $z = t$ , find

$$\frac{\partial u}{\partial r}, \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}.$$

2. If  $u = f(x, y, z)$ ,  $x = a_1x' + b_1y' + c_1z'$ ,  $y = a_2x' + b_2y' + c_2z'$ , and  $z = a_3x' + b_3y' + c_3z'$ , find  $\partial u / \partial x'$ ,  $\partial u / \partial y'$ ,  $\partial u / \partial z'$ .

3. Let  $u$  be a function of  $x, y$ . If  $x = r \cos \theta$  and  $y = r \sin \theta$ , find  $\partial u / \partial r$  and  $\partial u / \partial \theta$ , and show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial \theta}\right)^2$$

4. If the edges  $x, y, z$  of a rectangular parallelepiped are increasing at the rates 2, 3, 4 in./sec., at what rate is the volume increasing when  $x, y, z = 5, 6, 7$ ?

5. Representing the derivative of  $f(ax + by)$  with respect to  $ax + by$  by  $f'(ax + by)$ , show that

$$\frac{\partial f}{\partial x} = af', \quad \frac{\partial f}{\partial y} = bf', \quad \text{and therefore } b \frac{\partial f}{\partial x} = a \frac{\partial f}{\partial y}$$

6. Let  $u$  and  $v$  be functions of  $x, y$ . Show that if  $v = f(u)$ , then

$$\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} = 0$$

7. Show that if  $x, y, z$  are independent of  $t$ , then

$$\frac{d}{dt} f(x, ty, t^2 z) = y f_{(y)}(x, ty, t^2 z) + 2 t z f_{(x z)}(x, ty, t^2 z).$$

8. A function  $f(x, y, z)$  is said to be *homogeneous* with respect to  $x, y, z$  if it satisfies an identity of the form

$$f(tx, ty, tz) \equiv t^n f(x, y, z) \quad (1)$$

Thus  $f(x, y, z) = (x^3 + y^3)/z$  is homogeneous;  $f(tx, ty, tz) = t^2 f(x, y, z)$ .

By differentiating (1) with respect to  $t$  and setting  $t = 1$  in the result, show that

$$nf(x, y, z) \equiv x f_x(x, y, z) + y f_y(x, y, z) + z f_z(x, y, z) \quad (2)$$

This identity is called *Euler's theorem* respecting homogeneous functions. The definition and theorem apply to functions of any number of variables.

9. Show that the following functions are homogeneous and verify Euler's theorem for them:

$$1. x \sin^{-1}(y^2/x^2) \quad 2. y + x(\log x - \log y) \quad 3. xyz/(x + y + z)$$

$$10. \text{ If } z = \frac{1}{x} f\left(\frac{y}{x}\right), \text{ show that } x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} + z = 0$$

**218. Differentials of composite functions.** 1. Multiplying § 217 (1) by  $dt$  gives

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

which is the formula (8) of § 216. Hence this formula continues to hold good when, instead of being independent

variables,  $x, y$  are differentiable functions of another variable. The like is true when they are functions of two or more other variables.

Thus let  $u = f(x, y)$ , where  $x = \phi(s, t)$ ,  $y = \psi(s, t)$ , and  $f_x, f_y, \phi_s, \phi_t, \psi_s, \psi_t$  are continuous. Since  $u$  is a function of  $s, t$ , and  $s, t$  are independent variables,

$$du = \frac{\partial u}{\partial s} ds + \frac{\partial u}{\partial t} dt \quad (1)$$

$$dx = \frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt \quad (2) \quad dy = \frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt \quad (3)$$

Multiply (2) by  $\partial u / \partial x$  and (3) by  $\partial u / \partial y$ , and add; we get

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \left( \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} \right) ds + \left( \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} \right) dt \quad (4)$$

By § 217 (2), the coefficients of  $ds$  and  $dt$  in (4) are  $\partial u / \partial s$  and  $\partial u / \partial t$ . Hence, by (1), the right member of (4), and therefore the left, equals  $du$ ; that is,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad (5)$$

It also follows from this proof that in any given case we can find the equation (1) by substituting in (5) the values of  $dx$  and  $dy$  given by (2) and (3).

EXAMPLE 1. If  $u = 3x + 4y$ ,  $x = 2s - t$ ,  $y = 5t$ , find  $du$  in terms of  $ds, dt$ .

$$du = 3dx + 4dy = 3(2ds - dt) + 4(5dt) = 6ds + 17dt.$$

Hence  $\partial u / \partial s = 6$ ,  $\partial u / \partial t = 17$ . Verify this by using § 217 (2).

2. By similar reasoning, we can prove the general theorem:

If  $u = f(x, y, \dots)$ , then  $du = f_x dx + f_y dy + \dots$ , whether  $x, y, \dots$  are independent variables or functions of other variables, provided that all the partial derivatives involved exist and are continuous.

In particular, the formulas of differentiation, as

$$df(u) = f'(u) du \quad d(uv) = v du + u dv$$

hold good when  $u, v$  are independent or are functions of other variables.

$$\text{EXAMPLE 2. } d \log(x^2 + y) = \frac{d(x^2 + y)}{x^2 + y} = \frac{2x dx + dy}{x^2 + y} = \frac{2x dx}{x^2 + y} + \frac{dy}{x^2 + y}$$



3. When  $x, y, \dots$  are independent variables, and it has been shown in any way respecting a certain function  $f(x, y, \dots)$  that  $df = P(x, y, \dots) dx + Q(x, y, \dots) dy + \dots$ , it may be inferred that  $P = f_x, Q = f_y, \dots$ .

For we also have  $df = f_x dx + f_y dy + \dots$ . Hence

$$P dx + Q dy + \dots = f_x dx + f_y dy + \dots$$

Since  $x, y, \dots$  are independent,  $dx, dy, \dots$  are arbitrary. Hence we may equate all of them except  $dx$  to 0, which gives  $P dx = f_x dx$ , and therefore  $P = f_x$ ; and so on.

Thus, from Ex. 2 it follows for  $u = \log(x^2 + y)$  that

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y}, \quad \frac{\partial u}{\partial y} = \frac{1}{x^2 + y}.$$

### EXERCISE XLIII

1. From  $de^{xv^2} = e^{xv^2}d(xy^2)$ , find  $\partial e^{xv^2}/\partial x$  and  $\partial e^{xv^2}/\partial y$ .
2. Find the partial derivatives of  $\tan^{-1}(y/x)$  and  $\sin(2x - y + 3z)$ .
3. Find the total differentials of each of the following:
  1.  $(x^2 + y^2)^{1/2}$
  2.  $(x + y)/(x + z)$
  3.  $e^x \sin(y + 2z)$
4. If  $u = x^2 - 3y^2$ ,  $x = 4s + 5t$ ,  $y = t^2$ , find  $du$  in terms of  $ds, dt$ .
5. If  $u = xy/z$ ,  $x = 2 \cos \theta$ ,  $y = \sin \theta$ , and  $z = 3\phi$ , find  $du$  in terms of  $d\theta, d\phi$ .
6. Show that if  $x = r \cos \theta$ ,  $y = r \sin \theta$ , then  $x dy - y dx = r^2 d\theta$ .

**219. Partial derivatives of the second order.** The partial derivatives of  $\partial f/\partial x$ ,  $\partial f/\partial y$  are called the partial derivatives of the second order of  $f(x, y)$ .

They are represented as follows:

$$\begin{array}{ll} \frac{\partial}{\partial x} \frac{\partial f}{\partial x}, \text{ by } \frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}(x, y) & \frac{\partial}{\partial y} \frac{\partial f}{\partial x}, \text{ by } \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{yx}(x, y) \\ \frac{\partial}{\partial x} \frac{\partial f}{\partial y}, \text{ by } \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{xy}(x, y) & \frac{\partial}{\partial y} \frac{\partial f}{\partial y}, \text{ by } \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}(x, y) \end{array}$$

EXAMPLE. If  $u = x^2 y^3$ , show that

$$\frac{\partial^2 u}{\partial x \partial y} = 6xy^2, \quad \frac{\partial^2 u}{\partial y \partial x} = 6xy^2 \quad \therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

**220. Theorem.** *If  $f_x, f_y, f_{xy}, f_{yx}$  are continuous at the point  $(x, y)$  and in its neighborhood, then*

$$f_{xy}(x, y) = f_{yx}(x, y)$$

By hypothesis,  $f_x, f_y, f_{xy}, f_{yx}$  are continuous in a region  $S$  about  $(x, y)$  of the kind described in § 214, 6. Suppose that  $h, k$  are such that the point  $(x + h, y + k)$  belongs to  $S$ , and consider the expression

$$U = f(x + h, y + k) - f(x + h, y) - f(x, y + k) + f(x, y) \quad (1)$$

For convenience, set  $F(y) = f(x + h, y) - f(x, y)$  (2)  
we then have

$$\begin{aligned} U &= F(y + k) - F(y) \\ &= F_y(y + \theta_1 k)k \quad [\text{by § 97 (6)}] \\ &= [f_y(x + h, y + \theta_1 k) - f_y(x, y + \theta_1 k)]k \\ &= f_{xy}(x + \theta_2 h, y + \theta_1 k)hk \quad [\text{by § 97 (6)}] \quad (3) \end{aligned}$$

Similarly, by setting  $G(x) = f(x, y + k) - f(x, y)$ , we find

$$U = f_{yx}(x + \theta_3 h, y + \theta_4 k)kh \quad (4)$$

By (3) and (4), we have

$$f_{xy}(x + \theta_2 h, y + \theta_1 k) = f_{yx}(x + \theta_3 h, y + \theta_4 k) \quad (5)$$

and since  $f_{xy}, f_{yx}$  are continuous, it follows from (5), when  $h, k \rightarrow 0$ , that

$$f_{xy}(x, y) = f_{yx}(x, y) \quad (6)$$

**221. Partial derivatives of higher order.** It follows from § 220 that all the partial derivatives that can be got from  $u = f(x, y)$  by differentiating  $m$  times with respect to  $x$  and  $n$  times with respect to  $y$  in any order are equal, provided that all the partial derivatives involved in the process are continuous. Hence they may be represented by the same symbol. Using the symbols of operation  $\partial/\partial x, \partial/\partial y$ , we write

$$\left(\frac{\partial}{\partial x}\right)^m \left(\frac{\partial}{\partial y}\right)^n u = \frac{\partial^{m+n}}{\partial x^m \partial y^n} u = \frac{\partial^{m+n} u}{\partial x^m \partial y^n}$$

The like is true of the partial derivatives of  $u = f(x, y, z)$ .

### EXERCISE XLIV

1. Verify the theorem of § 220 for  $u = x^2 \sin 2y$  and  $u = xy^2z^3$ .
2. Show that if  $u = \log(x^2 + y^2)^{1/2}$ , then  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ .
3. Show that if  $u = (x^2 + y^2 + z^2)^{-1/2}$ , then  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ .

4. If  $u = x^2 - y^2$  and  $x = 3s + 2t$ ,  $y = -s + 4t$ , find  $\partial^2 u / \partial s^2$ , and  $\partial^2 u / \partial t^2$ .

5. Show that if  $u = f(y + ax) + \phi(y - ax)$ , then  $\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial y^2}$ .

6. If  $f(x, y)$  is homogeneous and of degree  $n$ , show that

$$n(n-1)f(x, y) = f_{xx}x^2 + 2f_{xy}xy + f_{yy}y^2$$

**222. Taylor's theorem.** By the following method, Taylor's theorem may be extended to functions of two or more variables.

Suppose that  $f(x, y)$  and its partial derivatives to those of order  $n$  are continuous in the rectangle  $R$  bounded by  $x = a$ ,  $x = a + h$ ,  $y = b$ ,  $y = b + k$ . It is required to develop  $f(a + h, b + k)$  in powers of  $h, k$ .

Let  $t$  denote a variable whose range is from 0 to 1, and set

$$F(t) = f(a + ht, b + kt) \quad (1)$$

From the continuity of  $f(x, y)$  and its partial derivatives it follows that  $F(t)$ ,  $F'(t)$ ,  $\dots$ ,  $F^{(n)}(t)$  are continuous in the  $t$ -interval  $(0, 1)$ . Hence we may expand  $F(t)$  in powers of  $t$  to the  $n$ th by Maclaurin's formula, § 188 (9), and then set  $t = 1$ ; which gives

$$F(1) = F(0) + F'(0) + \frac{F''(0)}{2!} + \dots + \frac{F^{(n-1)}(0)}{(n-1)!} + \frac{F^{(n)}(\theta)}{n!} \quad 0 < \theta < 1 \quad (2)$$

By (1), we have  $F(1) = f(a + h, b + k)$  and  $F(0) = f(a, b)$ .

Differentiating (1) with respect to  $t$ , and then setting  $t = 0$ , gives

$$F'(0) = \frac{\partial}{\partial a} f(a, b)h + \frac{\partial}{\partial b} f(a, b)k = \left( \frac{\partial}{\partial a} h + \frac{\partial}{\partial b} k \right) f(a, b) \quad (3)$$

Dealing similarly with  $F'(t)$  gives

$$F''(0) = \frac{\partial^2}{\partial a^2} f(a, b)h^2 + 2 \frac{\partial^2}{\partial a \partial b} f(a, b)hk + \frac{\partial^2}{\partial b^2} f(a, b)k^2 \quad (4)$$

which may be expressed symbolically in the form

$$F''(0) = \left( \frac{\partial}{\partial a} h + \frac{\partial}{\partial b} k \right)^2 f(a, b) \quad (5)$$

and so on. Hence, by substitution in (2),

$$\begin{aligned} f(a + h, b + k) &= f(a, b) + \left( \frac{\partial}{\partial a} h + \frac{\partial}{\partial b} k \right) f(a, b) + \dots \\ &+ \frac{1}{(n-1)!} \left( \frac{\partial}{\partial a} h + \frac{\partial}{\partial b} k \right)^{n-1} f(a, b) + \frac{1}{n!} \left( \frac{\partial}{\partial a} h + \frac{\partial}{\partial b} k \right)^n f(a + \theta h, b + \theta k) \end{aligned} \quad (6)$$

the terms after the first being homogeneous polynomials of degrees 1, 2,  $\dots$  in  $h, k$ .

EXAMPLE. The curve  $x^2y^3 = 4$  passes through the point  $O'(2, 1)$ . By aid of Taylor's theorem, find the equation of the curve referred to axes  $O'x', O'y'$  parallel to  $Ox, Oy$ . See page 237, Ex. 12.

**223. Mean value theorem for  $f(x, y)$ .** 1. In the formula § 222 (6), take  $n = 1$ , and transpose the term  $f(a, b)$ ; we get

$$f(a + h, b + k) - f(a, b) = f_x(a + \theta h, b + \theta k)h + f_y(a + \theta h, b + \theta k)k$$

or, replacing  $a + h, b + k$  by  $x, y$ , and  $a + \theta h, b + \theta k$  by  $x_1, y_1$ ,

$$f(x, y) - f(a, b) = f_x(x_1, y_1)(x - a) + f_y(x_1, y_1)(y - b) \quad (1)$$

where  $(x_1, y_1)$  is some point on the line segment joining the points  $(x, y)$  and  $(a, b)$ .

In like manner, if  $f(x, y, z)$ ,  $f_x, f_y, f_z$  are continuous in the parallelepiped bounded by planes taken through the points  $(a, b, c)$  and  $(x, y, z)$  parallel to the  $yz$ -,  $zx$ -, and  $xy$ -planes, there is a point  $(x_1, y_1, z_1)$  on the line segment joining  $(a, b, c)$  and  $(x, y, z)$  such that

$$f(x, y, z) - f(a, b, c) = f_x(x_1, y_1, z_1)(x - a) + f_y(x_1, y_1, z_1)(y - b) + f_z(x_1, y_1, z_1)(z - c) \quad (2)$$

2. It follows from (1) that if  $f_x$  and  $f_y$  are 0 throughout the rectangle  $R$  of § 222, then  $f(x, y)$  is a constant in  $R$ .

Hence two functions  $\phi(x, y)$  and  $\psi(x, y)$  differ by a constant only if  $\phi_x = \psi_x$  and  $\phi_y = \psi_y$  throughout  $R$ . For if we set  $f(x, y) = \phi(x) - \psi(x)$ , then  $f_x$  and  $f_y$  are 0 throughout  $R$ .

The corresponding theorems for functions of  $x, y, z$  follow from (2).

### IMPLICIT FUNCTIONS

**224. Definition of an implicit function.** Let  $f(x, y) = 0$  be a given equation in  $x, y$ , and let  $(a, b)$  be a point of its graph, so that  $f(a, b)$  is 0. Suppose that a rectangle  $R$  bounded by lines

$$x = a - k \quad x = a + k \quad y = b - l \quad y = b + l$$

can be found such that the portion of the graph of  $f(x, y) = 0$  within  $R$  is a curve arc  $C$  which is met by each parallel to  $Oy$

that crosses  $R$  in one and but one point. Then to each value  $x'$  of  $x$  between  $a - k$  and  $a + k$  corresponds a single value  $y'$  of  $y$  between  $b - l$  and  $b + l$  such that  $f(x', y')$  is 0. Hence in  $R$  the equation  $f(x, y) = 0$  defines  $y$  as a one-valued function of  $x$ , § 14. Let  $y = \phi(x)$  denote this function. It is said to be *defined implicitly* by the equation  $f(x, y) = 0$  and the point  $(a, b)$ . It may also be called the solution of  $f(x, y) = 0$  for  $y$  in terms of  $x$  at  $(a, b)$  and in its neighborhood; for  $f[x, \phi(x)] \equiv 0$ .

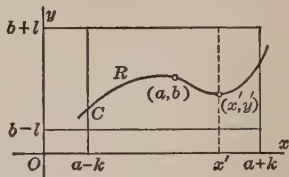


FIG. 118.

**225. Theorem.** *Let  $f(x, y)$ ,  $f_x$ ,  $f_y$  be continuous at the point  $P(a, b)$  and in its neighborhood. If  $f$  is 0 and  $f_y \neq 0$  at  $P$ , the equation  $f(x, y) = 0$  has a solution  $y = \phi(x)$  at  $P$  and in its neighborhood, and but one such solution.*

*Furthermore  $\phi'(x) = -f_x(x, y)/f_y(x, y)$ ; and  $\phi'(x)$  is continuous at  $P$  and in its neighborhood.*

1. By hypothesis,  $f$ ,  $f_x$ ,  $f_y$  are continuous in some region  $S$  about  $P$ , § 214, 6. We are to prove that in  $S$  there exists a rectangle  $R$ , of the kind described in § 224, in which the graph of  $f(x, y) = 0$  is met by parallels to  $Oy$  in single points.

First consider the  $y$ -function  $f(a, y)$ . It is continuous in  $S$ . It vanishes at  $y = b$  (since  $f(a, b) = 0$ ), and there changes sign (since its derivative at  $y = b$ , namely  $f_y(a, b)$ , is not 0). Hence, § 31, a positive number  $l$  can be found such that  $f(a, b - l)$  and  $f(a, b + l)$  have opposite signs.

Next consider the  $x$ -functions  $f(x, b - l)$  and  $f(x, b + l)$ . They are continuous in  $S$  and therefore, for values of  $x$  sufficiently near  $a$ , they have the same signs as  $f(a, b - l)$  and  $f(a, b + l)$  respectively (p. 22, footnote). Hence a positive number  $k$  can be found such that

$$\operatorname{sgn} f(x, b - l) = -\operatorname{sgn} f(x, b + l), \text{ when } |x - a| < k \quad (1)$$

Moreover  $f_y(x, y)$  is continuous in  $S$ . Therefore, since it is different from 0 at  $(a, b)$ , it is different from 0 at all points  $(x, y)$  sufficiently near  $(a, b)$ , that is, we may and shall take  $k, l$  so that they also satisfy the condition

$$f_y \neq 0 \text{ when } |x - a| < k, |y - b| < l \quad (2)$$

Finally, let  $x'$  be any value of  $x$  between  $a - k$  and  $a + k$ , and consider the  $y$ -function  $f(x', y)$ . By (1), it has opposite signs for  $y = b - l$  and  $y = b + l$ . It therefore vanishes for some value  $y'$  of  $y$  between  $b - l$  and  $b + l$ , § 18, 2. And there can be but one such value  $y'$  since, were there more than one,  $f_y(x', y)$  would vanish for some value of  $y$  between them, § 96, which is impossible, by (2).

We have thus proved that in the rectangle  $R$  bounded by the lines  $x = a - k$ ,  $x = a + k$ ,  $y = b - l$ ,  $y = b + l$ , to each value  $x'$  of  $x$  corresponds a single value  $y'$  of  $y$  such that  $f(x', y')$  is 0. Hence in  $R$  the equation  $f(x, y) = 0$  defines  $y$  as a one-valued function of  $x$ . This function  $y = \phi(x)$  is the solution, and the only solution, of the equation  $f(x, y) = 0$  for  $y$  in terms of  $x$  in  $R$ .

2. It remains to prove that

$$\phi'(x) = -f_x(x, y)/f_y(x, y)$$

Let  $(x', y')$  and  $(x' + \Delta x, y' + \Delta y)$  be points in  $R$  such that  $f(x', y') = 0$  and  $f(x' + \Delta x, y' + \Delta y) = 0$ , and therefore such that  $y' = \phi(x')$ , and  $y' + \Delta y = \phi(x' + \Delta x)$ .

Since  $f(x' + \Delta x, y' + \Delta y) - f(x', y') = 0$ , it follows from § 223 that

$$f_x(x' + \theta \Delta x, y' + \theta \Delta y) \Delta x + f_y(x' + \theta \Delta x, y' + \theta \Delta y) \Delta y = 0$$

Therefore, since  $f_x$  is continuous, and  $f_y$  continuous and  $\neq 0$ , in  $R$ ,

$$\frac{\Delta y}{\Delta x} = -\frac{f_x(x' + \theta \Delta x, y' + \theta \Delta y)}{f_y(x' + \theta \Delta x, y' + \theta \Delta y)} \quad \therefore \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = -\frac{f_x(x', y')}{f_y(x', y')}$$

But  $\lim_{\Delta x \rightarrow 0} (\Delta y / \Delta x) = \phi'(x')$ . Hence  $\phi'(x') = -f_x(x', y') / f_y(x', y')$ .

Moreover  $\phi'(x)$  is continuous in  $R$ ; for  $f_x, f_y$ , and  $\phi(x)$  are continuous, and  $\phi'(x) = -f_x[x, \phi(x)] / f_y[x, \phi(x)]$ .

By similar reasoning we can prove the following more general theorem.

**226. Theorem.** Let  $f(u, x, y, \dots), f_u, f_x, f_y \dots$  be continuous at the point  $P(u_1, a, b, \dots)$  and in its neighborhood. If  $f$  is 0 and  $f_u \neq 0$  at  $P$ , then in a certain region <sup>1</sup>  $R$  about  $P$  the equation  $f(u, x, y, \dots) = 0$  defines  $u$  as a one-valued function of  $x, y, \dots$ . This function  $u = \phi(x, y, \dots)$  is the solution and the only solution of  $f(u, x, y, \dots) = 0$  for  $u$  in terms of  $x, y, \dots$  in the region  $R$ .

Furthermore,  $\phi_x(x, y, \dots) = -f_x(u, x, y, \dots)/f_u(u, x, y, \dots)$ , and so on; and  $\phi_x, \phi_y, \dots$  are continuous in  $R$ .

**227. Differentiation of implicit functions.** 1. The function  $y = \phi(x)$  defined by  $f(x, y) = 0$ , at any point  $P(x, y)$  where  $f_y \neq 0$ , has finite derivatives  $dy/dx, \dots, d^n y/dx^n$  at  $P$  if  $f$  has finite partial derivatives to those of order  $n$  at  $P$ .

For it has already been proved that  $dy/dx = -f_x/f_y$ . Hence

$$\frac{d^2 y}{dx^2} = - \left[ \frac{\partial}{\partial x} \frac{f_x}{f_y} + \frac{\partial}{\partial y} \frac{f_x}{f_y} \cdot \frac{dy}{dx} \right] = - \frac{f_{xx}(f_y)^2 - 2f_{xy}f_x f_y + f_{yy}(f_x)^2}{(f_y)^3} \quad (1)$$

and repetitions of this process will give  $d^n y/dx^n$  in the form of a fraction whose numerator is an integral expression in partial derivatives of  $f$  to those of order  $n$  and whose denominator is some power of  $f_y$ .

Having proved that  $dy/dx, d^2 y/dx^2, \dots$  exist under the conditions just indicated, we may conclude, as in § 42, 2, that they satisfy the equations got from  $f(x, y) = 0$  by differentiating repeatedly with respect to  $x$ , regarding  $y$  as a function of  $x$ .

Thus, differentiating  $f(x, y) = 0$  gives  $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0$  (2)

Differentiating (2) gives

$$\frac{\partial^2 f}{\partial x^2} + 2 \frac{\partial^2 f}{\partial x \partial y} \frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2} \left( \frac{dy}{dx} \right)^2 + \frac{\partial f}{\partial y} \frac{d^2 y}{dx^2} = 0 \quad (3)$$

and so on. In (3) substitute  $dy/dx = -f_x/f_y$ , given by (2), and then solve for  $d^2 y/dx^2$ . We thus obtain (1).

2. Similarly, in the equation  $f(x, y, z) = 0$ , regard  $x, y$  as the independent variables, and  $z$  as standing for the function

<sup>1</sup> The region  $R$  consisting of all points  $(u, x, y, \dots)$  for which  $|x - a| < k$ ,  $|y - b| < k, \dots$ , and  $|u - u_1| < l$ , where  $k, l$  denote certain positive numbers.



$z = \phi(x, y)$  defined by the equation at any point  $P(x, y, z)$  where  $f_z \neq 0$ . Then  $\partial z/\partial x$ ,  $\partial z/\partial y$ ,  $\dots$  stand for  $\phi_x$ ,  $\phi_y$ ,  $\dots$ , and  $z$  has partial derivatives of order  $n$  at  $P$  if  $f$  has finite partial derivatives to those of order  $n$  at  $P$ . Furthermore, equations which determine  $\partial z/\partial x$ ,  $\partial z/\partial y$ ,  $\dots$  may be got from  $f(x, y, z) = 0$  by differentiating repeatedly with respect to  $x$  (regarding  $z$  as a function of  $x$ , and  $y$  as a constant), and with respect to  $y$  (regarding  $z$  as a function of  $y$ , and  $x$  as a constant).

Thus, 
$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial y} = 0,$$

which give 
$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}, \quad \frac{\partial z}{\partial y} = -\frac{f_y}{f_z}$$

EXAMPLE 1. If  $xyz + 2x + y = 2$  (1), find  $\frac{\partial z}{\partial x}$ ,  $\frac{\partial z}{\partial y}$ , and  $\frac{\partial^2 z}{\partial x \partial y}$  at  $(1, 2, -1)$ .

Differentiating (1) with respect to  $x$ ,  $yz + 2 + xy \frac{\partial z}{\partial x} = 0$  (2)

Differentiating (1) with respect to  $y$ ,  $xz + 1 + xy \frac{\partial z}{\partial y} = 0$  (3)

Differentiating (2) with respect to  $y$ ,

$$z + y \frac{\partial z}{\partial y} + x \frac{\partial z}{\partial x} + xy \frac{\partial^2 z}{\partial x \partial y} = 0 \quad (4)$$

Setting  $x, y, z = 1, 2, -1$  and solving, we find

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2}$$

EXAMPLE 2. If  $z^2 + zx + y = 0$ , find  $\partial^2 z/\partial x^2$  in terms of  $x, y, z$ .

3. In the equation  $f(x, y, z) = 0$ , any two of  $x, y, z$  may be taken as the independent variables. If  $y, z$  be so taken,  $f(x, y, z) = 0$  defines  $x$  as a function of  $y, z$  at every point where  $f$  is 0 and  $f_x \neq 0$ ; and all that has been said regarding the partial derivatives of  $z$  with respect to  $x$  and  $y$  is true of the partial derivatives of  $x$  with respect to  $y$  and  $z$ . Similarly, if  $x, z$  are taken as the independent variables.

EXAMPLE 3. If  $x^3 + xz - 2y = 0$ , find  $\partial^2 x/\partial y^2$  in terms of  $x, y, z$ .

EXAMPLE 4. If  $f(x, y, z) = 0$ , prove that  $\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial z} = 1$ ; also that

$$\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = -1.$$

**228. Expression of an implicit function by Taylor's series.**

1. Let  $y$  denote the function  $y = \phi(x)$  defined by  $f(x, y) = 0$  at any given point  $(a, b)$  where  $f$  is 0 and  $f_y \neq 0$ . Then, representing  $b = \phi(a)$  by  $y_a$ ,  $\phi'(a)$  by  $(dy/dx)_a$ , and so on, we have by § 188 (7),

$$y = y_a + \left(\frac{dy}{dx}\right)_a \frac{(x-a)}{1} + \left(\frac{d^2y}{dx^2}\right)_a \frac{(x-a)^2}{2!} + \dots \\ + \left(\frac{d^{n-1}y}{dx^{n-1}}\right)_a \frac{(x-a)^{n-1}}{(n-1)!} + \left(\frac{d^ny}{dx^n}\right)_{x_1} \frac{(x-a)^n}{n!} \quad (1)$$

where  $x_1$  denotes some number between  $a$  and  $x$ .

The expression holds good for every value of  $x$  between which and  $a$  the derivatives  $dy/dx$ ,  $\dots d^ny/dx^n$  are finite. For values of  $x$  sufficiently near  $a$ , we may use two or three terms of (1) to find the corresponding values of  $y$  approximately, § 193.

2. Similarly, by aid of Taylor's series for functions of two variables, § 222, we can find an expression for the function  $z = \phi(x, y)$  defined by  $f(x, y, z) = 0$  at a point  $(a, b, c)$  where  $f$  is 0 and  $f_z \neq 0$ , this expression being one in positive integral powers of  $x - a$ ,  $y - b$ .

**EXAMPLE 1.** Find to four terms the solution of  $x^2 + y^2 = 5$  for  $y$  at the point  $(2, 1)$ .

By successive differentiations of  $x^2 + y^2 - 5 = 0$ , we obtain

$$x + y \frac{dy}{dx} = 0, \quad 1 + \left(\frac{dy}{dx}\right)^2 + y \frac{d^2y}{dx^2} = 0, \quad 3 \frac{dy}{dx} \frac{d^2y}{dx^2} + y \frac{d^3y}{dx^3} = 0$$

Substituting  $x, y = 2, 1$ , we find  $\frac{dy}{dx} = -2$ ,  $\frac{d^2y}{dx^2} = -5$ ,  $\frac{d^3y}{dx^3} = -30$ .

$$\text{Hence, } y = 1 - 2(x-2) - 5 \frac{(x-2)^2}{2} - 30 \frac{(x-2)^3}{6} + \dots$$

**EXAMPLE 2.** Find to three terms the solution of  $y^3 + y^2 + y = x$  for  $y$  at the point  $(0, 0)$ .

**229. Functions defined by simultaneous equations. Functional determinants.** Let  $F_1$  and  $F_2$  denote functions of  $u, v$ . The determinant

$$J = \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{vmatrix} = \frac{\partial F_1}{\partial u} \frac{\partial F_2}{\partial v} - \frac{\partial F_1}{\partial v} \frac{\partial F_2}{\partial u} \quad (1)$$

is called the *functional determinant* or *Jacobian* of  $F_1, F_2$  with respect to  $u, v$ .

In general, if  $F_1, F_2, \dots, F_n$  are functions of  $u_1, u_2, \dots, u_n$ , (and it may be of other variables also), the functional determinant of  $F_1, F_2, \dots, F_n$  with respect to  $u_1, u_2, \dots, u_n$  is

$$J = \begin{vmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \dots & \frac{\partial F_1}{\partial u_n} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \dots & \frac{\partial F_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial u_1} & \frac{\partial F_n}{\partial u_2} & \dots & \frac{\partial F_n}{\partial u_n} \end{vmatrix} \quad (2)$$

**EXAMPLE.** If  $F_1 = a_1u + b_1v + c_1$ ,  $F_2 = a_2u + b_2v + c_2$ , then  $\partial F_1/\partial u = a_1, \dots$ ; hence

$$J = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$$

Show that the equations  $F_1 = 0, F_2 = 0$  have a solution, and but one, for  $u, v$  in terms of  $a_1, b_1, \dots, c_2$ , when, and only when,  $J \neq 0$ . Here  $J$  is the determinant  $\Delta$  of § 202.

**230. Theorem.** Let  $F_1, F_2, \dots, F_n$  be functions of the  $n$  variables  $u, v, \dots$ , and of any number of other variables  $x, y, \dots$ ; let  $J$  denote the functional determinant of  $F_1, F_2, \dots$  with respect to  $u, v, \dots$ ; and let the functions  $F_1, F_2, \dots, F_n$  and their partial derivatives of the first order be continuous at the point  $P(u_1, v_1, \dots; a, b, \dots)$  and in its neighborhood.

Suppose that at  $P$  the functions  $F_1, F_2, \dots, F_n$  all have the value 0, but  $J \neq 0$ . Then in some region  $R$  about  $P$  the equations

$$F_1 = 0, F_2 = 0, \dots, F_n = 0$$

define  $u, v, \dots$  as one-valued functions of  $x, y, \dots$ ; that is, a set of  $n$  functions

$u = \phi_1(x, y, \dots), \quad v = \phi_2(x, y, \dots), \quad w = \phi_3(x, y, \dots), \dots$  exists, and but one such set, which satisfy the equations  $F_1 = 0, F_2 = 0, \dots, F_n = 0$  identically in the region  $R$ , and have the values  $u_1, v_1, \dots$  when  $x, y, \dots = a, b, \dots$ .

Moreover  $\phi_1, \phi_2, \dots$  have continuous partial derivatives of the first order at  $P$  and in its neighborhood.

1. First consider the case of the two functions

$$F_1(u, v, x) \quad F_2(u, v, x), \quad \text{where} \quad J = \frac{\partial F_1}{\partial u} \frac{\partial F_2}{\partial v} - \frac{\partial F_1}{\partial v} \frac{\partial F_2}{\partial u}.$$

Since  $J \neq 0$  at  $P$ , at least one of  $\partial F_2 / \partial v, \partial F_2 / \partial u$  is not 0 at  $P$ . Suppose that  $\partial F_2 / \partial v \neq 0$ .

Then since  $F_2$  is 0 and  $\partial F_2 / \partial v \neq 0$  at  $P$ , the equation  $F_2 = 0$  has a solution for  $v$  in terms of  $u, x$  at  $P$  and in its neighborhood, § 226; let this solution be

$$v = V(u, x) \quad (1)$$

so that

$$F_2[u, V(u, x), x] \equiv 0 \quad (2)$$

Substitute (1) in  $F_1$ , and represent the resulting function of  $u, x$  by  $\Phi(u, x)$ ; so that

$$F_1[u, V(u, x), x] \equiv \Phi(u, x) \quad (3)$$

Since  $F_1$  is 0 at  $P$ , so also is  $\Phi$ ; and, as will be shown immediately,  $\Phi_u \neq 0$  at  $P$ . Hence the equation  $\Phi = 0$  has a solution  $u = \phi(x)$  at  $P$  and in its neighborhood, § 225. Substituting this in (1), and calling the result  $v = \psi(x)$ , we have

$$u = \phi(x) \quad v = \psi(x) \quad (4)$$

which is the solution of  $F_1 = 0, F_2 = 0$  for  $u, v$  at  $P$  and in its neighborhood.

We have assumed that  $\Phi_u \neq 0$ . To prove this, differentiate the identities (2) and (3) with respect to  $u$ . We obtain

$$\frac{\partial \Phi}{\partial u} \equiv \frac{\partial F_1}{\partial u} + \frac{\partial F_1}{\partial v} \frac{\partial V}{\partial u} \quad \dots \quad 0 \equiv \frac{\partial F_2}{\partial u} + \frac{\partial F_2}{\partial v} \frac{\partial V}{\partial u}$$

Eliminating  $\frac{\partial V}{\partial u}$  between these equations, we get

$$\frac{\partial F_2}{\partial v} \frac{\partial \Phi}{\partial u} \equiv J. \quad \text{But } \frac{\partial F_2}{\partial v}, J \neq 0. \quad \text{Hence } \frac{\partial \Phi}{\partial u} \neq 0.$$

2. We next prove that if the theorem is true for  $n - 1$  of the functions  $F_1, F_2, \dots, F_n$  with respect to  $n - 1$  of the variables  $u, v, \dots$ , it is

true for all  $n$  of  $F_1, F_2, \dots, F_n$  with respect to all  $n$  of  $u, v, \dots$ . It can then be inferred from 1. that the theorem is true when  $n = 3$ , hence when  $n = 4$ , and so on.

Representing the cofactors of the elements of the first column of  $J$  by  $J_1, J_2, \dots, J_n$ , we have, § 201, 1.,

$$J = \frac{\partial F_1}{\partial u} J_1 + \frac{\partial F_2}{\partial u} J_2 + \dots + \frac{\partial F_n}{\partial u} J_n$$

Hence, since  $J \neq 0$  at  $P$ , at least one of  $J_1, J_2, \dots, J_n$  is not 0 at  $P$ . Suppose  $J_1 \neq 0$ . Then, since  $J_1$  is the functional determinant of  $F_2, \dots, F_n$  with respect to  $v, w, \dots$ , the equations  $F_2 = 0, \dots, F_n = 0$  have, by hypothesis, a solution for  $v, w, \dots$  in terms of  $u, x, y, \dots$  at  $P$  and in its neighborhood. Let this solution be

$$v = V(u, x, y, \dots), \quad w = W(u, x, y, \dots), \dots \quad (5)$$

Substitute (5) in  $F_1$  and represent the resulting function of  $u, x, y, \dots$  by  $\Phi$ . Then

$$F_1(u, V, W, \dots, x, y, \dots) \equiv \Phi(u, x, y, \dots) \quad (6)$$

Also  $F_2(u, V, W, \dots) \equiv 0, \quad F_3(u, V, W, \dots) \equiv 0, \dots$

At  $P$ ,  $\Phi$  is 0 but  $\Phi_u \neq 0$ . For differentiating the identities (6) with respect to  $u$ ,

$$\frac{\partial \Phi}{\partial u} \equiv \frac{\partial F_1}{\partial u} + \frac{\partial F_1}{\partial v} \frac{\partial V}{\partial u} + \frac{\partial F_1}{\partial w} \frac{\partial W}{\partial u} + \dots,$$

$$0 \equiv \frac{\partial F_2}{\partial u} + \frac{\partial F_2}{\partial v} \frac{\partial V}{\partial u} + \frac{\partial F_2}{\partial w} \frac{\partial W}{\partial u} + \dots,$$

$$0 \equiv \frac{\partial F_3}{\partial u} + \frac{\partial F_3}{\partial v} \frac{\partial V}{\partial u} + \frac{\partial F_3}{\partial w} \frac{\partial W}{\partial u} + \dots$$

Multiply the first of these identities by  $J_1$ , the second by  $J_2$ , and so on, and add the results. We get, by the theorems of § 201,

$$J_1 \frac{\partial \Phi}{\partial u} \equiv J. \quad \text{But } J, J_1, \neq 0. \quad \text{Hence } \frac{\partial \Phi}{\partial u} \neq 0$$

Hence the equation  $\Phi(u, x, y, \dots) = 0$  has a solution  $u = \phi_1(x, y, \dots)$ , and but one, at  $P$  and in its neighborhood, § 226. Substituting this in (5), we have finally

$$u = \phi_1(x, y, \dots), \quad v = \phi_2(x, y, \dots), \quad w = \phi_3(x, y, \dots), \dots \quad (7)$$

which is the solution of  $F_1 = 0, \dots, F_n = 0$  for  $u, v, \dots$  at  $P$  and in its neighborhood.

That the functions  $\phi_1, \phi_2, \dots$  have continuous partial derivatives of the first order at  $P$  and in its neighborhood follows from the last clause of the theorem of § 226.

In applying this theorem to a given set of  $n$  equations in  $m(> n)$  variables, any  $m$  of the variables for which the corresponding functional determinant  $J$  is not 0 may be taken as dependent, the rest being independent.

**231. Differentiation of functions defined by simultaneous equations.** By § 230, the equations  $F_1(u, v, x, y) = 0$ ,  $F_2(u, v, x, y) = 0$  define  $u$  and  $v$  as functions of  $x, y$  at every point  $P$  where  $F_1 = 0$ ,  $F_2 = 0$  and  $J \neq 0$ ; and the partial derivatives  $\partial u/\partial x$ ,  $\partial u/\partial y$ ,  $\partial v/\partial x$ ,  $\partial v/\partial y$  exist at  $P$ . These partial derivatives and those of higher order may be found by a procedure analogous to that of § 227.

Differentiating  $F_1(u, v, x, y) = 0$ ,  $F_2(u, v, x, y) = 0$  with respect to  $x$

$$\frac{\partial F_1}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial F_1}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial F_1}{\partial x} = 0 \quad \frac{\partial F_2}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial F_2}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial F_2}{\partial x} = 0 \quad (1)$$

Since  $J \neq 0$ , we can solve this pair of equations for  $\partial u/\partial x$ ,  $\partial v/\partial x$ :

$$\frac{\partial u}{\partial x} = \left( \frac{\partial F_1}{\partial v} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial v} \right) / J \quad \frac{\partial v}{\partial x} = \left( \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial u} - \frac{\partial F_1}{\partial u} \frac{\partial F_2}{\partial x} \right) / J \quad (2)$$

The corresponding formulas for  $\partial u/\partial y$ ,  $\partial v/\partial y$  may be got by replacing  $x$  by  $y$  in (2).

In the pair of equations got by differentiating the pair (1) with respect to  $x$  (or  $y$ ) the determinant of the coefficients of  $\partial^2 u/\partial x^2$  and  $\partial^2 v/\partial x^2$  (or  $\partial^2 u/\partial x \partial y$  and  $\partial^2 v/\partial x \partial y$ ) is again  $J$ ; and so on. It therefore follows, by an obvious extension of the reasoning in § 227, that if  $J \neq 0$ , and the partial derivatives of  $F_1, F_2$  to those of order  $n$  exist, then the partial derivatives of  $u$  and  $v$  to those of order  $n$  also exist and are determined by the equations got by repeated differentiations of  $F_1 = 0$ ,  $F_2 = 0$  with respect to  $x$  and  $y$ .

**EXAMPLE.** Show that the pair of equations  $y^2 - z - x = 0$ ,  $z^2 + y - 2x = 0$  has a solution for  $y$  and  $z$  in terms of  $x$  at  $O$  and in its neighborhood, and express this solution by Taylor series to two terms each.

$$2y \frac{dy}{dx} - \frac{dz}{dx} - 1 = 0$$

$$\frac{dy}{dx} + 2z \frac{dz}{dx} - 2 = 0$$

$$\text{Hence, at } O, J = 1, \frac{dy}{dx} = 2, \frac{dz}{dx} = -1$$

$$2y \frac{d^2y}{dx^2} - \frac{d^2z}{dx^2} + 2 \left( \frac{dy}{dx} \right)^2 = 0$$

$$\frac{d^2y}{dx^2} + 2z \frac{d^2z}{dx^2} + 2 \left( \frac{dz}{dx} \right)^2 = 0 \quad \text{Hence, at } O, \frac{d^2y}{dx^2} = -2, \frac{d^2z}{dx^2} = 8$$

$$\text{Therefore,} \quad y = 2x - x^2 + \dots \quad z = -x + 4x^2 + \dots$$

**232. Interdependence of functions.** Two functions of  $x, y$  may be functions one of the other. Thus if  $u = (x + y)/y$  and  $v = (y/x)^2$ , then  $v = 1/(u - 1)^2$ .

Let the functions

$$u = \phi(x, y) \quad (1) \quad v = \psi(x, y) \quad (2)$$

and their partial derivatives of the first order be continuous at the point  $P$  and in some region  $R$  about  $P$ , and suppose that at least one of these partial derivatives, say  $\phi_y$ , is not 0 in  $R$ .

1. If  $v$  be a function of  $u$ , say  $v = f(u)$ , then in  $R$

$$J = \phi_x \psi_y - \phi_y \psi_x = 0$$

For, differentiating  $v = f(u)$  with respect to  $x$  and  $y$ ,

$$\psi_x - f'(u)\phi_x = 0 \quad \psi_y - f'(u)\phi_y = 0$$

Hence, eliminating  $f'(u)$ ,  $\phi_x \psi_y - \phi_y \psi_x = 0$

2. Conversely, if  $J = 0$  in  $R$ , then  $v$  is a function of  $u$  in  $R$ .

Since  $\phi_y \neq 0$  in  $R$ , the equation (1) has a solution for  $y$  in terms of  $u, x$  at any point  $Q$  of  $R$ , say  $y = Y(u, x)$  (3). It is to be proved that, when  $J = 0$ , then  $\psi[x, Y(u, x)]$  is a function of  $u$  only, and not of  $x$ .

This is equivalent to showing that when  $u$  and  $x$  are taken as the independent variables in (1) and (2), and  $J = 0$ , then  $\partial v / \partial x = 0$ .

But when  $u, x$  are the independent variables, differentiating (1) and (2) with respect to  $x$  gives

$$0 = \phi_x + \phi_y \frac{\partial y}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial x} = \psi_x + \psi_y \frac{\partial y}{\partial x}$$

Eliminating  $\frac{\partial y}{\partial x}$  between these equations gives

$$\frac{\partial v}{\partial x} \phi_y = -J. \quad \text{But } J = 0 \text{ and } \phi_y \neq 0. \quad \text{Hence, } \frac{\partial v}{\partial x} = 0$$



3. Under like conditions, and by similar reasoning, it can be shown that if

$$u = \phi(x, y, z) \quad v = \psi(x, y, z) \quad w = \chi(x, y, z)$$

the necessary and sufficient condition that one of the functions  $u, v, w$  be a function of the other two is that the functional determinant of  $\phi, \psi, \chi$  with respect to  $x, y, z$  be identically 0.

The theorem admits of extension to the general case of  $n$  functions of  $n$  independent variables.

### EXERCISE XLV

1. Find the indicated solutions of the following equations to the terms involving the second powers of the independent variable or variables.

(1)  $x^2 - y^2 + x - 2y = 0$  for  $y$  in terms of  $x$ , and  $x$  in terms of  $y$ , at  $(0, 0)$ .

(2)  $y^3 - 4y + 3x = 0$  for  $y$  in terms of  $x$ , at  $(1, 1)$ .

(3)  $z^2 + yz + z - y + x = 0$  for  $z$  in terms of  $x$  and  $y$ , at  $(0, 0, 0)$ .

(4)  $y^2 + 3z^2 - 4x = 0, z + 2y + x = 0$  for  $y$  and  $z$  in terms of  $x$ , at  $(1, -1, 1)$ .

2. Show that in  $u = xy, x + 2y - 3z = 4$ , any two of  $x, y, z, u$  may be chosen as the independent variables. Find  $\partial u / \partial x$  for each of the choices:  $x, y; x, z; x, u$ .

3. If  $u^3 + v^3 + x^3 - 3y = 0$  and  $u^2 + v^2 + y^2 + 2x = 0$ , find  $\partial u / \partial x, \partial v / \partial x$ .

4. If  $z = uv$ , and  $u^2 - v + x = 0, u + v^2 - y = 0$ , find  $\partial z / \partial x, \partial z / \partial y$ .

5. If  $f(ux, uy) = 0$ , find  $\partial u / \partial x, \partial u / \partial y$ .

6. If  $u$  is a function of  $x - y, y - z, z - x$ , show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

7. Show that if  $F(x, y, z, t) = 0$ , then  $\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial t} \cdot \frac{\partial t}{\partial x} = 1$ .

8. If  $f_1(u, v) = 0, f_2(v, y) = 0, f_3(y, x) = 0$ , find  $du/dx$ .

9. If  $x = \phi(u, v)$  and  $y = \psi(u, v)$ , show that

$$\frac{\partial x}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial x} = 1, \quad \frac{\partial y}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial x} = 0, \quad \text{and find } \frac{\partial u}{\partial x}$$

10. If  $x = r \cos \theta, y = r \sin \theta$ , show that  $\partial x / \partial r = \partial r / \partial x$ .

**11.** Let  $u$  and  $v$  be functions of  $x, y$  such that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Show that if we set  $x = r \cos \theta, y = r \sin \theta$ , we find

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

**12.** In  $f(x, y)$  set  $y = tx$ , and represent  $f(x, xt)$  by  $\phi(x, t)$ . Prove that at a point where  $f_x = 0, f_y = 0$ , we have  $\phi_x = 0, \phi_t = 0$ .

**13.** By expanding  $f(x, y)$  in powers of  $x - x_1, y - y_1$ , at the point  $(x_1, y_1)$ , § 222, show that if the line  $y - y_1 = m(x - x_1)$  touches the curve  $f(x, y) = 0$  at  $(x_1, y_1)$ , then  $f[x, y_1 + m(x - x_1)]$  is divisible by  $(x - x_1)^2$ .

## XXIV. APPLICATIONS OF PARTIAL DIFFERENTIATION

### SURFACES. CURVES IN SPACE

**233. Tangent plane to a surface.** Let  $S$  be a given surface, and  $f(x, y, z) = 0$  its equation; also let  $P(x', y', z')$  be a given point of  $S$  and  $Q(x' + \Delta x, y' + \Delta y, z' + \Delta z)$  any adjacent point of the surface, so that  $f(x', y', z') = 0$ , and  $f(x' + \Delta x, y' + \Delta y, z' + \Delta z) = 0$ .

Then by the mean value theorem § 223 (2),

$$f_x(x_1, y_1, z_1)\Delta x + f_y(x_1, y_1, z_1)\Delta y + f_z(x_1, y_1, z_1)\Delta z = 0 \quad (1)$$

where  $(x_1, y_1, z_1)$  is some point between  $P$  and  $Q$  on the line segment  $PQ$ .

The direction ratios of the line  $PQ$  are  $\Delta x : \Delta y : \Delta z$ . Hence, § 212, the equation (1) shows that the line  $PQ$  is perpendicular to the line  $L$  whose direction ratios are

$$f_x(x_1, y_1, z_1) : f_y(x_1, y_1, z_1) : f_z(x_1, y_1, z_1)$$

Suppose  $Q$  to move on  $S$  into coincidence with  $P$ . Then the line  $PQ$  becomes a line tangent to  $S$  at  $P$ , the ratios  $\Delta x : \Delta y : \Delta z$  become the direction ratios of this tangent line, and  $L$  becomes a line  $N$  perpendicular to this tangent and having the direction ratios

$$f_x(x', y', z') : f_y(x', y', z') : f_z(x', y', z') \quad (2)$$

Hence the line  $N$  which has the direction ratios (2) is perpendicular to every line which touches  $S$  at  $P$ . Therefore all these tangent lines lie in the plane perpendicular to  $N$  through  $P$ . This plane is defined as the *tangent plane* to  $S$  at  $P$ , and  $N$  is called the *normal* to  $S$  at  $P$ .

Therefore, by § 213, the equation of the tangent plane to  $S$  at  $P$  is

$$\frac{\partial f}{\partial x'}(x - x') + \frac{\partial f}{\partial y'}(y - y') + \frac{\partial f}{\partial z'}(z - z') = 0 \quad (3)$$

and, § 210, the equations of the normal are

$$\frac{x - x'}{\partial f / \partial x'} = \frac{y - y'}{\partial f / \partial y'} = \frac{z - z'}{\partial f / \partial z'} \quad (4)$$

When the equation  $f = 0$  is of the form  $z - \phi(x, y) = 0$ , then (3) becomes

$$z - z' = \frac{\partial z'}{\partial x'}(x - x') + \frac{\partial z'}{\partial y'}(y - y') \quad (5)$$

Observe that  $\partial z' / \partial x'$  and  $\partial z' / \partial y'$  are the slopes at  $P$  of the curves in which  $S$  is cut by the planes  $y = y'$  and  $x = x'$ .

In deriving (3) we have assumed that  $f$  and its partial derivatives  $f_x, f_y, f_z$  are continuous at  $P$  and in its neighborhood, that is, in a region of the kind described in § 214, 6 in the space about  $P$ ; also that at least one of  $f_x, f_y, f_z$  is not 0 at  $P$ . A point where these conditions are not satisfied is called a *singular point* of  $S$ ; for example, the vertex  $O$  of the conical surface  $x^2 + y^2 - z^2 = 0$ . Obviously (3) is illusory at a point where  $f_x, f_y, f_z$  are all 0.

**234. Tangents to space curves.** 1. Consider the equations

$$x = f(t) \qquad y = g(t) \qquad z = h(t) \quad (1)$$

where  $f, g, h$  denote functions having continuous derivatives which do not vanish simultaneously. To each value of  $t$  in the  $t$ -interval for which  $f, g, h$  are defined corresponds a single set of values of  $x, y, z$  and therefore a single point  $P$ . When  $t$  varies continuously,  $P$  traces a curve  $C$ . We call (1) the equations of  $C$  in terms of the parameter  $t$ . Compare § 56.

Let  $P(x', y', z')$ , corresponding to  $t = t'$ , be a given point of  $C$ ; and let  $Q(x' + \Delta x, y' + \Delta y, z' + \Delta z)$ , corresponding to  $t = t' + \Delta t$ , be any adjacent point of  $C$ . The direction ratios of the line  $PQ$  are  $\Delta x : \Delta y : \Delta z$ . Suppose that  $\Delta t \rightarrow 0$  and therefore that  $Q \rightarrow P$  along  $C$ . In the limit,  $PQ$  be-

comes the tangent to  $C$  at  $P$ , and  $\Delta x : \Delta y : \Delta z$  become the direction ratios of this tangent. But

$$\Delta x : \Delta y : \Delta z = \frac{\Delta x}{\Delta t} : \frac{\Delta y}{\Delta t} : \frac{\Delta z}{\Delta t} \rightarrow \frac{dx}{dt} : \frac{dy}{dt} : \frac{dz}{dt} \quad (2)$$

Therefore the equations of the tangent to  $C$  at  $P$  are

$$\frac{x - x'}{f'(t')} = \frac{y - y'}{g'(t')} = \frac{z - z'}{h'(t')} \quad (3)$$

**EXAMPLE 1.** The equations  $x = a \cos \theta$ ,  $y = a \sin \theta$ ,  $z = k\theta$  represent a *helix*. Show that the curve has the form of a spiral surrounding a right circular cylinder of radius  $a$  and with its axis along  $Oz$ . Show also that the curve meets the elements of the cylinder at a constant angle.

2. A space curve  $C$  may also be represented by the equations  $F_1 = 0$ ,  $F_2 = 0$  of two surfaces of which it is the intersection. By § 230, the equations  $F_1 = 0$ ,  $F_2 = 0$  have one and but one solution for  $y$  and  $z$  in terms of  $x$  at any point  $P$  of  $C$  where the functional determinant  $J$  of  $F_1$ ,  $F_2$  with respect to  $y$ ,  $z$  is not 0. Call this solution

$$y = f(x) \quad z = \phi(x) \quad (4)$$

The equations (4) are equivalent to (1) when  $t = x$  and represent  $C$  in the neighborhood of  $P$  in terms of the parameter  $x$ . By setting  $t = x$  in (2), we obtain for the direction ratios of the tangent at  $P$ ,

$$1 : \frac{dy}{dx} : \frac{dz}{dx} \quad \text{or} \quad dx : dy : dz \quad (5)$$

**EXAMPLE 2.** Find the direction ratios of the tangent to the circle  $x^2 + y^2 + z^2 - 3x = 0$ ,  $2x - 3y + 5z - 4 = 0$  at  $(1, 1, 1)$ .

Differentiating,  $(2x - 3)dx + 2ydy + 2zdz = 0$

$$2dx - 3dy + 5dz = 0$$

Hence at  $(1, 1, 1)$

$$-dx + 2dy + 2dz = 0$$

$$2dx - 3dy + 5dz = 0$$

Therefore  $dx : dy : dz = 16 : 9 : -1$

**235. Length of arc.** 1. Let  $C$  denote a curve arc between  $x = a$  and  $x = b$ , and let its equations be

$$y = f(x) \qquad z = \phi(x) \qquad (1)$$

where  $f'(x)$  and  $\phi'(x)$  are supposed continuous.

As in § 57, inscribe in  $C$  a polygon of  $n$  sides all of which  $\rightarrow 0$  when  $n \rightarrow \infty$ . We define the length of  $C$  as the limit approached by the perimeter of the polygon when  $n \rightarrow \infty$ . What follows shows that the limit exists.

Let  $PQ$  denote any side of the polygon, and  $\Delta x$ ,  $\Delta y$ ,  $\Delta z$  the projections of  $PQ$  on  $Ox$ ,  $Oy$ ,  $Oz$ . Then

$$PQ = [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]^{1/2}$$

By § 97 (6) there are two points,  $x_1$  and  $x_2$ , in  $\Delta x$  such that

$$\Delta y = f'(x_1) \Delta x \qquad \Delta z = \phi'(x_2) \Delta x$$

$$\text{Therefore } PQ = [1 + f'^2(x_1) + \phi'^2(x_2)]^{1/2} \Delta x \qquad (2)$$

$$\begin{aligned} \text{Hence }^1 \lim \Sigma PQ &= \lim \Sigma [1 + f'^2(x_1) + \phi'^2(x_2)]^{1/2} \Delta x \\ &= \lim \Sigma [1 + f'^2(x_1) + \phi'^2(x_1)]^{1/2} \Delta x \\ &= \int_a^b [1 + f'^2(x) + \phi'^2(x)]^{1/2} dx \end{aligned}$$

Therefore the length  $s_a^b$  of the arc  $C$  is given by the formula :

$$s_a^b = \int_a^b [1 + f'^2(x) + \phi'^2(x)]^{1/2} dx \qquad (3)$$

<sup>1</sup> For let  $M$  and  $m$  denote the greatest and least values of  $\phi'^2(x)$  in  $\Delta x$ . Then

$$\phi'^2(x_2) = \phi'^2(x_1) + \theta(M - m) \quad -1 < \theta < 1 \qquad (1)$$

If  $A$  and  $B$  denote any two numbers such that  $A > 1$  and  $A + B > 0$ , we have  $|(A + B)^{1/2} - A^{1/2}| = |B \div [(A + B)^{1/2} + A^{1/2}]| < |B|$  (2)

Hence, setting  $A = 1 + f'^2(x_1) + \phi'^2(x_1)$  and  $B = \theta(M - m)$ , we have by (1)

$$|[1 + f'^2(x_1) + \phi'^2(x_2)]^{1/2} - [1 + f'^2(x_1) + \phi'^2(x_1)]^{1/2}| < |\theta| \cdot |(M - m)|$$

Therefore

$$\lim \Sigma [1 + f'^2(x_1) + \phi'^2(x_2)]^{1/2} \Delta x = \lim \Sigma [1 + f'^2(x_1) + \phi'^2(x_1)]^{1/2} \Delta x,$$

since

$$\lim \Sigma |\theta| (M - m) \Delta x = 0.$$

Ch.-J. de la Vallée-Poussin, Cours d'Analyse Infinitésimale.

2. Replacing  $b$  by the  $x$  of any point  $P$  of  $C$ , we obtain

$$s = \int_a^x [1 + f'^2(x) + \phi'^2(x)]^{1/2} dx \quad (4)$$

which expresses the length  $s$  of the variable arc  $AP$  as a function of the  $x$  of its movable end point  $P$ . We have

$$\frac{ds}{dx} = [1 + f'^2(x) + \phi'^2(x)]^{1/2} \quad (5)$$

Hence  $ds/dx$  is positive at all points of  $C$ . This means that in giving the radical in (2) the  $+$  sign, we make the positive sense along the curve correspond to that along  $Ox$ .

3. We may regard the  $x, y, z$  of  $P$  as functions of the length  $s$  of the variable arc  $AP$ , and these functions have  $s$ -derivatives. For, by (4), (5),  $s$  is a one-valued, increasing, and differentiable function of  $x$ ; hence, §§ 52, 54,  $x$  is a one-valued, increasing, and differentiable function of  $s$ . And, by hypothesis,  $y$  and  $z$  are differentiable functions of  $x$ , and therefore of  $s$ , § 40.

4. If we multiply (5) by  $dx$  and set  $f'(x) dx = dy$ ,  $\phi'(x) dx = dz$ , we get

$$ds = [1 + f'^2(x) + \phi'^2(x)]^{1/2} dx = \pm [(dx)^2 + (dy)^2 + (dz)^2]^{1/2}$$

the  $\pm$  sign being necessary because  $dx$  may be positive or negative. Hence, if the  $x, y, z$  of  $P$  are given as functions of a parameter  $t$ , of the kind described in § 234, and we take as the positive sense on the curve that which corresponds to increasing values of  $t$ , we have for the length  $s$  of the arc  $AP$  from  $t = t_0$  to  $t = t$ :

$$s = \int_{t_0}^t \left[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \right]^{1/2} dt \quad (6)$$

Since, by hypothesis,  $dx/dt, dy/dt, dz/dt$  do not vanish simultaneously, it follows from this formula, as from (4), that the  $x, y, z$  of  $P$  are differentiable functions of  $s$ .

EXAMPLE 1. The length of  $y = x, z = \frac{2}{3}x^{3/2}$  between  $x = 0$  and  $x = 2$  is  $s = \int_0^2 (2 + x)^{1/2} dx = \frac{2}{3}(8 - 2\sqrt{2})$ .



**EXAMPLE 2.** The length of the helix,  $x = 3 \cos \theta$ ,  $y = 3 \sin \theta$ ,  $z = 4 \theta$ , between  $\theta = 0$  and  $\theta = \theta$  is  $s = \int_0^\theta [9 \sin^2 \theta + 9 \cos^2 \theta + 16]^{1/2} d\theta = 5 \theta$ . Hence also  $\theta = \frac{s}{5}$ ,  $x = 3 \cos \frac{s}{5}$ ,  $y = 3 \sin \frac{s}{5}$ ,  $z = \frac{4s}{5}$ .

**236. Theorem.** *The limit of the ratio of an arc to its chord when the arc  $\rightarrow 0$ , is 1.*

For let  $\Delta s$  and  $PQ$  denote the lengths of the arc and chord. By § 235, 3.,  $\Delta x \rightarrow 0$  when  $\Delta s \rightarrow 0$ , and by § 235 (2), (4) and § 96 (7), there are points  $x_1, x_2, x_3$  in  $\Delta x$  such that

$$PQ = [1 + f'^2(x_1) + \phi'^2(x_2)]^{1/2} \Delta x \quad \Delta s = [1 + f'^2(x_3) + \phi'^2(x_3)]^{1/2} \Delta x$$

$$\text{Hence} \quad \lim_{\Delta s \rightarrow 0} \frac{PQ}{\Delta s} = \lim_{\Delta x \rightarrow 0} \frac{[1 + f'(x_1)^2 + \phi'(x_2)^2]^{1/2}}{[1 + f'(x_3)^2 + \phi'(x_3)^2]^{1/2}} = 1$$

**237. Direction cosines of the tangent.** Let  $\cos \alpha$ ,  $\cos \beta$ ,  $\cos \gamma$  denote the direction cosines of the tangent at  $P$  to the path of  $P$ , in the direction of the motion. Taking  $s$  as the independent variable, give the  $s$  of  $P(x, y, z)$  the positive increment  $\Delta s$ , and let  $Q(x + \Delta x, y + \Delta y, z + \Delta z)$  be the point corresponding to  $s + \Delta s$ . The direction cosines of  $PQ$  are  $\Delta x/PQ$ ,  $\Delta y/PQ$ ,  $\Delta z/PQ$ , and when  $\Delta s \rightarrow 0$  they  $\rightarrow \cos \alpha$ ,  $\cos \beta$ ,  $\cos \gamma$ . Therefore, since  $\Delta x/PQ = (\Delta x/\Delta s)(\Delta s/PQ)$  and so on, and  $\Delta s/PQ \rightarrow 1$ , we have

$$\cos \alpha = \frac{dx}{ds} \quad \cos \beta = \frac{dy}{ds} \quad \cos \gamma = \frac{dz}{ds} \quad (1)$$

We have  $\frac{dx}{ds} = \frac{dx}{dt} / \frac{ds}{dt} = \frac{dx}{dt} / \left[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \right]^{1/2}$ , and so on.

**238. Directional derivatives.** Let  $u = f(x, y, z)$  be a given function of  $x, y, z$ . When, as in § 237,  $P$  moves along its path,  $s$  increasing, the value of  $u$  at  $P$  changes. The rate of change of  $u$  with respect to  $s$  at  $P$  is, § 237 (1),

$$\frac{du}{ds} = f_x \frac{dx}{ds} + f_y \frac{dy}{ds} + f_z \frac{dz}{ds} = f_x \cos \alpha + f_y \cos \beta + f_z \cos \gamma \quad (1)$$

Its value at any point  $P$  depends only on the values of  $f_x, f_y, f_z$  at  $P$  and the direction  $\alpha, \beta, \gamma$ . It is therefore called the *directional derivative* of  $f$  in the direction  $\alpha, \beta, \gamma$ .

EXAMPLE 1. The directional derivative of  $xyz$  in the direction for which  $\cos \alpha = -1/3$ ,  $\cos \beta = 2/3$ ,  $\cos \gamma = -2/3$  is

$$(-yz + 2xz - 2xy)/3$$

EXAMPLE 2. The point  $P$  is moving on the curve  $x = t$ ,  $y = t^2$ ,  $z = t^3$ ,  $t$  increasing. Show that when  $t = -1$  the rate of change of  $r = OP$  with respect to  $s$  is  $-\sqrt{42}/7$ .

EXAMPLE 3. The directional derivative of  $f(x, y, z)$  at  $P$  in the direction of the normal to the surface  $f(x, y, z) = c$  through  $P$  is called the *normal derivative* of  $f$  at  $P$  and is represented by  $df/dn$ . Show that  $df/dn = \pm (f_x^2 + f_y^2 + f_z^2)^{1/2}$ , and that it is greater numerically than any other directional derivative of  $f$  at  $P$ . The sign is  $+$  for one direction on the normal,  $-$  for the other. The vector whose  $x$ -,  $y$ -,  $z$ -components are  $f_x, f_y, f_z$  is called the *gradient* of  $f$ .

**239. Velocity and acceleration.** The discussion of curvilinear motion in §§ 82–87 is easily extended to space.

Let  $v_x, v_y, v_z$  denote the  $x$ -,  $y$ -,  $z$ -components of the velocity,  $v$  its magnitude, and  $\alpha, \beta, \gamma$  its direction angles; and let  $a_x, a_y, a_z, a$ , and  $\alpha', \beta', \gamma'$  have the corresponding meanings for the acceleration. Then

$$v_x = \frac{dx}{dt} \quad v_y = \frac{dy}{dt} \quad v_z = \frac{dz}{dt} \quad v = (v_x^2 + v_y^2 + v_z^2)^{1/2}$$

$$a_x = \frac{d^2x}{dt^2} \quad a_y = \frac{d^2y}{dt^2} \quad a_z = \frac{d^2z}{dt^2} \quad a = (a_x^2 + a_y^2 + a_z^2)^{1/2}$$

$$\cos \alpha = \frac{v_x}{v}, \quad \cos \beta = \frac{v_y}{v}, \quad \cos \gamma = \frac{v_z}{v}$$

$$\cos \alpha' = \frac{a_x}{a}, \quad \cos \beta' = \frac{a_y}{a}, \quad \cos \gamma' = \frac{a_z}{a}$$

EXAMPLE 1. Discuss the motion of a point on the helix  $x = 3 \cos t$ ,  $y = 3 \sin t$ ,  $z = 4t$ ,  $t$  denoting time.

$$v_x = -3 \sin t, \quad v_y = 3 \cos t, \quad v_z = 4 \quad \therefore v = 5$$

and  $\cos \alpha = -\frac{3}{5} \sin t, \quad \cos \beta = \frac{3}{5} \cos t, \quad \cos \gamma = \frac{4}{5}$

$$a_x = -3 \cos t, \quad a_y = -3 \sin t, \quad a_z = 0 \quad \therefore a = 3$$

and  $\cos \alpha' = -\cos t, \quad \cos \beta' = -\sin t, \quad \cos \gamma' = 0$

Hence the point moves with constant speed; the direction of the motion makes a constant angle with  $Oz$ ; and the acceleration is of constant magnitude and is always directed toward  $Oz$ .

EXAMPLE 2. A point is moving on the curve whose equations are  $xy + z = 0$ ,  $2x + 3y + z = 5$ . Find its velocity and acceleration vectors at  $(2, 1, -2)$ , if  $v_x = 2$  and  $a_x = 1$  at this point.

Differentiating both equations twice with respect to  $t$ , substituting the given values, and solving, we find

$$v_x, v_y, v_z = 2, -2, 2; \quad a_x, a_y, a_z = 1, -9, 25.$$

**240. The osculating plane.** Let  $C$  denote a given space curve, and  $P_1, P_2, P_3$  three points on  $C$ . In general, when  $P_2, P_3 \rightarrow P_1$  on  $C$ , the plane  $P_1P_2P_3$  approaches a definite limiting position  $E$ . The plane  $E$  is called the *osculating plane* to  $C$  at  $P_1$ . Its equation may be found as follows :

Let the equations of  $C$  be

$$x = f(t) \quad y = g(t) \quad z = h(t) \quad (1)$$

Let the equation of the plane  $P_1P_2P_3$  be

$$ax + by + cz + d = 0 \quad (2)$$

where  $a, b, c, d$  are undetermined constants, and consider the function

$$F(t) = a \cdot f(t) + b \cdot g(t) + c \cdot h(t) + d \quad (3)$$

Let  $P_1, P_2, P_3$  correspond to  $t = t_1, t_2, t_3$  in (1), and suppose that  $t_1 < t_2 < t_3$ . Then,  $P_1, P_2, P_3$  being in the plane (2),  $F(t)$  vanishes at  $t = t_1, t_2, t_3$ . Hence, by Rolle's theorem, § 96,  $F'(t)$  vanishes at a point  $t'_1$  between  $t_1$  and  $t_2$ , and also at a point  $t'_2$  between  $t_2$  and  $t_3$ ; and since  $F'(t)$  vanishes at  $t'_1$  and  $t'_2$ ,  $F''(t)$  vanishes at a point  $t''_1$  between  $t'_1$  and  $t'_2$ .

When  $t_2, t_3 \rightarrow t_1$ , then  $t'_1, t'_2, t''_1 \rightarrow t_1$ . Hence when  $t_2, t_3 \rightarrow t_1$ , and (2) becomes the equation of the osculating plane at  $P_1$ , the coefficients  $a, b, c, d$  satisfy the equations  $F(t_1) = 0, F'(t_1) = 0, F''(t_1) = 0$ , or setting  $x_1 = f(t_1), y_1 = g(t_1), z_1 = h(t_1)$ , the equations

$$\begin{aligned} ax_1 + by_1 + cz_1 + d &= 0 & a \cdot f'(t_1) + b \cdot g'(t_1) + c \cdot h'(t_1) &= 0 \\ a \cdot f''(t_1) + b \cdot g''(t_1) + c \cdot h''(t_1) &= 0 \end{aligned} \quad (4)$$

Hence the equation of the osculating plane may be found by eliminating  $a, b, c, d$  between (2) and (4). It is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ f'(t_1) & g'(t_1) & h'(t_1) \\ f''(t_1) & g''(t_1) & h''(t_1) \end{vmatrix} = 0 \quad (5)$$

When the equations of  $C$  are of the form  $F_1(x, y, z) = 0, F_2(x, y, z) = 0$ , we may take  $x$  as the parameter  $t$ . The second and third rows of (5) are then

$$1, \quad \frac{dy_1}{dx_1}, \quad \frac{dz_1}{dx_1}; \quad 0, \quad \frac{d^2y_1}{dx_1^2}, \quad \frac{d^2z_1}{dx_1^2}$$

Evidently the tangent to the curve  $C$  at  $P_1$  is in the osculating plane  $E$ . The normal to  $C$  at  $P_1$  which lies in  $E$  is called the *principal normal*, and the normal to  $E$  at  $P_1$  is called the *binormal* to  $C$  at  $P_1$ .

Thus, for the curve  $y = x^2, z = x^3$  at  $O$ , Fig. 119, the plane  $z = 0$  is the osculating plane  $E$ ,  $Ox$  is the tangent,  $Oy$  the principal normal, and  $Oz$  the binormal. The curve passes through  $E$  at  $O$ , as was to be expected from the definition of  $E$ . Its projections on the  $xy$ -,  $xz$ -, and  $yz$ -planes are the curves  $y = x^2$ ,  $z = x^3$ , and  $z^2 = y^3$ , the second of which has a point of inflection and the third a cusp at  $O$ .

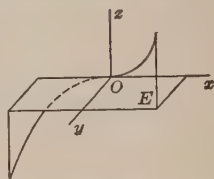


FIG. 119.

EXAMPLE. Find the equation of the osculating plane to the curve  $x = t, y = t^2, z = t^3$  at the point  $t = 1$ ; also the equations of the tangent, binormal, and principal normal at this point.

**241. Curvature and torsion.** In the equation § 240 (5), suppose the parameter  $t$  to be  $s$ , the length of the arc  $AP$  from some fixed point  $A$  on the curve  $C$  to the moving point  $P$ . Let  $\phi$  and  $\tau$  denote the angles which the tangent and the binormal to  $C$  at  $P$  make with some fixed direction. The  $s$ -rates of change of  $\phi$  and  $\tau$ , as  $s$  increases and  $P$  moves along  $C$ , are called the *curvature* and the *torsion* of  $C$  at  $P$ . Hence by definition

$$\text{Curvature} = \frac{d\phi}{ds} \qquad \text{Torsion} = \frac{d\tau}{ds} \qquad (1)$$

The reciprocals of these numbers are called the *radius of curvature* and the *radius of torsion*.

Let  $l, m, n$  denote the direction cosines of the tangent at  $P$ . Then

$$\cos \Delta\phi = l(l + \Delta l) + m(m + \Delta m) + n(n + \Delta n)$$

$$\text{But } l^2 + m^2 + n^2 = 1 \qquad (l + \Delta l)^2 + (m + \Delta m)^2 + (n + \Delta n)^2 = 1$$

$$\text{Hence } (\Delta l)^2 + (\Delta m)^2 + (\Delta n)^2 = 2(1 - \cos \Delta\phi) = [2 \sin (\Delta\phi/2)]^2$$

Therefore

$$\left(\frac{d\phi}{ds}\right)^2 = \lim_{\Delta s \rightarrow 0} \left[ \frac{2 \sin (\Delta\phi/2)}{\Delta s} \right]^2 = \left(\frac{dl}{ds}\right)^2 + \left(\frac{dm}{ds}\right)^2 + \left(\frac{dn}{ds}\right)^2 \qquad (2)$$

Similarly, if  $l_1, m_1, n_1$  be the direction cosines of the binormal,

$$\left(\frac{d\tau}{ds}\right)^2 = \left(\frac{dl_1}{ds}\right)^2 + \left(\frac{dm_1}{ds}\right)^2 + \left(\frac{dn_1}{ds}\right)^2 \quad (3)$$

**242. Indicatrix. Surface curvature.** 1. Take any non-singular point of a surface  $S$  as the origin  $O$ , and the tangent plane to  $S$  at  $O$  as the  $xy$ -plane. For points of  $S$  in the neighborhood of  $O$  the expression for  $z$  in terms of  $x, y$  given by Taylor's theorem, § 228, 2, will be of the form

$$z = \frac{1}{2}(Ax^2 + 2Hxy + By^2) + R \quad (1)$$

in which we suppose that  $A, H, B$  are not all 0. It will be shown later, § 245, that, generally speaking, we can find a region about  $O$  in the  $xy$ -plane in which the value of  $R$  is as small as we please as compared with that of the bracketed term. And by a proper choice of the  $x$ -,  $y$ -axes we can give the bracketed expression the form  $ax^2 + by^2$ . Hence in the neighborhood of  $O$ , the surface  $S$  closely approximates a quadric surface

$$z = \frac{1}{2}(ax^2 + by^2) \quad (2)$$

and the sections of  $S$  by planes  $z = k$  near the tangent plane  $z = 0$  are similar to the conic

$$1 = \frac{1}{2}(ax^2 + by^2) \quad (3)$$

which is therefore called the *indicatrix* of  $S$  at  $O$ .

When  $a$  and  $b$  have the same sign, the indicatrix is an ellipse, real or imaginary, and, by (2), according as this sign is  $+$  or  $-$ ,  $S$  will lie wholly above or wholly below the plane  $z = 0$  in the neighborhood of  $O$ . We then call  $O$  an *elliptic point*. Thus any point on a sphere.

When  $a$  and  $b$  have contrary signs, the indicatrix is a hyperbola whose asymptotes are the two lines represented by  $ax^2 + by^2 = 0, z = 0$ . It then follows from (2) that in one of the two pairs of opposite regions bounded by the planes  $ax^2 + by^2 = 0, S$  near  $O$  will lie above the plane  $z = 0$ , and in the other pair below it, and that  $S$  will cut the plane  $z = 0$  in

curves which touch the lines  $ax^2 + by^2 = 0$ ,  $z = 0$  at  $O$ . We call  $O$  a *hyperbolic point*. Thus any point on a hyperbolic paraboloid (Fig. 112).

When  $b$  or  $a$  is 0, the indicatrix is a pair of parallel lines, and (2) represents a parabolic cylinder which touches the plane  $z = 0$  along  $Ox$  or  $Oy$ . Hence  $O$  is called a *parabolic point* of  $S$ . But it will be shown later that when  $a$  or  $b$  is 0, the form of  $S$  near  $O$  may also depend on terms in the expansion of  $z$  which involve higher powers of  $x, y$ .

2. The curves in which  $S$  is cut by planes through the normal  $Oz$  are called the *normal sections* of  $S$  at  $O$ . It can be proved that one of the numbers  $a, b$  in (2) is the curvature of the section of greatest curvature at  $O$  and the other that of the section of least curvature.<sup>1</sup> The numbers  $1/a$  and  $1/b$  are called the *principal radii* of curvature of  $S$  at  $O$ . The product  $ab$  is called the *total curvature* of  $S$  at  $O$ . When  $ab$  is + everywhere on  $S$ ,  $S$  is called a *surface of positive curvature*; when  $-$ , a *surface of negative curvature*.

### EXERCISE XLVI

1. Find the tangent planes and normals to the surfaces

(1)  $z = x^2 + xy - 2y^2$  at  $(1, 2, -5)$

(2)  $x^2y + y^2z + z^2x = 1$  at  $(-1, 1, 0)$

2. Find the points of the surface  $x^2 - y^2 + 2yz + 2x = 0$  where the tangent plane is parallel to the  $xy$ -plane.

3. Find the equation of the cylindrical surface parallel to  $Oz$  and touching the surface  $2xz + z^2 + y = 0$ .

<sup>1</sup> Let the normal plane which makes the angle  $\theta$  with  $Ox$  cut the  $xy$ -plane in  $OD$ , and  $S$  in the curve  $C$ . Let  $P$  be any point of  $C$  near  $O$ , and let  $OPR$  be the circle which touches  $C$  at  $O$  and passes through  $P$ . When  $P \rightarrow O$  on  $C$ ,  $OPR$  becomes the circle of curvature of  $C$  at  $O$ , § 97, Ex. 2. Therefore, since  $OD^2 = DP \cdot DQ$ , we have  $\lim DP/OD^2 = 1/2\rho$ , where  $\rho$  is the radius of curvature of  $C$  at  $O$ ; and since  $z = DP$ ,  $x = OD \cos \theta$ ,  $y = OD \sin \theta$ , we have, by (2),  $1/\rho = a \cos^2 \theta + b \sin^2 \theta$ . This is Euler's formula for the curvature of any normal section. Evidently the greatest and least values of  $1/\rho$  are  $a$  and  $b$ , or  $b$  and  $a$ .

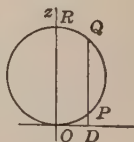


FIG. 120.



4. Find the tangent line and normal plane to the curve  $xy + yz = 4$ ,  $x^2 + y^2 + z^2 = 14$  at  $(-1, 2, 3)$ .

5. At what angles does the curve  $x = t$ ,  $y = t^2$ ,  $z = t^3$  meet the plane  $3x - 7y + 2z = 0$ ?

6. Two surfaces  $F_1 = 0$ ,  $F_2 = 0$  are said to be *orthogonal* at a point of intersection  $P$  if their normals at  $P$  are perpendicular. Show that at  $P$  we then have

$$\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial x} + \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial y} + \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial z} = 0$$

7. What is the condition that the curve  $F_1 = 0$ ,  $F_2 = 0$  touch the surface  $F_3 = 0$  at  $P$ ?

8. Show that the condition that  $F_1 = 0$ ,  $F_2 = 0$ ,  $F_3 = 0$  touch the same line at  $P$  is that  $F_1$ ,  $F_2$ ,  $F_3$  and the functional determinant of  $F_1$ ,  $F_2$ ,  $F_3$  with respect to  $x$ ,  $y$ ,  $z$  all be 0 at  $P$ .

9. Show that  $x^2 + x - 2y + z = 0$ ,  $y^2 - 2x + y + z = 0$ , and  $z^2 + x + y - 2z = 0$  touch the same line at  $O$ .

10. Find the length of the arc of  $2y = x^2$ ,  $3z = 4x^{3/2}$  between  $x = 0$  and  $x = 1$ .

11. Show that as a moving point  $P$  passes through a point  $Q$  of a surface  $F = 0$  where  $F_x$ ,  $F_y$ ,  $F_z$  are not all 0, the value of  $F$  at  $P$  changes sign.

12. Find the osculating plane to the curve  $y = x^3$ ,  $z = x^4$  at  $(1, 1, 1)$ ; also the tangent, principal normal, and binormal.

13. Find the osculating plane to the curve  $x^2 + y^2 + z^2 = 14$ ,  $yz + zx + xy = 11$  at  $(1, 2, 3)$ .

14. Show that as  $P$  moves on the curve  $C$ , the acceleration vector  $PA$  lies always in the osculating plane at  $P$ ; and that if the speed is constant,  $PA$  lies along the principal normal at  $P$ .

15. Prove the following relations between the  $s$ - and  $t$ -derivatives in curvilinear motion:

$$(1) \frac{ds}{dt} = \frac{dx}{ds} \frac{dx}{dt} + \frac{dy}{ds} \frac{dy}{dt} + \frac{dz}{ds} \frac{dz}{dt} \quad (2) \frac{d^2x}{dt^2} = \frac{dx}{ds} \frac{d^2s}{dt^2} + \frac{d^2x}{ds^2} \left( \frac{ds}{dt} \right)^2$$

16. If a particle of mass  $m$  be at  $O$ , and  $r = OP$ , then  $m/r$  is called the *potential* at  $P$  due to  $m$ . Prove that its directional derivative in any direction is the component in that direction of the force with which  $m$  attracts a unit particle at  $P$ .

17. Find the curvature and torsion at any point of the helix whose equations are  $x = a \cos \theta$ ,  $y = a \sin \theta$ ,  $z = k\theta$ .



18. Find the total curvature at  $O$  of the surface  $z = 3x^2 - 4y^2$ ; of  $z = x^2 - 4xy + 3y^2$ .

19. If a plane through  $OD$  in Fig. 120, and making the angle  $\phi$  with  $Oz$ , cuts  $S$  in  $C'$ , the radius of curvature of  $C'$  at  $O$  equals that of the normal section  $C$  multiplied by  $\cos \phi$ . Prove this theorem.

## ENVELOPES

**243. Envelopes.** In the  $xy$ -plane, an equation  $f(x, y, a) = 0$  which involves an arbitrary constant or parameter  $a$  represents an infinite set or family of curves, there being one curve for each value of  $a$ . Let  $S$  denote this family of curves. It may be that a curve  $E$  exists which all the curves of  $S$  touch; we then call  $E$  the *envelope* of  $S$ .

The tangents to any given curve  $C$  constitute a family of lines of the type  $S$ , and the curve is its envelope. Let  $t_1$  be a particular tangent, and  $t$  any neighboring tangent. When  $t$  rolls on  $C$  into coincidence with  $t_1$ , the point  $P$  where  $t$  cuts  $t_1$  approaches a definite point  $A$  of  $t_1$  as limit, namely the point where  $t_1$  touches  $C$ . This fact suggests the following method of finding the envelope of any family of curves  $f(x, y, a) = 0$ , if it have one.

We assume that the values of  $x, y, a$  under consideration belong to a region  $R$ , § 214, 2. in which  $f(x, y, a)$  and its partial derivatives of the first and second orders with respect to  $x, y, a$  are continuous.

Suppose that a representative curve  $(a)$  of  $S$  is met by a neighboring curve  $(a + \Delta a)$  in a point  $P$ , and that when  $\Delta a \rightarrow 0$ ,  $P$  approaches a definite non-singular point  $A$  of  $(a)$  as limit.

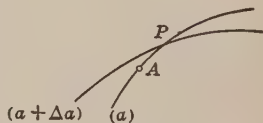


FIG. 121.

By the mean value theorem, § 97 (6), we have

$$f(x, y, a + \Delta a) - f(x, y, a) = f_a(x, y, a + \theta \Delta a) \Delta a \quad (1)$$

Therefore, since the  $x, y$  of  $P$  satisfy  $f(x, y, a) = 0$  and  $f(x, y, a + \Delta a) = 0$ , they also satisfy  $f_a(x, y, a + \theta \Delta a) = 0$ .

When  $\Delta a \rightarrow 0$ , then  $f_a(x, y, a + \theta \Delta a) \rightarrow f_a(x, y, a)$ , and  $P \rightarrow A$ . Hence the  $x, y$  of  $A$  satisfy the equations

$$f(x, y, a) = 0 \quad (2) \quad \text{and} \quad f_a(x, y, a) = 0 \quad (3)$$

and the locus  $L$  of all these limit points satisfies the equation got by eliminating  $a$  between (2) and (3).

We are to prove that, if  $\partial f_a / \partial a \neq 0$  on  $L$ , then  $L$  touches each curve  $(a)$  of  $S$  at the limit point or points  $A$  on  $(a)$  and is therefore the envelope of  $S$ .

Unless  $\partial f_a / \partial a$  is 0 at  $A$ , the equation  $f_a(x, y, a) = 0$  has a solution for  $a$  in terms of  $x, y$  at  $A$  and in its neighborhood, § 226; let this solution be

$$a = \phi(x, y) \quad (4)$$

If we substitute (4) in  $f(x, y, a) = 0$ , we obtain an equation

$$f[x, y, \phi(x, y)] = 0 \quad (5)$$

which represents  $L$  at  $A$  and in its neighborhood.

Hence the slope of the tangent to  $L$  at  $A$  is that given by

$$f_x dx + f_y dy + f_a d\phi = 0 \quad (6)$$

But since  $f_a$  is 0 at  $A$  and  $d\phi$  is finite, this equation reduces to

$$f_x dx + f_y dy = 0 \quad (7)$$

which also gives the slope of the curve  $(a)$  at  $A$ . Hence  $L$  touches  $(a)$  at  $A$ , as was to be proved.<sup>1</sup> Therefore

*The envelope of  $S$ , if there be one, satisfies the equations  $f = 0$ ,  $f_a = 0$  and therefore also the equation  $E(x, y) = 0$  got by eliminating  $a$  between  $f = 0$  and  $f_a = 0$ .*

The proof requires that the curves  $(a)$  and  $(a + \Delta a)$  meet. But the intersections need not be real points. The proof also holds good when they are imaginary points.

<sup>1</sup> Observe that this conclusion would not follow were  $A$  a singular point of  $(a)$  since then both  $f_x$  and  $f_y$  would be 0 at  $A$ . It can be easily proved that if all the curves of  $S$  have singular points, the locus  $L'$  of these points also satisfies the equations  $f = 0$  and  $f_a = 0$ ; but ordinarily  $L'$  does not touch the curves of  $S$ .

One can also prove directly that if a curve  $C$  envelopes the curves  $f(x, y, a) = 0$ , then the  $x, y$  of any point  $P$  of  $C$  satisfy the equations  $f = 0, f_a = 0$ . For the  $x, y$  of  $P$  are functions of  $a$ , say  $x = g(a), y = h(a)$ , such that

$$f[g(a), h(a), a] \equiv 0 \quad (8) \quad \text{and} \quad h'(a)/g'(a) = -f_x/f_y \quad (9)$$

Differentiating (8) with respect to  $a$  and using (9) gives  $f_a = 0$ .

**EXAMPLE 1.** Find the envelope of the family of circles

$$5x^2 + 5y^2 - 10ax + 4a^2 = 0 \quad (1)$$

Here  $f_a = 0$  is

$$-10x + 8a = 0 \quad (2)$$

Eliminating  $a$  between (1), (2),

$$4y^2 - x^2 = 0$$

Hence the envelope is the pair of lines  $y = x/2, y = -x/2$ .

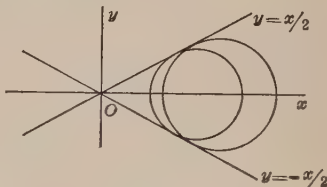


FIG. 122.

**EXAMPLE 2.** Find the envelope of the circles through  $O$  with centers on  $y^2 = x$ .

A circle through  $O$  with center at  $(a, b)$  has the equation

$$x^2 + y^2 - 2ax - 2by = 0$$

But since  $(a, b)$  is on  $y^2 = x$ , we have  $b^2 = a$ . Hence

$$x^2 + y^2 - 2b^2x - 2by = 0$$

Eliminating  $b$  between  $f = 0$  and  $f_b = 0$  gives

$$y^2(2x + 1) + 2x^3 = 0$$

**EXAMPLE 3.** Find the envelope of a line of constant length  $k$  whose ends move on  $Ox, Oy$ .

The intercept equation of the line is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad (1) \quad \text{where } a^2 + b^2 = k^2 \quad (2)$$

Regarding  $b$  as a function of  $a$ , and differentiating (1), (2),

$$\frac{x}{a^2} + \frac{y}{b^2} \frac{db}{da} = 0, \quad a + b \frac{db'}{da} = 0 \quad \therefore \frac{x}{a^3} = \frac{y}{b^3} \quad \therefore \frac{x^{1/3}}{a} = \frac{y^{1/3}}{b} \quad (3)$$

The elimination of  $a, b$  between (1), (2), (3) gives  $x^{2/3} + y^{2/3} = k^{2/3}$ .

**244. Envelopes of surfaces.** 1. An equation of the form  $f(x, y, z, a) = 0$  in which  $a$  is an arbitrary constant or parameter represents a family of surfaces  $S$ . Suppose that a representative surface  $(a)$  of  $S$  is cut by a neighboring surface

$(a + \Delta a)$  in a curve and that when  $\Delta a \rightarrow 0$  this curve approaches a definite limiting position  $C$ . We then call  $C$  a *characteristic* of  $S$ . Let  $E$  denote the surface formed of all the characteristics of  $S$ . Under conditions similar to those stated in § 243 and by the reasoning of that section it can be proved that this surface  $E$  satisfies the equations

$$f = 0 \qquad \frac{\partial f}{\partial a} = 0 \qquad (1)$$

and therefore the equation  $E(x, y, z) = 0$  got by eliminating  $a$  between  $f = 0$  and  $f_a = 0$ ; also that  $E$  touches each surface  $(a)$  of  $S$  all along the characteristic  $C$  of  $(a)$ . Hence  $E$  is called the *envelope* of  $S$ .

When  $f$  is of the first degree in  $x, y, z$ , and therefore  $S$  is a family of planes, the characteristics are straight lines and the envelope is called a *developable surface*. The osculating planes to a given curve in space form such a "one-parameter" family  $S$ ; the envelope is the surface formed of the tangent lines to the given curve.

2. An equation  $f(x, y, z, a, b) = 0$ , in which  $a$  and  $b$  are arbitrary constants, represents a doubly infinite or *two-parameter family* of surfaces  $S$ . Under conditions analogous to those of § 243, it can be proved that a surface  $E$  exists, called the *envelope* of  $S$ , which touches each surface of  $S$  in one or more isolated points; and that  $E$  satisfies the equations

$$f = 0 \qquad \frac{\partial f}{\partial a} = 0 \qquad \frac{\partial f}{\partial b} = 0 \qquad (2)$$

and therefore the equation  $E(x, y, z) = 0$  got by eliminating  $a, b$  between these equations.

EXAMPLE 1. Show that the envelope of the one-parameter family of spheres  $5x^2 + 5y^2 + 5z^2 - 10ax + 4a^2 = 0$  is the conical surface  $4(y^2 + z^2) - x^2 = 0$ .

EXAMPLE 2. Show that the envelope of the two-parameter family of spheres  $(x - a)^2 + (y - b)^2 + z^2 = 2a + 2b$  is the surface

$$z^2 = 2(x + y + 1).$$

## EXERCISE XLVII

1. Find the envelopes of the following families of curves:

1.  $2x^2 + 2y^2 - 4\alpha x + \alpha^2 = 0$

4.  $(x - \alpha)^2 + (y - \alpha)^2 = 2$

2.  $y^2 - \alpha x + \alpha^2 = 0$

5.  $(x - \alpha)^2 + (y - \alpha)^2 = 2\alpha$

3.  $y = \alpha x + 1/\alpha$

6.  $\alpha y^2 = x(x + \alpha)^2$

2. Circles are described on the double ordinates of the parabola  $y^2 = 4x$  as diameters; show that the envelope of these circles is the parabola  $y^2 = 4(x + 1)$ .

3. The envelope of circles through  $O$  and with centers on  $y = x^2 - x$  is  $2y(x^2 + y^2) + (x - y)^2 = 0$ .

4. If  $P$  is on  $xy = c^2$ , the envelope of circles with diameter  $OP$  is  $(x^2 + y^2)^2 = 4c^2xy$ .

5. The envelope of circles which touch  $Ox$  and have centers on  $y = x^2$  is  $y(2x^2 + 2y^2 - y) = 0$ .

6. A straight line cuts  $Ox, Oy$  at  $A, B$ ; find its envelope when it moves so that

1.  $OAB$  is constant

2.  $OA + OB$  is constant.

7. Let  $A(a, 0)$  and  $B(0, b)$  be fixed points on  $Ox$  and  $Oy$ ,  $P$  a variable point on the line segment  $AB$ , and  $PM, PN$  the perpendiculars from  $P$  on  $Ox, Oy$ . Show that the equation of the envelope of  $MN$  is

$$(x/a)^{1/2} + (y/b)^{1/2} = 1.$$

8. Let  $A$  be a fixed point on  $Ox$ , and  $P$  a variable point on  $Oy$ . Show that the envelope of the line through  $P$  perpendicular to  $AP$  is a parabola with vertex  $O$  and focus  $A$ .

9. The point  $P$  is on  $y^2 = x$ ;  $PQ$  is perpendicular to  $OP$ ; show that the envelope of  $PQ$  is  $27y^2 = 4(x - 1)^3$ .

10. Prove that the envelope of the ellipses  $x^2/\alpha^2 + y^2/\beta^2 = 1$  for which  $\alpha\beta = k$ , is the pair of hyperbolas  $4x^2y^2 = k^2$ .

11. The locus of the centers of curvature of a curve  $y = f(x)$  is called its *evolute*, § 81. The evolute is the envelope of the normals to the curve. Prove this by showing that if, taking the equation of the normal,  $(dy_1/dx_1)(y - y_1) + (x - x_1) = 0$ , we regard  $x_1, y_1$  as parameters connected by  $y_1 = f(x_1)$ , and differentiate with respect to  $x_1$ , we are led to the parametric equations of the evolute, § 80 (4).

12. By the method of Ex. 11 prove that the evolute of  $y = x^2$  is  $27x^2 = 2(2y - 1)^3$ .

13. The slope equation of the normal to the parabola  $y^2 = 4\alpha x$  is  $y = m(x - 2\alpha) - \alpha m^3$ ; find the evolute.

## MAXIMA AND MINIMA

**245. Maximum and minimum values.** 1. A function  $f(x, y)$  is said to have a *maximum value* at the point  $(a, b)$  if  $f(a, b)$  is its greatest value in some region  $S$ , § 214, 6., about  $(a, b)$ ; that is, if

$$f(a + h, b + k) - f(a, b) < 0 \quad (1)$$

for all values of  $h$  and  $k$  (except 0, 0) which are numerically less than some positive number  $d$ .

In like manner,  $f(x, y)$  is said to have a *minimum value* at the point  $(a, b)$  if

$$f(a + h, b + k) - f(a, b) > 0 \quad (2)$$

for all values of  $h, k$  such that  $|h|, |k| < d$ .

Similar definitions apply to functions of more than two variables.

2. In a region  $S$  in which  $f(x, y), f_x, f_y$  are continuous,  $f(x, y)$  can be a maximum or minimum only at points where

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0 \quad (3)$$

For the statement that  $f(x, y)$  has a maximum or minimum value at  $(a, b)$  implies that the  $x$ -function  $f(x, b)$  has a maximum or minimum value at  $x = a$ , and therefore that  $f_x(a, b) = 0$ , § 45. Similarly,  $f_y(a, b) = 0$ .

A similar theorem holds good for  $f(x, y, z)$ , and so on.

3. At a point  $(a, b)$  where  $f_x, f_y$  are 0,  $f(x, y)$  may be a maximum, or a minimum, or neither. We have the following test:

Suppose that  $f(x, y)$  and its partial derivatives to those of the third order are continuous at  $(a, b)$  and in its neighborhood, that is, in some region  $S'$  about  $(a, b)$ . Then, since  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$ , we have by Taylor's Theorem, § 222, if  $(a + h, b + k)$  is in  $S'$ ,

$$f(a + h, b + k) - f(a, b) = \frac{1}{2}(Ah^2 + 2Bhk + Ck^2) + \frac{1}{6}R \quad (4)$$

$$\text{where} \quad A = \frac{\partial^2}{\partial a^2} f(a, b) \quad B = \frac{\partial^2}{\partial a \partial b} f(a, b) \quad C = \frac{\partial^2}{\partial b^2} f(a, b)$$

$$\text{and} \quad R = \left( \frac{\partial}{\partial a} h + \frac{\partial}{\partial b} k \right)^3 f(a + \theta h, b + \theta k) \quad 0 < \theta < 1$$

The expression  $Ah^2 + 2Bhk + Ck^2$  has a constant sign for all values of  $h, k$  ( $\neq 0$ ) if its factors are imaginary, that is, if  $B^2 - AC < 0$  or  $AC > B^2$  (which implies that  $A$  and  $C$  have the same sign); and this constant sign is that of  $A$ , or  $C$ . Moreover in this case we can find<sup>1</sup> a positive number  $d$  such that when  $|h|, |k| < d$ , then

$$\frac{1}{2} |Ah^2 + 2Bhk + Ck^2| > \frac{1}{8} |R| \quad (5)$$

and therefore

$$\operatorname{sgn} [f(a+h, b+k) - f(a, b)] = \operatorname{sgn} (Ah^2 + 2Bhk + Ck^2). \quad \text{Hence}$$

If  $AC > B^2$ , then  $f(a, b)$  is a maximum or minimum: namely a maximum if  $A < 0$ , a minimum if  $A > 0$ .

If  $AC < B^2$ , then  $Ah^2 + 2Bhk + Ck^2$  has real and distinct factors, and if  $AC = B^2$ , it has equal factors. In the first case, its sign is not constant and therefore  $f(a, b)$  is not a maximum or minimum. In the second case, the sign of  $f(a+h, b+k) - f(a, b)$  depends also on terms in higher powers of  $h, k$ .

The corresponding test for  $f(x, y, z)$  at points where  $f_x, f_y, f_z = 0$  is somewhat complicated and will not be given here.

EXAMPLE 1. Find the maxima and minima of  $z = x^3 + y^2 - 3x$ .

$$f_x = 3x^2 - 3, \quad f_y = 2y; \quad f_{xx} = 6x, \quad f_{xy} = 0, \quad f_{yy} = 2$$

The equations  $f_x = 0, f_y = 0$  have the solutions  $(1, 0)$  and  $(-1, 0)$ . At  $(1, 0)$  we have  $AC > B^2$  and  $A > 0$ ; hence  $f(1, 0) = -2$  is a minimum. At  $(-1, 0)$  we have  $AC < B^2$ ; hence  $f(-1, 0) = 2$  is neither a maximum nor minimum.

The surface represented by  $z = x^3 + y^2 - 3x$  touches the plane  $z = -2$  at the point  $(1, 0, -2)$  and lies above it in the neighborhood of the point. The surface also touches the plane  $z = 2$  at the point  $(-1, 0, 2)$  but lies partly above and partly below it in the neighborhood of the point. Compare § 242.

EXAMPLE 2. Test  $z = x^2 + xy + y^2 + y^3$  for maxima and minima.

EXAMPLE 3. For  $f(x, y) = (x - y)^2 - x^3$  show that  $f_x, f_y$  are 0 at  $(0, 0)$ , but that  $f(0, 0)$  is not a maximum or minimum.

<sup>1</sup> For, setting  $h = r \cos \phi, k = r \sin \phi$  gives

$$|Ah^2 + 2Bhk + Ck^2| = r^2 |F(\phi)| \quad \text{and} \quad |R| = r^3 |G(\phi)|$$

where  $F(\phi)$  and  $G(\phi)$  denote polynomials in  $\sin \phi, \cos \phi$ . Let  $m$  denote the least value of  $|F(\phi)|$ ; and  $M$  the greatest value of  $|G(\phi)|$  in  $S'$ . Then the inequality (5) will be satisfied when

$$\frac{1}{2} mr^2 > \frac{1}{8} Mr^3, \quad \text{that is, when } r < \frac{3m}{M}$$



4. In many cases it is unnecessary to use the test in 3. because of the following obvious theorem. (Compare § 47.)

Let  $S$  denote a given region in the  $xy$ -plane and  $C$  its boundary, and let  $u = f(x, y)$ ,  $f_x, f_y$  be continuous in  $S$ . Then  $u$  has a greatest value in  $S$ , § 18, 1., and it takes this value either on  $C$  or at some interior point or points where  $f_x, f_y = 0$ . Hence

*If there is one and but one point  $(a, b)$  within  $S$  at which  $f_x, f_y = 0$ , and if  $u$  has a greater value at  $(a, b)$  than anywhere on  $C$ , then  $f(a, b)$  is the greatest value of  $u$  in  $S$ . Similarly for the least value of  $u$  in  $S$ .*

There are corresponding theorems for  $f(x, y, z)$ , and so on.

**EXAMPLE 4.** The plane  $x/a + y/b + z/c = 1$ , where  $a, b, c > 0$ , cuts  $Ox, Oy, Oz$  at  $A, B, C$ . Find when the volume  $u$  of the rectangular parallelepiped which has three faces in the  $yz$ -,  $zx$ -, and  $xy$ -planes and a corner  $P(x, y, z)$  in the triangle  $ABC$ , is greatest.

$$u = xyz \quad (1) \quad \text{where} \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad (2)$$

Because of (2), but two of  $x, y, z$  are independent; let these be  $x, y$ .

The  $x, y$ -function  $u = cxy(1 - x/a - y/b)$ , being continuous in the triangle  $OAB$ , has a greatest value in  $OAB$ . Furthermore  $u$  is positive within  $OAB$  and is 0 on the boundary. Hence if there be an interior point, and but one, where  $\partial u/\partial x, \partial u/\partial y = 0$ , then  $u$  takes its greatest value at that point.

Differentiating (1), (2) with respect to  $x$  and  $y$ , and setting  $\partial u/\partial x, \partial u/\partial y = 0$ ,

$$yz + xy \frac{\partial z}{\partial x} = 0 \quad \frac{1}{a} + \frac{1}{c} \frac{\partial z}{\partial x} = 0 \quad (3)$$

$$xz + xy \frac{\partial z}{\partial y} = 0 \quad \frac{1}{b} + \frac{1}{c} \frac{\partial z}{\partial y} = 0 \quad (4)$$

Eliminating  $\partial z/\partial x$  between the equations (3) gives  $y(z/c - x/a) = 0$ , and therefore, since  $y$  is  $\neq 0$  within  $OAB$ ,  $z/c - x/a = 0$ .

Similarly, it follows from the equations (4) that  $z/c - y/b = 0$ .

Hence  $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$  and therefore by (2)  $x = \frac{a}{3}, y = \frac{b}{3}, z = \frac{c}{3}$

Therefore the greatest value of  $u$  is  $abc/27$ .

**EXAMPLE 5.** Show that if  $u = f(x, y, z)$  is a rational integral function, and when not 0 is positive, and if  $f_x, f_y, f_z$  are 0 at one and but one point  $(a, b, c)$ , then  $f(a, b, c)$  is the least value of  $u$ .

**EXAMPLE 6.** Find the point the sum of the squares of whose distances from three given points  $(a_1, b_1, c_1)$ ,  $(a_2, b_2, c_2)$ ,  $(a_3, b_3, c_3)$  is least.

**246. Implicit functions.** 1. In § 245, Ex. 4, an illustration is given of the method of finding the maximum or minimum values, if any, of a function of the type  $u = f(x, y, z)$  when the variables  $x, y, z$  are connected by some relation  $\phi(x, y, z) = 0$ . The method may be extended to the case in which  $u$  is given as a function of  $n$  variables connected by  $m(< n)$  equations.

**EXAMPLE 1.** Find the major and minor axes of the ellipse in which the ellipsoid  $x^2 + 2y^2 + z^2 = 60$  is cut by the plane  $x + y - z = 0$ .

We seek the least and greatest values of

$$u = x^2 + y^2 + z^2 \quad (1)$$

when  $x, y, z$  are connected by

$$x^2 + 2y^2 + z^2 = 60 \quad (2) \quad \text{and} \quad x + y - z = 0 \quad (3)$$

The equations (2), (3) define  $y$  and  $z$  as functions of  $x$ . Hence differentiating (1), (2), (3) with respect to  $x$  and setting  $du/dx = 0$ , we have

$$x + y \frac{dy}{dx} + z \frac{dz}{dx} = 0 \quad x + 2y \frac{dy}{dx} + z \frac{dz}{dx} = 0 \quad 1 + \frac{dy}{dx} - \frac{dz}{dx} = 0$$

Eliminating  $dy/dx$  and  $dz/dx$ , we get  $y(x + z) = 0 \therefore y = 0$  (4) or  $x + z = 0$  (5)

Solving (2), (3), (4), we find  $x^2 = 30, y^2 = 0, z^2 = 30 \therefore$ , by (1),  $u = 60$ .

Solving (2), (3), (5), we find  $x^2 = 6, y^2 = 24, z^2 = 6 \therefore$ , by (1),  $u = 36$ .

Hence the lengths of the major and minor semiaxes are  $2\sqrt{15}$  and 6. Their direction ratios are  $1 : 0 : 1$  and  $-1 : 2 : 1$ .

2. The following example illustrates the procedure when the variable whose maximum or minimum is sought is involved implicitly in one or more of the given equations.

**EXAMPLE 2.** Find the points of the circle  $x^2 + y^2 + z^2 = 6$  (1),  $x + y + z = 0$  (2) which are at the greatest distances from the  $xy$ -plane.

We seek the points of the circle where  $|z|$  is greatest. The equations (1) and (2) define any two of  $x, y, z$  as functions of the third. To solve our problem we must choose  $z$  as one of these dependent variables. Let  $y$  be the other, so that  $x$  is the independent variable. Then differentiating with respect to  $x$ , we have

$$x + y \frac{dy}{dx} + z \frac{dz}{dx} = 0 \qquad 1 + \frac{dy}{dx} + \frac{dz}{dx} = 0$$

Setting  $dz/dx = 0$ , and then eliminating  $dy/dx$ , we get  $x - y = 0$  (3)

Solving the equations (1), (2), (3), we find the points  $(-1, -1, 2)$  and  $(1, 1, -2)$ .

Hence the greatest distance of any point of the circle from the  $xy$ -plane is 2.

### EXERCISE XLVIII

1. Find the maximum and minimum values of the following:

$$(1) \ x^2 + 2xy + 2y^2 + 4x - 6y \qquad (2) \ x^3 - y^3 + x^2 + y^2$$

2. Show that the cube is the greatest rectangular parallelepiped of given surface area.

3. For what rectangular parallelepiped of given volume is the sum of the edges least?

4. Find the shortest distance between the lines  $y = 2x$ ,  $z = 3x$  and  $y = x + 3$ ,  $z = x$ .

5. Find the rectangular parallelepiped of greatest volume that can be inscribed in the ellipsoid  $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ .

6. What is the minimum distance of a point of the curve  $xyz = 1$ ,  $x + 2y - z = 0$  from the  $xy$ -plane?

7. If  $f(x, y, z) = 0$  be given, show that the maximum or minimum values of  $z$ , if any, are determined by the equations  $f = 0, f_x = 0, f_y = 0$ . What is the geometrical meaning of these conditions?

8. If  $(x + y)^2 + (y + z)^2 + (z + x)^2 = 3$ , show that the greatest and least values of  $z$  are  $3/2$  and  $-3/2$ .

9. Divide a given positive number  $a$  into three parts  $x, y, z$  such that  $x^m y^n z^p$  may be a maximum,  $m, n, p$  denoting positive constants.

10. A torpedo has the shape of a cylinder with conical ends. For given surface area what dimensions will give the greatest volume?

11. A house of given cubical content is to be constructed on a square base and with a gable roof. For what dimensions will the combined areas of the walls and roof be least?

12. Let  $ABC$  be a given triangle in the  $xy$ -plane. Respecting pyramids of given altitude  $h$  which have  $ABC$  for base, prove that the one of least surface area has its apex vertically above the center of the inscribed circle of  $ABC$ .

13. Let  $a, b, c$  denote the lengths of the sides of a given triangle,  $\Delta$  its area, and  $x, y, z$  the perpendicular distances of an inner point from the sides. Show that the least value of  $x^2 + y^2 + z^2$  is  $4\Delta^2/(a^2 + b^2 + c^2)$ .

14. The plane  $x + y + z = 0$  cuts the hyperbolic paraboloid  $z = xy$  in an hyperbola one branch of which passes through  $O$ , the other not. Find the point of this second branch which is nearest to  $O$ .

15. By the method of maxima and minima, find the point of the plane  $ax + by + cz + d = 0$  which is nearest to  $O$ ; also the point of the line of intersection of the planes  $ax + by + cz + d = 0$ ,  $a'x + b'y + c'z + d' = 0$  which is nearest to  $O$ .

16. The points  $P$  and  $P'$  are on two non-intersecting algebraic surfaces  $S$  and  $S'$ ; show that when  $PP'$  is shortest it is normal to both  $S$  and  $S'$ .

17. Prove the corresponding theorem for two non-intersecting space curves  $C$  and  $C'$ .

18. Show that a polygon of  $n$  sides inscribed in a circle is greatest when it is regular.

19. Prove the following rule, called *Lagrange's method of multipliers* :

Let  $u = f(x, y, z)$ , where  $\phi(x, y, z) = 0$ . To find the values of  $x, y, z$ , if there be any, which make  $u$  a maximum or minimum, form the expression  $F = f + \lambda\phi$ , in which  $\lambda$  is an undetermined constant, and then solve the equations  $F_x = 0, F_y = 0, F_z = 0, \phi = 0$  for  $x, y, z$ .

## XXV. DIFFERENTIAL EQUATIONS

**247. Differential equations.** An equation which involves derivatives or differentials is called a *differential equation*.

Examples of such equations have been given in § 110. Other examples are :

$$1. \ x \frac{dy}{dx} - y = 0 \text{ or } x \, dy - y \, dx = 0 \quad 2. \ z \, dx + x \, dy + y \, dz = 0$$

$$3. \ \frac{d^2y}{dx^2} + 4 \, y = 0 \quad 4. \ \left(x - y \frac{dy}{dx}\right)^2 = y \quad 5. \ \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z.$$

If partial derivatives occur in the equation, it is called a *partial differential equation* (5.); if not, it is called an *ordinary differential equation* (1.-4.).

The order of the derivative of highest order in a differential equation is called the *order of the equation*. Thus 1., 2., 4., 5. are of the first order, and 3. is of the second order.

When an equation is rational and integral with respect to all the derivatives which occur in it, its degree with respect to the derivative of highest order is called the *degree of the equation*. Thus 1., 2., 3., 5. are of the first degree, and 4. is of the second degree.

**248. Solutions of differential equations.** 1. Let  $F = 0$  denote a given differential equation in  $x$ ,  $y$ , and  $x$ -derivatives of  $y$ . Any function  $y$  of  $x$ , whether expressed in the form  $y = \phi(x)$  or the form  $f(x, y) = 0$ , which satisfies  $F = 0$  identically, is called a *solution* of  $F = 0$ .

**EXAMPLE 1.** Prove that  $y = e^{3x}$  is a solution of  $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$ .

If  $y = e^{3x}$ , then  $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = (9 - 3 - 6)e^{3x} = 0$ .

EXAMPLE 2. Show that  $y^2 - x + 1 = 0$  (1) is a solution of

$$4y \left( \frac{dy}{dx} \right)^2 - 2x \frac{dy}{dx} + y = 0 \quad (2)$$

Differentiating (1) gives  $2y \frac{dy}{dx} - 1 = 0$  or  $\frac{dy}{dx} = \frac{1}{2y}$  (3). Substituting (3) in (2) gives  $\frac{1}{y} - \frac{x}{y} + y = \frac{y^2 - x + 1}{y}$ , which  $= 0$  because of (1).

2. From the equation  $d^2y/dx^2 = 0$  (1) we obtain, by integration,  $y = C_1x + C_2$  (2), where  $C_1, C_2$  are arbitrary constants. This is the *general solution* of (1). Solutions, as  $y = 3x, y = x - 5$ , got by assigning particular values to the constants  $C_1, C_2$  in (2) are called *particular solutions* of (1).

Similarly the general solution of  $x dy/dx + y = 0$  is  $xy = C$  (3).

We shall be chiefly concerned with differential equations  $F = 0$  whose general solutions can, like (2) and (3), be obtained by the process of integration. We call them *integrable equations*. If such an equation is of the  $n$ th order, the number of integrations required to obtain its general solution will be  $n$ , and this solution will therefore involve  $n$  arbitrary constants.

3. If  $f = 0$  denote an equation in  $x, y$  which involves  $n$  arbitrary constants algebraically, we can obtain the differential equation  $F = 0$  of which it is the general solution by algebraically eliminating the  $n$  constants between  $f = 0$  and the  $n$  equations got by differentiating  $f = 0$   $n$  times with respect to  $x$ .

EXAMPLE 3. Find the differential equation whose general solution is  $x^2 + y^2 + 2Cx = 0$ .

$$\text{Differentiating} \quad x^2 + y^2 + 2Cx = 0 \quad (1)$$

$$\text{gives} \quad x + y \frac{dy}{dx} + C = 0 \quad (2)$$

Eliminating  $C$  between (1) and (2) gives

$$x^2 - y^2 + 2xy \frac{dy}{dx} = 0 \quad (3)$$

Geometrically both (1) and (3) represent the set of all circles which touch  $Oy$  at  $O$

EXAMPLE 4. Find the differential equations of which the following are general solutions :

1.  $y = C_1 + C_2x + C_3x^2$
2.  $y = C_1 \sin 2x + C_2 \cos 2x$
3.  $Ax^2 + By^2 - 2x = 0$
4.  $(y - B)^2 = 4Ax$

4. To illustrate the sense in which a differential equation  $F = 0$  in general, whether integrable or not, may be said to have solutions, consider the equation

$$\frac{dy}{dx} = f(x, y) \quad (1)$$

in a region  $S$  of the  $xy$ -plane in which  $f$  and its partial derivatives are real, one-valued, and continuous. Take any point  $P(a, b)$  in  $S$ . The value of  $dy/dx$  at  $P$  is  $f(a, b)$ , and by successive differentiations of (1) we can find the values of  $d^2y/dx^2$ ,  $d^3y/dx^3$ ,  $\dots$  at  $P$ . Hence (1) and  $P$  determine a definite Taylor series :

$$y = b + \left(\frac{dy}{dx}\right)_{a,b} \frac{x - a}{1} + \left(\frac{d^2y}{dx^2}\right)_{a,b} \frac{(x - a)^2}{2!} + \dots \quad (2)$$

It is proved later that for all values of  $x$  within a certain distance  $l$  of  $a$ , this series converges and satisfies (1) identically. Hence the function which it defines is a particular solution of (1). It may be called the solution of (1) at  $P$ .

If in (2) we replace  $b$  by an arbitrary constant  $C$ , we obtain an equation of the form  $y = \phi(x, C)$  which represents the general solution of (1) within a certain rectangle  $R$  whose center is  $P$  and whose sides are parallel to  $Ox$  and  $Oy$ . Geometrically, it represents a set of curves, called *integral curves* of (1), one of which, and but one, passes through each inner point of  $R$ . A point  $Q$  moving across  $R$  in such a manner that the tangent of its direction angle always equals the value of  $f(x, y)$  at  $Q$  will trace one of these curves.

For regions in which  $\phi_C \neq 0$  we may regard  $y = \phi(x, C)$  as expressible in the form  $F(x, y) = C$ , § 226.



EXAMPLE 5. Given the equation  $\frac{dy}{dx} = x + y$ . At the point  $(0, C)$  we have

$$\begin{aligned}\frac{dy}{dx} &= x + y = C, & \frac{d^2y}{dx^2} &= 1 + \frac{dy}{dx} = 1 + C, \\ \frac{d^3y}{dx^3} &= \frac{d^2y}{dx^2} = 1 + C, \dots, & \frac{d^ny}{dx^n} &= \frac{d^{n-1}y}{dx^{n-1}} = 1 + C.\end{aligned}$$

Hence, by (2),

$$y = C + Cx + (C + 1) \left[ \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots \right]$$

This series converges for  $|x| < \infty$ , whatever the value of  $C$  may be. Hence it is the general solution of the given equation. It is equivalent to

$$y = (C + 1)e^x - (x + 1)$$

4. We may deal in a similar manner with an equation of the second order

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right) \quad (3)$$

Assign to  $x$  a particular value  $a$ , and arbitrary values  $C$ ,  $C'$  to  $y$ ,  $dy/dx$ , and then by aid of (3) compute  $d^2y/dx^2$ ,  $d^3y/dx^3$ ,  $\dots$  in terms of  $C$ ,  $C'$  and form the corresponding Taylor series for  $y$  in powers of  $x - a$ . We thus obtain an equation of the type  $y = \phi(x, C, C')$  which represents the general solution of (3) for values of  $x$  within a certain distance  $l$  of  $a$ , and values of  $C$ ,  $C'$  between certain limits.

This discussion may be extended to equations of any order  $n$  and shows the sense in which it is true in general that the general solution of a differential equation of the  $n$ th order involves  $n$  arbitrary constants.

EXAMPLE 6. Find the solution of  $\frac{d^2y}{dx^2} = y$  at  $x, y, \frac{dy}{dx} = 0, C, C'$ .

$$\text{Here } y = \frac{d^2y}{dx^2} = \frac{d^4y}{dx^4} = \frac{d^6y}{dx^6} = \dots = C, \quad \frac{dy}{dx} = \frac{d^3y}{dx^3} = \frac{d^5y}{dx^5} = \dots = C'$$

$$\text{Hence } y = C \left[ 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right] + C' \left[ x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right]$$

The series are convergent when  $|x| < \infty$ . Hence this is the general solution.

**249. Graphical representation.** The form and arrangement of the curves represented by a given differential equation  $dy/dx = f(x, y)$  may sometimes be inferred quite readily from the equation itself.

**EXAMPLE.** Find the form of the integral curves of  $\frac{dy}{dx} = y(1-x)$ .

$$\frac{dy}{dx} = y(1-x) \quad (1) \quad \therefore \frac{d^2y}{dx^2} = \frac{dy}{dx}(1-x) - y = yx(x-2) \quad (2)$$

Since  $y = 0$  is a solution,  $Ox$  is one of the curves. Since (1) remains unchanged when  $y$  is replaced by  $-y$ , the family is symmetric to  $Ox$ .

For  $y > 0$ , we have

$$\frac{dy}{dx} \begin{matrix} \leq \\ > \end{matrix} 0 \text{ as } 1-x \begin{matrix} \leq \\ > \end{matrix} 0 \quad (3), \quad \frac{d^2y}{dx^2} \begin{matrix} \leq \\ > \end{matrix} 0 \text{ as } x(x-2) \begin{matrix} \leq \\ > \end{matrix} 0 \quad (4)$$

Hence above  $Ox$  the curves have maximum points on  $x = 1(l)$ , and points of inflection on  $x = 0(l_1)$  and on  $x = 2(l_2)$ , and are convex downward between  $l_1$  and  $l_2$  and are elsewhere convex upward. No curve can meet  $Ox$ , since for points on  $Ox$ ,  $y = 0$  is the only solution; hence each curve tends to parallelism with  $Ox$  at the left and right. The curves below  $Ox$  are symmetric with those above  $Ox$ .

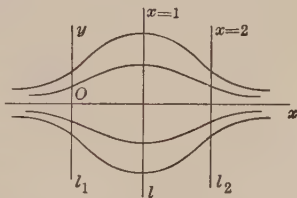


FIG. 123.

### EXERCISE XLIX

1. Find the differential equations of the following families of curves:

1. All straight lines of slope  $\pm 2$ .      2. All circles through  $O$ .

2. Find the equation of the family of curves defined by  $\frac{d^2y}{dx^2} = 6x$ ; also of the set of these curves which pass through the point  $(1, -1)$ , and of the single curve which has the slope 3 at  $(1, -1)$ .

3. The slope of the radius vector from  $O$  to  $P(x, y)$  is  $y/x$ . Hence show that

(1)  $\frac{dy}{dx} = \frac{y}{x}$  represents the pencil of straight lines through  $O$ .

(2)  $\frac{dy}{dx} = -\frac{x}{y}$  represents the family of circles whose center is at  $O$ .

(3)  $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$  represents the family of circles through  $O$  and there

tangent to  $Ox$ .

4. Express the particular solution of  $dy/dx = x^2 + y^2$  whose graph passes through the point  $(1, -1)$  by a Taylor series to the term involving  $(x - 1)^4$ .

5. Sketch the family of curves represented by  $dy/dx = x + y$ , showing that the line  $x + y + 1 = 0$  is one of them and that the others approach this line asymptotically.

### EQUATIONS OF THE FIRST ORDER AND DEGREE

**250. When the variables can be separated.** An equation of the first order and degree  $dy/dx = f(x, y)$  may be written in the differential form  $M dx + N dy = 0$ . It may happen that  $M$  and  $N$  are of the form  $M = X_1 Y_1$ ,  $N = X_2 Y_2$ , where  $X_1, X_2$  denote functions of  $x$  only, and  $Y_1, Y_2$  functions of  $y$  only. It is then possible, by dividing by  $Y_1 X_2$ , to reduce the equation to the form

$$\frac{X_1}{X_2} dx + \frac{Y_2}{Y_1} dy = 0$$

whose solution is  $\int \frac{X_1}{X_2} dx + \int \frac{Y_2}{Y_1} dy = C$  (1)

We call (1) the solution whether the integrations can be effected in finite terms or not.

**EXAMPLE 1.** Solve

$$\cos x \sin y dx + \sin x \cos y dy = 0$$

Dividing by  $\sin y \sin x$ , we get

$$\cot x dx + \cot y dy = 0$$

Integrating,  $\log \sin x + \log \sin y = \text{const.}$

$$\therefore \log \sin x \sin y = \text{const.} \quad \therefore \sin x \sin y = C.$$

**EXAMPLE 2.** Solve the following equations:

1.  $x^2(y-1) dx + (xy+y) dy = 0$

2.  $(y^2+1) dx - y(x^2+1) dy = 0$

3.  $\frac{dy}{dx} = 2xy$

4.  $(e^{x+v} - e^x) dx + (e^{x+v} + e^v) dy = 0$

5.  $x\sqrt{1-y^2} dx + y\sqrt{1-x^2} dy = 0$

**251. Homogeneous equations.** When  $M$  and  $N$  are homogeneous functions of  $x$  and  $y$ , and of the same degree,  $M dx + N dy = 0$  can be solved by aid of the substitution

$$y = vx$$

$$dy = v dx + x dv$$

For in the resulting equation in  $x$  and  $v$ ,  $M/N$  reduces to a function of  $v$  only, and the variables are therefore separable, as in § 250.

**EXAMPLE 1.** Solve  $2xy dx + (y^2 - x^2) dy = 0$

Substituting  $y = vx$ ,  $2x^2v dx + (v^2x^2 - x^2)(v dx + x dv) = 0$   
which reduces to  $(v^3 + v) dx + (v^2 - 1)x dv = 0$

Separating variables and integrating,  $\int \frac{dx}{x} + \int \frac{v^2 - 1}{v^3 + v} dv = \text{const.}$  (a)

By the method of partial fractions, § 118,  $\frac{v^2 - 1}{v^3 + v} = -\frac{1}{v} + \frac{2v}{v^2 + 1}$  (b)

Hence (a) gives  $\log x - \log v + \log (v^2 + 1) = \text{const.}$

Therefore  $\frac{(v^2 + 1)x}{v} = C$ , or setting  $v = y/x$  and simplifying,  
 $x^2 + y^2 = Cy$

**EXAMPLE 2.** Solve the following equations:

1.  $(2x + y) dx + (x + y) dy = 0$       2.  $2xy dx + (y^2 - 2x^2) dy = 0$

3.  $y^2 dx = (xy - x^2) dy.$       4.  $(\sqrt{xy} - x) dy + y dx = 0$

**EXAMPLE 3.** Solve  $(x - y + 3) dx + (x + y + 1) dy = 0$  (a)

The lines  $x - y + 3 = 0$  and  $x + y + 1 = 0$  meet at the point  $(-2, 1)$ . The substitution  $x = x' - 2$ ,  $y = y' + 1$  in (a) gives the homogeneous equation

$$(x' - y') dx' + (x' + y') dy' = 0 \quad (b)$$

Solving (b), as in Ex. 1, we get

$$\tan^{-1} \frac{y'}{x'} + \frac{1}{2} \log (x'^2 + y'^2) = C$$

Hence the solution of (a) is

$$\tan^{-1} \frac{y - 1}{x + 2} + \frac{1}{2} \log [(x + 2)^2 + (y - 1)^2] = C$$

All equations of the type  $(ax + by + c) dx + (a'x + b'y + c') dy = 0$  can be solved by this method with the exception of those in which  $a'x + b'y = k(ax + by)$ , where  $k$  is a constant. In this exceptional case, the solution may be obtained by aid of the substitution

$$u = ax + by$$

**EXAMPLE 4.** Solve (1)  $(2x + y - 4) dx + (x + y - 1) dy = 0.$

(2)  $(2x + 3y) dx - (2x + 3y - 2) dy = 0.$

**252. Linear equations.** 1. A differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad (1)$$

where  $P, Q$  are functions of  $x$ , is called a *linear equation* of the first order.

Let  $\int P dx$  denote any particular integral of  $P$ . When the first member of (1) is multiplied by  $e^{\int P dx}$ , it becomes the derivative of  $ye^{\int P dx}$ ; for

$$\frac{d}{dx} ye^{\int P dx} = \frac{dy}{dx} e^{\int P dx} + yPe^{\int P dx} = e^{\int P dx} \left[ \frac{dy}{dx} + Py \right]$$

Hence we can find the solution of (1) by *multiplying both members by  $e^{\int P dx}$  and then integrating with respect to  $x$* , the result being

$$ye^{\int P dx} = \int Qe^{\int P dx} dx + C \quad (2)$$

**EXAMPLE 1.** Solve  $\frac{dy}{dx} + \frac{2}{x}y = x^3$ .

$$\int P dx = \int \frac{2}{x} dx = 2 \log x = \log x^2 \quad \therefore e^{\int P dx} = e^{\log x^2} = x^2.$$

Hence, by (2), we have

$$yx^2 = \int x^5 dx + C = \frac{x^6}{6} + C, \quad \text{or} \quad y = \frac{x^4}{6} + \frac{C}{x^2}.$$

2. It follows from (1) and (2) that the solution of

$$\frac{d}{dx} \phi(y) + P\phi(y) = Q \quad \text{is} \quad e^{\int P dx} \phi(y) = \int Qe^{\int P dx} dx + C \quad (3)$$

**EXAMPLE 2.** Solve  $x \cos y \frac{dy}{dx} + (x+1) \sin y = e^x$ .

The equation may be written

$$\begin{aligned} \frac{d}{dx} \sin y + \left(1 + \frac{1}{x}\right) \sin y &= \frac{e^x}{x} \\ \int P dx &= \int \left(1 + \frac{1}{x}\right) dx = x + \log x \quad \therefore e^{\int P dx} = xe^x. \end{aligned}$$

Therefore the solution is

$$xe^x \sin y = \int \frac{e^x}{x} \cdot xe^x dx + C, \quad \text{that is,} \quad xe^x \sin y = \frac{e^{2x}}{2} + C.$$

**EXAMPLE 3.** Show that the equation  $\frac{dy}{dx} + Py = Qy^n$  may be reduced to the form  $\frac{d}{dx} y^{-n+1} + (-n+1)Py^{-n+1} = (-n+1)Q$ ; and indicate its solution.

**EXAMPLE 4.** Show that the solution of  $(x^2 + 1)\frac{dy}{dx} - xy = x^4y^3$  is  $5(x^2 + 1) + (2x^5 + C)y^2 = 0$ .

**EXAMPLE 5.** Solve the following equations:

$$1. \frac{dy}{dx} = x + y \quad 2. \frac{dy}{dx} = x^2 + y \quad 3. \frac{dx}{dy} = x + y$$

$$4. \sin x \frac{dy}{dx} + 2y \cos x = \sin 2x \quad 5. x^2 \frac{dy}{dx} + (1 - 2x)y = x^2$$

$$6. \frac{dy}{dx} + \frac{y}{\sqrt{x^2 + 1}} = \sqrt{x^2 + 1} \quad 7. x \frac{dy}{dx} + y = y^2 \log^2 x$$

$$8. dy + (ax + by + c) dx = 0 \quad 9. \frac{dy}{dx} + \frac{y}{(x-1)(x-2)} = x^2 - 1$$

10.  $ydx + (2x - y^2) dy = 0$ , which can be reduced to the form

$$\frac{dx}{dy} + P(y) \cdot x = Q(y)$$

**253. Exact differential equations.** When  $M dx + N dy$  is the total differential of some function  $u$  of  $x$  and  $y$ , § 216,  $M dx + N dy = 0$  is called an *exact differential equation*. It is then equivalent to  $du = 0$  and has the solution  $u = C$ .

1. Evidently if  $M dx + N dy$  can be reduced to a sum of known differentials, as

$$(1) d(xy) = y dx + x dy \quad (2) d(x^2 + y^2) = 2(x dx + y dy)$$

$$(3) d\frac{y}{x} = \frac{x dy - y dx}{x^2},$$

then  $M dx + N dy = 0$  is exact, and its solution is obtainable immediately.

**EXAMPLE 1.** Solve  $(2x - y) dx + (2y - x) dy = 0$ .

Regrouping terms,  $2(x dx + y dy) - (y dx + x dy) = 0$

$$\therefore d(x^2 + y^2) - dxy = 0 \quad \therefore x^2 + y^2 - xy = C.$$

EXAMPLE 2. Solve  $(3x^2y - y)dx + (3x^3 + x)dy = 0$ .

Regrouping terms, we obtain

$$3x^2(ydx + xdy) + (xdy - ydx) = 0$$

This is not an exact equation, but multiplying by  $1/x^2$ , we get

$$3(ydx + xdy) + \frac{xdy - ydx}{x^2} = 0$$

Hence  $3d(xy) + d\frac{y}{x} = 0 \quad \therefore 3xy + \frac{y}{x} = C$

2. If, as in Ex. 2,  $Mdx + Ndy = 0$  becomes an exact differential equation when multiplied by some function  $\mu$ , then  $\mu$  is called an *integrating factor* of  $Mdx + Ndy = 0$ .

EXAMPLE 3. Show that  $xdy - ydx = 0$  has the integrating factors  $1/x^2, 1/y^2, 1/xy, 1/(x^2 + y^2)$ .

EXAMPLE 4. Solve the following equations:

1.  $xdy - ydx = (x^2 + y^2)x dx$       2.  $ydx - xdy + y^2x dx = 0$

3.  $(xdy + ydx)(y + 1) + x^2y^2dy = 0$       4.  $x\frac{dy}{dx} = y + (x^2 + y^2)^{1/2}$

3. The necessary and sufficient condition that  $Mdx + Ndy = 0$  be an exact differential equation is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (1)$$

For  $Mdx + Ndy$  is a total differential when and only when a function  $u$  exists such that

$$\frac{\partial u}{\partial x} = M \quad (2)$$

$$\frac{\partial u}{\partial y} = N \quad (3)$$

(1) It follows from (2) and (3), if  $M, N$  have continuous first partial derivatives, that

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial M}{\partial y}, \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial N}{\partial x}, \quad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence (1) is the necessary condition that  $Mdx + Ndy = 0$  be an exact equation.

(2) But (1) is also the sufficient condition. The function  $\int M dx$  got by integrating  $M$  with respect to  $x$ , regarding  $y$  as constant, satisfies

(2). But

$$d \int M dx = M dx + \left[ \frac{\partial}{\partial y} \int M dx \right] dy$$

$$\therefore M dx + N dy = d \int M dx + \left[ N - \frac{\partial}{\partial y} \int M dx \right] dy$$



Hence  $M dx + N dy$  is a total differential if  $N - \frac{\partial}{\partial y} \int M dx$  is a function of  $y$  only, that is, if its partial derivative with respect to  $x$  is 0. But

$$\frac{\partial}{\partial x} \left[ N - \frac{\partial}{\partial y} \int M dx \right] = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y} \frac{\partial}{\partial x} \int M dx = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

and this is 0 when

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence when  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ , then  $M dx + N dy = 0$  is exact, and its solution is

$$\int M dx + \int \left[ N - \frac{\partial}{\partial y} \int M dx \right] dy = C \quad (4)$$

EXAMPLE 5. Solve  $(6x + 2y + 4) dx + (2x - 2y + 8) dy = 0$ .

Here  $\partial M / \partial y = 2$ ,  $\partial N / \partial x = 2$ ; hence the equation is exact.

$$\begin{aligned} \int M dx &= 3x^2 + 2xy + 4x, & \frac{\partial}{\partial y} \int M dx &= 2x \\ N - \frac{\partial}{\partial y} \int M dx &= -2y + 8 \end{aligned}$$

Hence the solution is  $3x^2 + 2xy + 4x - y^2 + 8y = C$ .

EXAMPLE 6. Show that the following are exact differential equations and solve them.

1.  $(x + 2y) dx + (2x + 3y) dy$
2.  $(ax + hy + g) dx + (hx + by + f) dy$
3.  $(2xy - y^3) dx + (x^2 - 3xy^2) dy$
4.  $2xy^3 dx + \frac{3x^2y^3 + 1}{y} dy$

**254. Applications.** 1. The families of curves represented by the equations

$$F\left(x, y, \frac{dy}{dx}\right) = 0 \quad (1) \qquad F\left(x, y, -\frac{dx}{dy}\right) = 0 \quad (2)$$

are orthogonal. For the values of  $dy/dx$  given by (2) at any point  $P(x, y)$  are the negative reciprocals of those given by (1); hence every curve of the family (2) cuts all curves of the family (1) at right angles, or is an *orthogonal trajectory* of the family (1); and vice versa. (See also § 110, Ex. 3.)

EXAMPLE 1. Show that the orthogonal trajectories of the family  $y^2 = Cx^3$  are the ellipses  $x^2/3 + y^2/4 = C$ .

2. The following examples illustrate the method of finding curves possessing assigned properties which can be expressed by differential equations.

EXAMPLE 2. Using polar coordinates, find the curves for which the angle  $\psi$  between the radius vector of any point and the tangent at the point is  $n$  times the vectorial angle  $\theta$ .

$$\text{By § 75 (1),} \quad r \left/ \frac{dr}{d\theta} \right. = \tan n\theta \quad \therefore \frac{dr}{r} = \cot n\theta d\theta \quad \therefore r^n = C \sin n\theta$$

EXAMPLE 3. Find the curves for which the  $y$ -intercept of the tangent at any point  $P$  equals  $OP$ .

From the equation § 34 (4) it follows that the  $y$ -intercept of the tangent at the point  $P(x_1, y_1)$  is  $y_1 - (dy_1/dx_1)x_1$ . It may be  $\pm$ , but  $OP = (x_1^2 + y_1^2)^{1/2}$  is always  $+$ . Therefore, omitting subscripts, we

$$\text{have} \quad y - \frac{dy}{dx} x = \pm (x^2 + y^2)^{1/2}$$

$$\text{whose solution is} \quad x^2 + 2Cy - C^2 = 0 \quad (1)$$

Verify this result at the point (3, 4) for one of the curves (1) through this point.

EXAMPLE 4. Find the curves for which the length of the arc between  $x = a$  and  $x = x$  is  $k$  times the area bounded by the arc, the ordinates at  $x = a$  and  $x = x$ , and  $Ox$ .

$$\text{We have} \quad \int_a^x \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} dx = k \int_a^x y dx$$

$$\text{or differentiating,} \quad \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} = ky$$

By solving this equation algebraically for  $dy/dx$  and then separating the variables and integrating, we obtain

$$y = \frac{1}{2k} \left( Ce^{kx} + \frac{1}{C} e^{-kx} \right)$$

which represents a family of catenaries.

### EXERCISE L

Solve the following equations:

1.  $(y^3 + xy^3) dx + (x^3 - yx^3) dy = 0$

2.  $2xy dx + (y^2 - 3x^2) dy = 0$

3.  $\left( x - y \cos \frac{y}{x} \right) dx + x \cos \frac{y}{x} dy = 0$

4.  $\frac{dy}{dx} + y \cos x = \frac{\sin 2x}{2}$

5.  $3x^2 \frac{dx}{dy} - ax^3 = y + 1$

6.  $(x + y)^2 dy - dx = 0$  [Set  $u = x + y$ ]
7.  $x(x - 1) \frac{dy}{dx} + (2x - 1)y = x(x^2 - 1)$
8.  $x dy + y dx = y^2 \log x dx$       9.  $(x^2 y^2 + y) dx - x dy = 0$
10.  $(x^2 + y^2) dx - x^2(x dy - y dx) = 0$
11. Find the orthogonal trajectories of the following families of curves
1.  $x^2 + Cy^2 = 1$       2.  $x^2 + y^2 + 2Cx = 0$       3.  $y^m = Cx^n$
12. Find the curves which make the angle  $\pi/4$  with the curves  $xy = C$ .
13. Find the curves for which the length of the normal equals that of its  $x$ -intercept.
14. Find the curves for which the subnormal equals the  $y$ -intercept of the tangent.
15. Let  $AB$  be a curve arc and  $A'B'$  its projection on  $Ox$ , and let  $ABB'A'$  be revolved about  $Ox$ . For what curve will the area generated by  $AB$  equal the volume generated by  $ABB'A'$ ?
16. At the surface  $S$  of a certain convex lens whose axis is on  $Ox$ , rays parallel to  $Ox$  are refracted toward  $O$ . If the normal to  $S$  at  $P$  makes the angle  $\alpha$  with  $Ox$ , and  $\beta = x OP$ , and the index of refraction is  $n$ , then  $n = \sin \alpha / \sin (\alpha - \beta)$ . Find the curve which will generate  $S$  when revolved about  $Ox$ .
17. Find the curve (a *tractrix*) whose tangent has the constant length  $l$ .
18. Show that if both  $\phi(x, y) = C$ , and  $\psi(x, y) = C'$  are solutions of  $M dx + N dy = 0$ , then  $N\phi_x - M\phi_y = 0$ ,  $N\psi_x - M\psi_y = 0$ ; hence  $\phi_x\psi_y - \phi_y\psi_x = 0$ , and therefore  $\psi$  is a function of  $\phi$ , § 232.
19. Show that if  $F(x, y) = C$  is a solution of  $M dx + N dy = 0$ , then  $F_x/M$  equals  $F_y/N$ , and is an integrating factor of the equation.
20. As a hare running along  $Oy$  passes  $O$ , a hound, standing on  $Ox$  at the distance  $a$  to the right of  $O$ , starts in pursuit of him. The hound continually changes his direction so as always to face the hare, and runs twice as fast as the hare does. Show that if  $p = dy/dx$  denote the slope at any point  $(x, y)$  of the hound's path  $C$ , then  $\int_a^x (1 + p^2)^{1/2} dx = 2(y - px)$  and therefore, since also  $ds/dx$  is  $-$  and  $y, p = 0$  when  $x = a$ , the equation of  $C$  is

$$y = \frac{1}{3} \frac{x^{3/2}}{a^{1/2}} - a^{1/2} x^{1/2} + \frac{2}{3} a$$

21. A boat in being rowed across a river is kept directed toward a point on the shore directly opposite the starting point. If the rates of the current and the rowing are constant,  $a$  and  $b$  respectively, what is the path of the boat?

## EQUATIONS OF THE FIRST ORDER OF HIGHER DEGREE

**255. Geometrical interpretation.** An equation of the form  $f(x, y, C) = 0$ , which is rational and integral and of the  $n$ th degree with respect to the arbitrary constant  $C$ , in general represents an infinite set of curves,  $n$  of which (real or imaginary) will pass through any assigned point  $(x_1, y_1)$ .

If we eliminate  $C$  between the equations

$$f = 0 \text{ and } \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} p = 0, \text{ where } p = \frac{dy}{dx} \quad (1)$$

we get an equation of the form  $\phi(x, y, p) = 0$ , of the  $n$ th degree with respect to  $p$ , which represents the same set of curves.

Conversely, any differential equation  $F(x, y, p) = 0$ , of the first order and  $n$ th degree, represents an infinite set of curves,  $n$  of which (real or imaginary) will pass through any assigned point  $(x_1, y_1)$  and there have the  $n$  slopes got by solving  $F(x_1, y_1, p) = 0$  algebraically for  $p$ . Generally speaking, this set of curves is not one that can be represented by an equation  $f(x, y, C) = 0$  which is rational and integral with respect to  $C$ . But when it can be so represented, the equation  $f(x, y, C) = 0$  is of the  $n$ th degree in  $C$ .

**EXAMPLE.** The equation  $(x - C)^2 + y^2 = 1$  represents the infinite set of circles of radius 1 with centers on  $Ox$ . Two of the circles pass through any point  $P$  whose ordinate is numerically less than 1.

Differentiating  $(x - C)^2 + y^2 = 1$   
gives  $x + yp - C = 0$

Hence, eliminating  $C$ ,  $p^2y^2 + y^2 - 1 = 0$   
a differential equation of the first order and second degree which also represents the set.

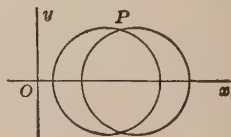


FIG. 124.

**256. Equations solvable by first solving algebraically for  $p$ .** It is sometimes possible to solve an equation  $F(x, y, p) = 0$  algebraically for  $p$  in terms of  $x, y$ , and then to find the general

solutions of the resulting differential equations of the first degree  $p = \phi_1(x, y)$ ,  $p = \phi_2(x, y)$ ,  $\dots$ . If these solutions are  $f_1(x, y, C) = 0$ ,  $f_2(x, y, C) = 0$ ,  $\dots$ , the general solution of  $F(x, y, p) = 0$  is

$$f_1(x, y, C) \cdot f_2(x, y, C) \cdots = 0$$

**EXAMPLE 1.** Solve the equation  $xp^2 - (x^2 - y)p - xy = 0$ .

Solving  $xp^2 - (x^2 - y)p - xy = 0$  (1) for  $p$  gives  $p = x$  (2) and  $px + y = 0$  (3). The solutions of (2) and (3) are

$$y - \frac{x^2}{2} + C = 0 \text{ and } xy + C = 0$$

$$\therefore \text{ that of (1) is } \left(y - \frac{x^2}{2} + C\right)(xy + C) = 0$$

**EXAMPLE 2.** Solve  $yp^2 + 2xp - y = 0$

$$p = \frac{-x \pm (x^2 + y^2)^{1/2}}{y} \quad \therefore \frac{y dy + x dx}{\pm (x^2 + y^2)^{1/2}} = dx$$

$$\therefore \pm (x^2 + y^2)^{1/2} = x + C$$

Hence the solution is  $(x^2 + y^2) = (x + C)^2$ , or  $y^2 = 2Cx + C^2$ .

**EXAMPLE 3.** Solve 1.  $(xp + y)^2 = x^3$  2.  $xp^2 - 2yp - x = 0$   
3.  $(1 - x^2)p^2 = 4$

**257. Equations solvable by first solving algebraically for  $y$  or  $x$ .** 1. When the equation  $F(x, y, p) = 0$  can be solved algebraically for  $y$ , it can be resolved into a set of one or more equations of the type

$$y = \psi(x, p) \quad (1)$$

It may be possible to solve (1) as follows: Differentiating with respect to  $x$ , we get

$$p = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial p} \frac{dp}{dx} \quad (2)$$

which does not involve  $y$ , and is an equation of the first order in  $p$ . Suppose that its solution is

$$\phi(x, p) = C \quad (3)$$

For any particular value of  $C$ , (1) and (3) may be interpreted as the parametric equations of a curve in terms of

the parameter  $p$ . The set of all such curves represents the solution of the equation (1).

To prove this, one must show for any curve represented by the parametric equations

$$y = \psi(x, t), \quad \phi(x, t) = C \quad (4)$$

that at the point  $t = p$ , its slope  $dy/dx$  is  $p$ . But from (4)

$$\frac{dy}{dt} = \frac{\partial \psi}{\partial x} \frac{dx}{dt} + \frac{\partial \psi}{\partial t} \quad \frac{\partial \phi}{\partial x} \frac{dx}{dt} + \frac{\partial \phi}{\partial t} = 0$$

$$\text{Hence} \quad \frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt} = \left[ \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial t} \right] \bigg/ \frac{\partial \phi}{\partial t} \quad (5)$$

On the other hand, substituting in (2) the value of  $dp/dx$  got by differentiating (3), we find

$$p = \left[ \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial p} \right] \bigg/ \frac{\partial \phi}{\partial p} \quad (6)$$

At the point  $t = p$ , the second member of (5) is identical with that of (6); hence at this point the slope  $dy/dx$  of the curve (4) equals  $p$ .

**EXAMPLE 1.** Solve the equation  $y = x + p^2$ .

Differentiating with respect to  $x$  gives

$$p = 1 + 2p \frac{dp}{dx}$$

The general solution of this equation is

$$x = 2p + \log(p-1)^2 + C$$

Hence the general solution of the given equation is

$$x = 2p + \log(p-1)^2 + C \quad y = 2p + p^2 + \log(p-1)^2 + C$$

2. One may deal in a similar manner with an equation  $F(x, y, p) = 0$  which can be solved algebraically for  $x$ . In differentiating we then replace  $dx/dy$  by  $1/p$ .

**EXAMPLE 2.** Solve  $p^2 + py - x = 0$

$$\text{Solving for } x \quad x = py + p^2 \quad (a)$$

Differentiating with respect to  $y$ ,

$$\frac{1}{p} = p + (2p + y) \frac{dp}{dy}$$

which reduces to the linear form

$$\frac{dy}{dp} - \frac{py}{1-p^2} = \frac{2p^2}{1-p^2}$$

Solving this equation,

$$y(1-p^2)^{1/2} = \sin^{-1}p - p\sqrt{1-p^2} + C \quad (b)$$

Hence (a) and (b) are the parametric equations of the required solution.

3. An equation of the type  $y = px + f(p)$  is called a *Clairaut equation*. Its general solution is  $y = Cx + f(C)$ .

For differentiating  $y = px + f(p)$  (1) with respect to  $x$  gives

$$\frac{dp}{dx} [x + f'(p)] = 0 \quad \therefore \frac{dp}{dx} = 0 \quad \text{or} \quad x + f'(p) = 0$$

The general solution of  $dp/dx = 0$  is  $p = C$ ; hence that of (1) is

$$y = Cx + f(C) \quad (2)$$

Since  $x + f'(p) = 0$  does not involve  $dp/dx$ , it contributes nothing to the general solution of (1). Nevertheless if  $p$  be regarded as a parameter  $t$ , it can be shown by the reasoning in 1. that the curve

$$x = -f'(t) \quad y = tx + f(t) \quad (3)$$

satisfies (1) regarded as a differential equation. The curve (3) is the envelope of the lines represented by the general solution (2), and (3) is called a *singular solution* of (1), § 258.

**258. Singular solutions.** 1. Let  $F(x, y, p) = 0$  (1) denote a given equation of the second or a higher degree, and  $f(x, y, C) = 0$  (2) its general solution. It may happen that the family of curves  $K$  represented by (1) or (2) is one which has an envelope, § 243, in other words, that a curve  $\phi(x, y) = 0$  (3) exists which is touched at each of its points by one of the curves  $K$ . In that case, the slope at each point  $(x, y)$  of  $\phi(x, y) = 0$  is the same as that of one of the curves  $K$  at the point, and therefore satisfies (1). Hence  $\phi(x, y) = 0$  is a solution of (1). It is called a *singular solution*, because, generally speaking, it is not included among the solutions got by assigning particular values to  $C$  in the general solution

$$f(x, y, C) = 0.$$

**EXAMPLE.** The equation  $p^2 - y = 0$  (1) has the general solution

$$4y = (x + C)^2 \quad (2)$$

The family of parabolas (2) has the envelope

$$y = 0 \quad (3)$$

Hence  $y = 0$  should be a singular solution of (1); and on testing it, we find that it is such a solution. For when  $y = 0$ , then  $p = 0$ ; and the values  $y = 0$ ,  $p = 0$  satisfy (1). But  $y = 0$  is not one of the curves (2).

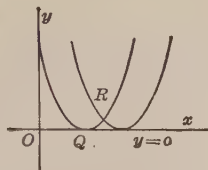


FIG. 125.



2. If the family of curves  $K$  represented by  $F(x, y, p) = 0$  has an envelope  $\phi(x, y) = 0$ , then at any point  $Q$  on  $\phi(x, y) = 0$  two of the values of  $p$  given algebraically by  $F(x, y, p) = 0$  are equal. For let (1) and (2) be curves of  $K$ . Their slopes at  $R$  are two of the values of  $p$  given by  $F(x, y, p) = 0$  at  $R$ , and when  $R \rightarrow Q$  along (1) and therefore (2)  $\rightarrow$  (1), both slopes  $\rightarrow$  the slope of (1) and  $\phi(x, y) = 0$  at  $Q$ .<sup>1</sup>

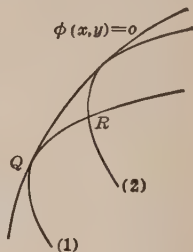


FIG. 126.

But when  $p = p_1$  is a double root of  $F = 0$  it is also a root of  $\partial F / \partial p = 0$ , § 50, 2. Therefore

*All points  $(x, y)$  at which  $F(x, y, p) = 0$  gives two equal values of  $p$  satisfy the equation  $D(x, y) = 0$  got by eliminating  $p$  between*

$$F(x, y, p) = 0 \text{ and } \frac{\partial F}{\partial p} = 0$$

Hence the only possible singular solutions of  $F(x, y, p) = 0$  are such as have equations of the form  $E(x, y) = 0$  where  $E$  denotes  $D$  or some factor of  $D$ . If any such  $E(x, y) = 0$  satisfies  $F(x, y, p) = 0$ , it is a singular solution. If no  $E(x, y) = 0$  satisfies  $F(x, y, p) = 0$ , there is no singular solution. The first case is exceptional, the second general.

*Ordinarily the family of curves  $K$  represented by the equation  $F(x, y, p) = 0$  has no envelope, but the curves  $K$  have cusps, and  $D(x, y) = 0$  represents the locus of these cusps.*

For an equation of the second degree with respect to  $p$ , say  $Ap^2 + Bp + C = 0$ , we have  $D(x, y) = B^2 - 4AC$ .

EXAMPLE. Find the singular solution of  $xp^2 - yp + a = 0$ , if there be any.

$D = y^2 - 4ax = 0$  will be found to satisfy the equation; hence it is the singular solution. The general solution is  $y = Cx + a/C$ .

<sup>1</sup>This reasoning also holds good when the point of intersection of (2) with (1) is imaginary. See page 273.

## EXERCISE LI

Solve the following equations, also finding singular solutions if any.

1.  $y = p^2x + p^3$

2.  $x = 2p + p^2$

3.  $y - px + p^3 = 0$

4.  $xp^2 - 2yp + x = 0$

5.  $p^2 + px - y = 0$

6.  $y + np^2 + ax^{n+1}p^n = 0$

7.  $x = py + ap^2$

8.  $yp^2 - 2xp + y = 0$

9.  $y + 3xp + x^4p^3 = 0$

10.  $y = 2px + ayp^2$  [set  $z = y^2$ ]

11. In the case of  $p^2 - x = 0$ , show that  $D = x = 0$  represents a locus of cusps.

12. The product of the perpendicular distances of the points  $(-a, 0)$ ,  $(a, 0)$  from the tangent to a curve  $C$  is a constant  $c$ . Show that the singular solution of  $(y - px)^2 = c(1 + p^2) + a^2p^2$  is the equation of  $C$  and find it.

13. Find the curve for which the sum of the  $x$ - and  $y$ -intercepts of the tangent is constant.

## LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

**259. Linear equations.** 1. A differential equation of any order  $n$  is said to be *linear* if it is of the first degree with respect to  $y$  and its derivatives. The general form of such an equation is :

$$\frac{d^ny}{dx^n} + X_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + X_{n-1} \frac{dy}{dx} + X_n y = X \quad (1)$$

where  $X_1, X_2, \cdots X_n$  and  $X$  denote functions of  $x$  only, or constants.

Represent  $d/dx, d^2/dx^2, \cdots$  by  $D, D^2, \cdots$ , and the expression  $D^n + X_1 D^{n-1} + \cdots + X_n$  by  $f(D)$ , and let  $f(D)y$  mean the result of applying the operation represented by  $f(D)$  to  $y$ ; we may then write (1) in the form

$$(D^n + X_1 D^{n-1} + \cdots + X_{n-1} D + X_n)y = X$$

$$\text{or} \quad f(D)y = X \quad (2)$$

2. Consider the *auxiliary equation* got by replacing  $X$  by 0 in (2), namely

$$(D^n + X_1 D^{n-1} + \cdots + X_{n-1} D + X_n) y = 0$$

$$\text{or} \quad f(D)y = 0 \quad (3)$$

If  $y_1, y_2$  are functions of  $x$ , and  $c_1, c_2$  are arbitrary constants, then  $D^r(c_1 y_1 + c_2 y_2) = c_1 D^r y_1 + c_2 D^r y_2$ , and therefore

$$f(D)[c_1 y_1 + c_2 y_2] = c_1 f(D)y_1 + c_2 f(D)y_2$$

The like is true for  $c_1 y_1 + c_2 y_2 + c_3 y_3$ , and so on. Hence

If  $y = y_1, y = y_2, \cdots, y = y_n$  are particular solutions of (3), so that  $f(D)y_1 \equiv 0$  and so on, and if  $c_1, c_2, \cdots, c_n$  are arbitrary constants, then

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \quad (4)$$

is also a solution of (3).

Suppose that  $y_1, y_2, \cdots, y_n$  are *linearly independent*, that is, that no particular set of constants  $c'_1, c'_2, \cdots, c'_n$  exists (except 0, 0,  $\cdots$ , 0) such that  $c'_1 y_1 + \cdots + c'_n y_n \equiv 0$ . It is then not possible, by combining terms, to reduce the number  $n$  of arbitrary constants in (4), and (4) is the *general solution* of (3). For it can be proved that every particular solution of (3) can be got by assigning proper values to  $c_1, c_2, \cdots, c_n$  in (4). We call (4) the *complementary function* of the given equation (1).

3. Again if  $y_1, y_2, \cdots, y_n$  have the same meanings as in (4), and if  $y = Y$  denotes a particular solution of the given equation (1), the general solution of (1) is

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + Y \quad (5)$$

For  $f(D)[c_1 y_1 + \cdots + c_n y_n + Y] = f(D)Y = X$ ; and every particular solution of (1) can be got by assigning proper values to  $c_1, c_2, \cdots, c_n$ .

Hence the problem of solving a linear equation  $f(D)y = X$  reduces to that of finding  $n$  linearly independent particular

solutions of the auxiliary equation  $f(D)y = 0$ , and one particular solution of the equation  $f(D)y = X$  itself. When the coefficients  $X_1, X_2, \dots, X_n$  are constants,  $b_1, b_2, \dots, b_n$ , this can be done by the methods explained in the following sections.

## 260. The equation $f(D)y = 0$ with constant coefficients.

1. Suppose that

$$f(D) = D^n + b_1 D^{n-1} + \dots + b_{n-1} D + b_n \quad (1)$$

where  $b_1, b_2, \dots, b_n$  denote real constants; also that the roots of  $f(D) = 0$ , regarded as an algebraic equation in  $D$ , are  $m_1, m_2, \dots, m_n$ . We then have, algebraically,

$$f(D) \equiv (D - m_1)(D - m_2) \dots (D - m_n) \quad (2)$$

But since  $m_1, m_2, \dots, m_n$  are constants, it is also true, when  $D$  stands for  $d/dx$ , and  $y$  for a function of  $x$ , that

$$f(D)y \equiv (D - m_1)(D - m_2) \dots (D - m_n)y \quad (3)$$

it being understood that the right member means that first the operation  $D - m_n$  is to be applied to  $y$ , then  $D - m_{n-1}$  to the result, and so on.

$$\begin{aligned} \text{Thus, if } f(D) &= D^2 + b_1 D + b_2 = (D - m_1)(D - m_2) \\ \text{we have } (D - m_1)(D - m_2)y &= (D - m_1)(Dy - m_2 y) \\ &= D^2 y - (m_1 + m_2)Dy + m_1 m_2 y \end{aligned}$$

$$\text{that is, } (D - m_1)(D - m_2)y = (D^2 + b_1 D + b_2)y$$

The factors in (2), and therefore in (3), may be arranged in any order.

2. *The solution of each of the equations  $(D - m_1)y = 0, \dots$  is a solution of the equation  $f(D)y = 0$ .*

For let  $D - a$  denote any one of the factors  $D - m_1, \dots$  in (3), and  $\phi(D)$  the product of the remaining factors. Then if  $y = y_1$  is a solution of  $(D - a)y = 0$ , we have

$$f(D)y_1 = \phi(D)(D - a)y_1 = \phi(D)0 = 0$$

By § 252, the solution of  $(D - a)y = 0$  is  $y = ce^{ax}$ . Hence, § 259 (4),

When the roots  $m_1, m_2, \dots, m_n$  are distinct, the general solution of  $f(D)y = 0$  is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x} \quad (4)$$

EXAMPLE 1. Solve the equation  $D^3 y - D^2 y - 2 D y = 0$ .

The roots of  $D^3 - D^2 - 2 D = 0$  are 0, 2, -1. Hence the general solution is  $y = c_1 + c_2 e^{2x} + c_3 e^{-x}$ .

3. Suppose that  $r (> 1)$  of the roots equal  $a$ . The corresponding terms of (4) can then be combined in a single term  $ce^{ax}$  and therefore (4), since it involves less than  $n$  arbitrary constants, is not the general solution. But  $(D - a)^r$  is a factor of  $f(D)$ , and it can be proved as in 2. that the solution of  $(D - a)^r y = 0$  is a solution of  $f(D)y = 0$ .

The solution of  $(D - a)^r y = 0$  is

$$y = e^{ax}(c_0 + c_1 x + \dots + c_{r-1} x^{r-1})$$

For  $(D - a)e^{ax}\phi(x) = e^{ax}\phi'(x)$

Hence  $(D - a)^r e^{ax}\phi(x) = e^{ax}\phi^{(r)}(x)$

But when  $\phi(x) = c_0 + c_1 x + \dots + c_{r-1} x^{r-1}$ , then  $\phi^{(r)}(x) = 0$ . Therefore

When  $r$  of the roots equal  $a$ , the group of the corresponding terms of (4) is to be replaced by

$$e^{ax}(c_0 + c_1 x + \dots + c_{r-1} x^{r-1}) \quad (5)$$

EXAMPLE 2. Solve the equation  $D^3 y - 3 D y + 2 y = 0$ .

$D^3 - 3 D + 2 = (D - 1)^2(D + 2)$ . Hence  $y = e^x(c_1 + c_2 x) + c_3 e^{-2x}$

4. The equation  $f(D) = 0$  may have pairs of conjugate imaginary roots, as  $m_1 = \alpha + i\beta$  and  $m_2 = \alpha - i\beta$ , where  $i = \sqrt{-1}$ , § 19, 3. The corresponding terms  $c_1 e^{m_1 x}$  and  $c_2 e^{m_2 x}$  of (4) then have imaginary exponents. But from the definition of  $e^{x+iy}$  given in the chapter on functions of the complex variable it follows that  $e^{(\alpha \pm i\beta)x} = e^{\alpha x} \cdot e^{\pm i\beta x}$  and that

$$e^{i\beta x} = \cos \beta x + i \sin \beta x \quad e^{-i\beta x} = \cos \beta x - i \sin \beta x \quad (6)$$

and multiplying these equations by  $c_1 e^{ax}$  and  $c_2 e^{ax}$  and adding, we get

$$c_1 e^{m_1 x} + c_2 e^{m_2 x} = e^{ax} (c'_1 \cos \beta x + c'_2 \sin \beta x) \quad (7)$$

where  $c'_1 = c_1 + c_2$  and  $c'_2 = ic_1 - ic_2$ .

When the roots  $m_1, m_2$  occur  $r$  times, replace the constants  $c'_1, c'_2$  in (7) by polynomials in  $x$ , like (5), of degree  $r - 1$  and having arbitrary coefficients.

By adding and subtracting the formulas (6), we get

$$\cos \beta x = \frac{e^{i\beta x} + e^{-i\beta x}}{2} \quad \sin \beta x = \frac{e^{i\beta x} - e^{-i\beta x}}{2i} \quad (8)$$

**EXAMPLE 3.** Solve the equation  $D^3 y - 8y = 0$ .

The roots of  $D^3 - 8 = 0$  are  $2, -1 \pm i\sqrt{3}$ .

Hence  $y = c_1 e^{2x} + e^{-x}(c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x)$

**EXAMPLE 4.** Solve the equation  $D^4 y + 2D^2 y + y = 0$ .

The roots of  $D^4 + 2D^2 + 1 = 0$  are  $i, i, -i, -i$ .

Hence  $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$

**EXAMPLE 5.** Solve the following equations:

1.  $\frac{d^2 y}{dx^2} - \frac{dy}{dx} - 12y = 0$
2.  $6\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 2y = 0$
3.  $\frac{d^3 y}{dx^3} + 2\frac{d^2 y}{dx^2} + 5\frac{dy}{dx} = 0$
4.  $\frac{d^4 y}{dx^4} + \frac{d^2 y}{dx^2} - 12y = 0$
5.  $a\frac{d^2 y}{dx^2} - (a^2 + 1)\frac{dy}{dx} + ay = 0$
6.  $\frac{d^3 y}{dx^3} - 3\frac{d^2 y}{dx^2} + 4y = 0$
7.  $D^2(2D - 1)(D + 1)^3 y = 0$
8.  $(D^2 - 2D + 10)^2 y = 0$

**261.** The equation  $f(D)y = X$ . Let  $1/f(D)$  denote the inverse of the operation  $f(D)$ , that is, the operation which  $f(D)$  undoes, so that

$$f(D) \left[ \frac{1}{f(D)} X \right] = X \quad (1)$$

Using this symbol  $1/f(D)$ , and § 260 (2), we may express the solution of the equation  $f(D)y = X$  in the form

$$y = \frac{1}{f(D)} X = \frac{1}{D - m_1} \cdot \frac{1}{D - m_2} \cdots \frac{1}{D - m_n} X \quad (2)$$

But if  $a$  denote any constant, we can find  $[1/(D - a)]X$  as follows :

Set  $z = [1/(D - a)]X$ . Then

$$(D - a)z = (D - a) \frac{1}{D - a} X, \text{ or } \frac{dz}{dx} - az = X$$

But this linear equation gives  $z = e^{ax} \int e^{-ax} X dx$ . Hence

$$\frac{1}{D - a} X = e^{ax} \int e^{-ax} X dx \quad (3)$$

The substitution of (3) in (2) for  $a = m_n, m_{n-1}, \dots m_1$  gives

$$y = e^{m_1 x} \int e^{(-m_1 + m_2)x} \dots \int e^{(-m_{n-1} + m_n)x} \int e^{-m_n x} X (dx)^n \quad (4)$$

When constants of integration are introduced in performing the indicated integrations, (4) gives the general solution<sup>1</sup> of  $f(D)y = X$ . But in applying (4) we shall omit these constants. It then gives a particular solution,  $y = Y$ , of  $f(D)y = X$ . We obtain the general solution by adding  $Y$  to the complementary function obtained by the methods of § 260. See § 259 (5).

EXAMPLE 1. Solve  $D^2y - 5Dy + 6y = e^x$ .

Since  $D^2 - 5D + 6 = (D - 2)(D - 3)$ , we have

$$Y = e^{2x} \int e^{(-2+3)x} \int e^{-3x} \cdot e^x (dx)^2 = e^{2x} \int e^x \int e^{-2x} (dx)^2 = e^{2x} \int e^x \frac{e^{-2x}}{(-2)} dx = \frac{e^x}{2}$$

Hence the general solution is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{e^x}{2}$$

EXAMPLE 2. Solve  $(D - a)^r y = e^{ax}$ .

$$Y = e^{ax} \int e^{(-a+a)x} \dots \int e^{(-a+a)x} \int e^{-ax} \cdot e^{ax} (dx)^r = e^{ax} \int \dots \int (dx)^r = e^{ax} \frac{x^r}{r!}$$

Hence the general solution is

$$y = e^{ax} \left( c_1 + c_2 x + \dots + c_{r-1} x^{r-1} + \frac{x^r}{r!} \right)$$

EXAMPLE 3. Solve the following equations :

$$1. (D^2 - D - 6)y = e^{2x} + e^{4x} \quad 2. (D - 5)^3 y = e^{5x}$$

$$3. (D^2 - 4)y = x \quad 4. (D^3 - 5D^2 + 6D)y = e^x$$

In the following cases there are simpler methods for finding the particular solution  $[1/f(D)]X$ .

<sup>1</sup>Since there are  $n$  integrations there are  $n$  constants and they enter linearly, as was stated in § 259.



**262.** When  $X = e^{ax}$ . Since  $De^{ax} = ae^{ax}$ ,  $D^2e^{ax} = a^2e^{ax}$ , and so on, we have

$$f(D)e^{ax} = f(a)e^{ax}, \text{ and therefore } \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax} \quad (1)$$

the second formula being true since the results got by applying  $f(D)$  to its two members are equal by the first formula.

When  $f(a)$  is 0, (1) fails. But we then have  $f(D) = (D - a)^r \phi(D)$  where  $\phi(a) \neq 0$ ; hence, § 261, Ex. 2,

$$\frac{1}{f(D)}e^{ax} = \frac{1}{(D - a)^r} \frac{1}{\phi(D)}e^{ax} = \frac{1}{(D - a)^r} \frac{e^{ax}}{\phi(a)} = \frac{e^{ax}}{\phi(a)} \frac{x^r}{r!}$$

EXAMPLE. Solve  $(D^2 - 1)^2 y = e^{2x} + e^x$ .

$$\begin{aligned} \frac{1}{(D^2 - 1)^2} e^{2x} &= \frac{1}{(2^2 - 1)^2} e^{2x} = \frac{e^{2x}}{9} \\ \frac{1}{(D - 1)^2} \frac{1}{(D + 1)^2} e^x &= \frac{1}{(D - 1)^2} \frac{e^x}{4} = \frac{e^x x^2}{4 \cdot 2} \end{aligned}$$

Hence the general solution is

$$y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)e^{-x} + \frac{e^{2x}}{9} + \frac{e^x x^2}{8}$$

**263.** When  $X$  is a polynomial in  $x$  of degree  $m$ . It is easily shown that in this case, when  $D$  is not a factor of  $f(D)$ , the  $Y$  given by § 261 (4) is of the form

$$A_0 x^m + A_1 x^{m-1} + \cdots + A_m \quad (a)$$

and when  $D^r$  is a factor of  $f(D)$ , of the form

$$x^r (A_0 x^m + A_1 x^{m-1} + \cdots + A_m) \quad (b)$$

Hence  $Y$  may be found by substituting (a) or (b) for  $y$  in  $f(D)y = X$ , and then determining the values of  $A_0, A_1, \dots, A_m$  which make the resulting equation an identity.

EXAMPLE 1. Solve  $D^2 y + y = x^3 + x^2$ .

To find  $Y$ , substitute  $y = Ax^3 + Bx^2 + Cx + E$ , which gives

$$Ax^3 + Bx^2 + (C + 6A)x + (E + 2B) = x^3 + x^2$$

$$\therefore A = 1, B = 1, C = -6, E = -2$$

Hence  $Y = x^3 + x^2 - 6x - 2$ , and the general solution is

$$y = c_1 \cos x + c_2 \sin x + x^3 + x^2 - 6x - 2$$

EXAMPLE 2. Solve  $D^2y - Dy = 8x^3$ . To find  $Y$ , substitute

$$y = Ax^4 + Bx^3 + Cx^2 + Ex$$

EXAMPLE 3. Solve

1.  $(D^3 - 8)y = x^2$

2.  $(D^2 - 2D + 1)y = x^2 + x$

**264. When  $X$  is  $\sin ax$  or  $\cos ax$ .** Observe that

$$D^2 \sin ax = -a^2 \sin ax \quad D^2 \cos ax = -a^2 \cos ax.$$

Hence in finding  $[1/f(D)]X$ , when  $X$  is  $\sin ax$  or  $\cos ax$ , we may replace  $D^2$  by  $-a^2$ .

EXAMPLE 1. For the equation  $(D^2 + 1)y = \sin 3x$ , we have

$$Y = \frac{1}{D^2 + 1} \sin 3x = \frac{1}{-3^2 + 1} \sin 3x = -\frac{\sin 3x}{8}$$

EXAMPLE 2. Find  $Y$  for the equation  $(D^2 + 2D + 5)y = \sin 2x$ .

$$\begin{aligned} \frac{1}{D^2 + 2D + 5} \sin 2x &= \frac{1}{2D + 1} \sin 2x = \frac{2D - 1}{4D^2 - 1} \sin 2x \\ &= \frac{2D - 1}{-17} \sin 2x = -\frac{1}{17} (4 \cos 2x - \sin 2x) \end{aligned}$$

EXAMPLE 3. Solve

1.  $(D^2 - 5D + 6)y = \sin 3x + \cos 4x$

2.  $(D^4 - D^2)y = \sin x$

This method fails when  $D^2 + a^2$  is a factor of  $f(D)$ . We may then express  $\sin ax$  or  $\cos ax$  in terms of  $e^{iax}$  and  $e^{-iax}$ , § 260 (8), and then proceed as in § 262.

**265. When  $X$  is  $e^{ax}V(x)$ .** Since  $D = a + (D - a)$  we can express  $f(D)$  in the form  $F(D - a)$ , § 186. By § 260, 3., we have  $(D - a)^r e^{ax}V = e^{ax}D^r V$ , and therefore

$$F(D - a)e^{ax}V = e^{ax}F(D)V$$

But  $F(D - a) = f(D)$ , and therefore  $F(D) = f(D + a)$ .

Hence

$$f(D)e^{ax}V = e^{ax}f(D + a)V$$

$$\text{and therefore} \quad \frac{1}{f(D)} e^{ax}V = e^{ax} \frac{1}{f(D + a)} V \quad (1)$$

the second formula being true because the results got by applying  $f(D)$  to its two members are equal by the first.

EXAMPLE 1. Find a particular solution of  $(D^2 + 1)y = e^x \sin 2x$ .

$$\begin{aligned} Y &= \frac{1}{D^2 + 1} e^x \sin 2x = e^x \frac{1}{(D + 1)^2 + 1} \sin 2x \\ &= e^x \frac{1}{D^2 + 2D + 2} \sin 2x = -\frac{e^x}{10} (2 \cos 2x + \sin 2x) \end{aligned}$$

EXAMPLE 2. Solve

$$1. D^2y + 5Dy + 4y = e^{-x}x^2 \qquad 2. D^3y = e^x(\sin x + \cos x)$$

**266. Variation of parameters.** The following example illustrates a method of deriving the general solution from the complementary function. It is applicable to linear equations with constant or variable coefficients.

EXAMPLE 1. The complementary function of

$$D^2y + y = \tan x \quad (1) \quad \text{is} \quad y = C_1 \sin x + C_2 \cos x \quad (2)$$

Let us inquire whether if  $C_1, C_2$  be regarded as functions of  $x$  instead of constants, they can be so determined that (2) will represent the general solution of (1).

Differentiating (2),

$$Dy = C_1 \cos x - C_2 \sin x + \frac{dC_1}{dx} \sin x + \frac{dC_2}{dx} \cos x$$

Suppose  $C_1, C_2$  to be such functions that

$$\frac{dC_1}{dx} \sin x + \frac{dC_2}{dx} \cos x = 0 \quad (3)$$

so that (as when  $C_1, C_2$  are constants)

$$Dy = C_1 \cos x - C_2 \sin x \quad (4)$$

Differentiating (4) gives

$$D^2y = -C_1 \sin x - C_2 \cos x + \frac{dC_1}{dx} \cos x - \frac{dC_2}{dx} \sin x \quad (5)$$

Substituting (5) and (2) in (1), we obtain

$$\frac{dC_1}{dx} \cos x - \frac{dC_2}{dx} \sin x = \tan x \quad (6)$$

Hence (2) will satisfy (1) if  $C_1$  and  $C_2$  satisfy (3) and (6).

Solving (3), (6) algebraically for  $\frac{dC_1}{dx}, \frac{dC_2}{dx}$ , we get

$$\frac{dC_1}{dx} = \sin x \qquad \frac{dC_2}{dx} = -\sin x \tan x \quad (7)$$

Integrating the equations (7) gives

$$C_1 = -\cos x + c_1 \quad C_2 = -\log(\sec x + \tan x) + \sin x + c_2 \quad (8)$$

where  $c_1, c_2$  are constants. And substituting (8) in (2), we get the general solution of (1)

$$y = c_1 \sin x + c_2 \cos x - \cos x \log(\sec x + \tan x) \quad (9)$$

**EXAMPLE 2.** The complementary function of

$$x^2 D^2 y - 2x Dy + 2y = x^3 \quad \text{is} \quad y = C_1 x + C_2 x^2$$

By the method of variation of parameters show that the general solution is  $y = c_1 x + c_2 x^2 + x^3/2$ .

An equation like this, in which  $f(D)$  is a sum of terms of the type  $x^r D^r$ , is sometimes called a *homogeneous equation*. It can be reduced to a linear equation with constant coefficients by the substitution  $x = e^t$ .

### EXERCISE LII

Solve the following equations:

1.  $D^3(2D-5)(D^2+D+1)^2y=0$
2.  $(2D^2-D-6)y=e^{3x}$
3.  $(D-2)^2y=2e^x+e^{2x}$
4.  $(D^2+D)y=e^x+3$
5.  $(D^2+7D+12)y=6x+10$
6.  $(D^2+2D+2)y=x^2$
7.  $(D^2+2D)y=x^2$
8.  $(D+4)y=8x^3+2x^2+5$
9.  $(D^2+D)y=\sin 2x+\cos x$
10.  $(D^3+D^2-D-1)y=\sin x$
11.  $(D^2+1)y=\sin x$
12.  $(D^2+D-6)y=e^x x^2$
13.  $D^2y=e^{3x}\cos 2x$
14.  $(D^2+1)y=\sec x$  (by § 266)
15. Find the integral curve of  $D^2y-2Dy+2y=\cos x$  which touches  $Ox$  at  $O$ .

### EQUATIONS OF THE SECOND ORDER

**267. Equations of the second order.** 1. Any equation of the second order in  $y$  which does not involve  $y$  explicitly may be reduced to an equation of the first order in  $p$  by the substitution

$$\frac{dy}{dx} = p \quad \frac{d^2y}{dx^2} = \frac{dp}{dx} \quad (1)$$

2. An equation of the second order in  $y$  which does not involve  $x$  explicitly may be reduced to one of the first order in  $p$  by the substitution

$$\frac{dy}{dx} = p \qquad \frac{d^2y}{dx^2} = \frac{dp}{dy} p \qquad (2)$$

For 
$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy} \frac{dy}{dx} = \frac{dp}{dy} p$$

EXAMPLE 1. Substituting (2) in  $y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$

gives 
$$yp \frac{dp}{dy} + p^2 = 1$$

Hence  $(1 - p^2)y^2 = c_1 \quad \therefore \frac{dy}{dx} = \frac{(y^2 - c_1)^{1/2}}{y} \quad \therefore y^2 - (x + c_2)^2 = c_1$

3. Let  $f(D)y = X$  be any linear equation of the second order, and let  $y = y_1$  be a solution of  $f(D)y = 0$ . It will be found that the substitution  $y = y_1 v$  reduces  $f(D)y = X$  to a linear equation in  $v$  which lacks the  $v$  term and is therefore of the type 1.

EXAMPLE 2. Solve  $(x^2 + 1) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = (x^2 + 1)^2$ .

The auxiliary equation has the solution  $y = x$ .

Substituting  $y = vx$  gives

$$\frac{d^2v}{dx^2} + \frac{2}{x(x^2 + 1)} \frac{dv}{dx} = \frac{x^2 + 1}{x}$$

whose solution is 
$$v = \frac{x^3}{6} + \frac{x}{2} + c_1 \left(x - \frac{1}{x}\right) + c_2$$

Hence 
$$y = \frac{x^4}{6} + \frac{x^2}{2} + c_1(x^2 - 1) + c_2x$$

**268. Equations of motion.** A particle  $P$  of mass  $m$  is moving in a straight line under the action of forces the sum of whose components along the line is  $F$ . Let  $\alpha$  denote the acceleration of the motion, and take as the unit of force the force which gives unit mass unit acceleration. Then, if  $F$

denote the measure of the force  $F$  in terms of this unit, we have by Newton's laws of motion

$$F = m\alpha \quad (1)$$

Taking the line in which  $P$  moves as  $x$ -axis, (1) may be written

$$m \frac{d^2x}{dt^2} = F \quad \text{or} \quad mv \frac{dv}{dx} = F \quad (2)$$

Suppose  $F$  to be given as a constant or a known function of  $t, x, v$ . The differential equation (2), and any one set of corresponding values of  $t, x, v$  will then completely determine the motion of  $P$ .

In like manner, if  $P$  move in space under the action of forces the sums of whose  $x$ -,  $y$ -,  $z$ -components are  $X, Y, Z$ , then

$$m \frac{d^2x}{dt^2} = X \quad m \frac{d^2y}{dt^2} = Y \quad m \frac{d^2z}{dt^2} = Z \quad (3)$$

These equations and the position and velocity of  $P$  at any given instant  $t = t_0$  completely determine the motion of  $P$ .

EXAMPLE. A body of mass  $m$  is projected vertically upward with the initial velocity  $v_0$ . Discuss the motion on the supposition that the resistance of the air is proportional to the speed.

Take the vertical as  $x$ -axis, the point of departure as the origin, and the downward direction as positive. We then have

$$m \frac{d^2x}{dt^2} = mg - kv \quad (1)$$

where  $g$  is the acceleration of gravity and  $k$  is some positive constant.

Hence dividing by  $m$ , and setting  $k/m = \mu$ ,

$$\frac{d^2x}{dt^2} + \mu \frac{dx}{dt} = g \quad (2)$$

The complementary function of (2) is  $x = c_1 + c_2 e^{-\mu t}$ ; and  $x = (g/\mu)t$  is a particular integral. Hence the complete solution is

$$x = c_1 + c_2 e^{-\mu t} + \frac{g}{\mu} t \quad (3)$$

and this gives

$$v = -c_2 \mu e^{-\mu t} + \frac{g}{\mu} \quad (4)$$

When  $t = 0$ , then  $x = 0$  and  $v = v_0$

hence 
$$0 = c_1 + c_2 \quad \text{and} \quad v_0 = -c_2\mu + \frac{g}{\mu}$$

Solving these equations for  $c_1$ ,  $c_2$  and substituting in (3), we get finally

$$x = \left( \frac{v_0}{\mu} - \frac{g}{\mu^2} \right) + \frac{g}{\mu} t + \left( \frac{g}{\mu^2} - \frac{v_0}{\mu} \right) e^{-\mu t} \quad (5)$$

which gives the height  $x$  at any instant  $t$ .

**269. Simple harmonic motion.** This name is given to the motion of a particle  $P$  on a line under the action of a force which is directed toward a fixed point  $O$  on the line and is proportional to the distance of  $P$  from  $O$ . Taking the line as  $x$ -axis and  $O$  as origin, we can reduce the equation of motion to the form  $d^2x/dt^2 = -k^2x$ , or

$$\frac{d^2x}{dt^2} + k^2x = 0 \quad (1)$$

By § 260 (7), the general solution of this equation is

$$x = c_1 \cos kt + c_2 \sin kt \quad (2)$$

or 
$$x = a \cos (kt - \beta) \quad (3)$$

where

$$a = (c_1^2 + c_2^2)^{1/2} \quad \cos \beta = \frac{c_1}{(c_1^2 + c_2^2)^{1/2}} \quad \sin \beta = \frac{c_2}{(c_1^2 + c_2^2)^{1/2}}$$

If the position and velocity of  $P$  at any instant  $t = t_0$  be given, we can find  $c_1$ ,  $c_2$  or  $a$ ,  $\beta$  from (2) or (3) and the corresponding equation for  $v$  as in the example in § 268.

As (3) shows, the motion is *oscillatory*. As  $t$  increases from the value  $\beta/k$ ,  $P$  moves backward from  $x = a$  to  $x = -a$ , then forward to  $x = a$ , and so on. The time or *period* of each complete oscillation is  $2\pi/k$ . The distance  $a$  is called the *amplitude*.

**EXAMPLE 1.** When (3) is  $x = 3 \cos 2t$ , we have  $v = -6 \sin 2t$ . Find  $x$  and  $v$  for each of the following values of  $t$ :  $0, \pi/4, \pi/2, 3\pi/4, \pi$ .

**EXAMPLE 2.** If  $d^2x/dt^2 + 4x = 0$ , and  $x = 3$ ,  $v = 8$  when  $t = 0$ , show that  $x = 3 \cos 2t + 4 \sin 2t = 5 \cos (2t - .94)$



**270. Damped harmonic motion.** Suppose that a body in simple harmonic motion is subjected to a resistance which is proportional to the speed. Its equation of motion can then be reduced to the form

$$\frac{d^2x}{dt^2} + 2\mu \frac{dx}{dt} + k^2x = 0 \quad (1)$$

where  $\mu$  denotes a positive constant. The roots of

$$D^2 + 2\mu D + k^2 = 0 \quad \text{are} \quad -\mu \pm \sqrt{\mu^2 - k^2}$$

Suppose  $\mu^2 < k^2$  and set  $k^2 - \mu^2 = \gamma^2$ . Then, § 260 (7), the general solution of the equation (1) is

$$x = e^{-\mu t} [c_1 \cos \gamma t + c_2 \sin \gamma t] = Ce^{-\mu t} (\cos \gamma t - \delta) \quad (2)$$

where  $C$  and  $\delta$  denote constants whose values can be found when any set of corresponding values of  $t$ ,  $x$ ,  $v$  is given.

Hence the motion continues to be oscillatory. But since  $\gamma < k$ , the period is increased; and since  $e^{-\mu t}$  decreases as  $t$  increases, the amplitude continually decreases.

**271. Forced vibrations.** Suppose that a body in simple harmonic motion is subjected to a disturbing force which is itself periodic, being such that the reduced equation of motion becomes

$$\frac{d^2x}{dt^2} + k^2x = h \cos \gamma t \quad (1)$$

where  $h$  and  $\gamma$  are known constants. By §§ 264, 269, the solution of (1) is

$$x = a \cos (kt - \beta) + \frac{h}{k^2 - \gamma^2} \cos \gamma t \quad (2)$$

The motion is compounded of two simple harmonic motions. The case of most interest is that in which  $k$  and  $\gamma$  are nearly equal. The amplitude  $h/(k^2 - \gamma^2)$  is then very large and the oscillations or vibrations of the body are therefore violent. It is for this reason that soldiers break step when crossing a bridge, lest their steps be in tune with the natural oscillations of the bridge.

**272. Curvilinear motion about a center of force.** A particle  $P$  moving about  $O$  under the action of a force directed toward  $O$  and inversely proportional to  $OP^2$  describes a conic of which  $O$  is a focus.

For take any rectangular axes  $Ox, Oy$ . Let  $r$  denote the length of the vector  $OP$  and  $\theta$  its direction angle referred to  $Ox$ . Also let  $PT$  denote a vector perpendicular to  $OP$  and having the direction angle

$\theta + \pi/2$ . Finally let  $A_x, A_y, A_{OP}, A_{PT}$  denote the components of the acceleration in the directions  $Ox, Oy, OP, PT$  respectively. Then

$$A_{OP} = A_x \cos \theta + A_y \sin \theta \quad A_{PT} = -A_x \sin \theta + A_y \cos \theta$$

which, when  $A_x, A_y$  are expressed in terms of  $r, \theta, t$  by aid of the relations  $x = r \cos \theta, y = r \sin \theta$ , reduce to the form

$$A_{OP} = \frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2, \quad A_{PT} = 2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2 \theta}{dt^2}$$

By hypothesis,  $A_{OP} = -\mu/r^2$ , where  $\mu$  denotes some positive constant, and  $A_{PT} = 0$ . Hence the equations of motion of  $P$  are

$$\frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 = -\frac{\mu}{r^2} \quad (1) \quad \text{and} \quad 2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2 \theta}{dt^2} = 0 \quad (2)$$

$$\text{The solution of (2) is} \quad r^2 \frac{d\theta}{dt} = h \quad (3)$$

where  $h$  is a constant. For convenience, set

$$r = \frac{1}{u} \quad (4) \quad \text{and therefore} \quad \frac{d\theta}{dt} = hu^2 \quad (5)$$

$$\text{Then} \quad \frac{dr}{dt} = -\frac{1}{u^2} \frac{du}{dt} = -\frac{1}{u^2} \frac{du}{d\theta} \frac{d\theta}{dt} = -h \frac{du}{d\theta}$$

$$\text{and therefore} \quad \frac{d^2 r}{dt^2} = -h \frac{d^2 u}{d\theta^2} \frac{d\theta}{dt} = -h^2 u^2 \frac{d^2 u}{d\theta^2} \quad (6)$$

Substituting (4), (5), (6) in (1) and reducing, we get

$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu}{h^2} \quad (7)$$

The general solution of (7) can be expressed in the form

$$u = -c \cos(\theta - \epsilon) + \frac{\mu}{h^2}$$

where  $c$  and  $\epsilon$  are constants. Therefore, since  $r = 1/u$ ,

$$r = \frac{h^2/\mu}{1 - c(h^2/\mu) \cos(\theta - \epsilon)} \quad (8)$$

and this is the equation, in polar coordinates, of a conic which has a focus at  $O$ , whose principal axis makes the angle  $\epsilon$  with  $Ox$ , whose eccentricity is  $ch^2/\mu$ , and whose latus rectum is  $2h^2/\mu$  (p. 80, Ex. 6).

### EXERCISE LIII

Solve the following equations:

$$1. \quad \frac{d^2 y}{dx^2} = 1 + \left( \frac{dy}{dx} \right)^2 \quad 2. \quad \frac{d^2 y}{dx^2} - 2y \frac{dy}{dx} = 0$$

3. A stone weighing  $m$  pounds is sent sliding along the ice with the initial velocity  $v_0$  ft./sec. If the coefficient of friction between the stone and the ice is  $\mu$ , the equation of the stone's motion is

$$m \, d^2x/dt^2 = -mg\mu$$

Show that the stone will come to rest when  $t = v_0/g\mu$  and that the distance it has traveled will be  $v_0^2/2g\mu$ .

4. Show that, if under the earth's attraction and that force only a body  $P$  were to move toward the earth, its equation of motion would be  $v \, dv/dx = -R^2g/x^2$ , where  $R$  is the radius of the earth,  $g$  the attraction of gravity at the earth's surface, and  $x$  the distance of  $P$  from the earth's center. Show also that were  $P$  to start from rest at an infinite distance, it would reach the earth's surface with the velocity  $\sqrt{2gR}$ , about seven miles per second.

5. It can be shown that the attraction of a solid sphere  $S$  on a particle  $P$  within  $S$  is directed toward the center  $O$  of  $S$ , and is proportional to  $OP$ . Show that, were  $P$  to move in a straight tube passing through the earth's center  $O$ , its equation of motion would be  $d^2x/dt^2 = -gx/R$ , where  $x = OP$ . Show also that, if  $P$  were to start from rest at the earth's surface, it would reach the center in about  $21\frac{1}{4}$  minutes.

6. A particle moving on  $Ox$  is attracted toward  $O$  by a force of magnitude  $k/x^3$ . If it starts from rest at  $x = a$ , how long will it be before it reaches  $O$ ?

7. Discuss the motion of a particle whose initial velocity is  $v_0$ , under the action of a resistance proportional to the speed, and of this force only.

8. Referring to § 272, show that from the equation  $r^2 \, d\theta/dt = h$  it follows that the radius vector  $OP$  sweeps out equal areas in equal times.

9. Suppose that the path or orbit of  $P$ , § 272 (8), is an ellipse. The time  $T$  of a circuit of the orbit is then called the *periodic time*. If  $a$  and  $b$  are the major and minor semi-axes of the ellipse, its area is  $\pi ab$  and its latus rectum is  $2b^2/a$ . Hence show that  $T = 2\pi a^{3/2}/\mu^{1/2}$ , and from this deduce Kepler's third law: In the motions of the planets about the sun, the squares of the periodic times of the several orbits are proportional to the cubes of their major axes.

10. Using the notation of § 272, find the general equation of the path of  $P$  on the supposition that  $A_{OP} = -\mu/r^3$ ,  $A_{PT} = 0$ .

11. In case  $A_{OP} = -k^2r$ ,  $A_{PT} = 0$ , show that

$$\frac{d^2x}{dt^2} + k^2x = 0 \qquad \frac{d^2y}{dt^2} + k^2y = 0$$

and therefore that the projections of  $P$  on  $Ox$  and  $Oy$  are in simple harmonic motion. Show that the orbit of  $P$  is an ellipse with its center at  $O$  and that the periodic time is  $2\pi/k$ .

12. A particle  $P$  of mass  $m$  is constrained to move on a given curve. If  $P$  is under the action of a force whose  $x$ -,  $y$ -,  $z$ -components are  $X$ ,  $Y$ ,  $Z$ , and this force only, show that its equation of motion is

$$\frac{d^2s}{dt^2} = X \frac{dx}{ds} + Y \frac{dy}{ds} + Z \frac{dz}{ds}$$

13. Show that the equation of motion of a particle  $P$  of mass  $m$  moving under the action of gravity from  $A$  to  $B$  along a given curve  $C$  is  $v dv/ds = g dy/ds$ , and therefore that the speed which it will have at  $B$  is independent of the path.

14. A particle  $P$  of mass  $m$  which is attached to a fixed point  $O$  by a thin rod of length  $l$  (a simple pendulum) is swinging in a vertical circle. Show that its equation of motion is  $l d^2\theta/dt^2 = -g \sin \theta$ , where  $\theta$  is the angle which  $OP$  makes with the vertical; therefore, that for small oscillations the motion is approximately simple harmonic.

### ORDINARY DIFFERENTIAL EQUATIONS IN THREE VARIABLES

273. **Exact equations.** A differential equation in  $x$ ,  $y$ ,  $z$  of the form

$$P dx + Q dy + R dz = 0 \qquad (1)$$

is called *exact* when its first member is the total differential of some function  $u(x, y, z)$ . The equation may then be written  $du = 0$ , and its general solution is  $u = C$ .

*The necessary and sufficient condition that (1) be an exact equation is that*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \qquad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \qquad (2)$$

This can be proved by reasoning like that in § 253. Another proof that (2) is the sufficient condition is given in § 294.

**274. Integrable equations.** 1. We say that the equation

$$P dx + Q dy + R dz = 0 \quad (1)$$

is *integrable* if its first member is an exact differential or becomes one when multiplied by some integrating factor  $\mu$ .

Thus,  $yz dx + zx dy - xy dz = 0$  is integrable. For when multiplied by  $1/xyz$  it becomes  $dx/x + dy/y - dz/z = 0$ , or  $d \log xy/z = 0$ , and has the solution  $xy = Cz$ .

2. The necessary and sufficient condition that (1) be integrable is that  $P, Q, R$  satisfy the identity<sup>1</sup>

$$P \left( \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) + Q \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) + R \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \equiv 0 \quad (2)$$

For if an integrating factor  $\mu$  exists such that

$$\mu(P dx + Q dy + R dz) = du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

then  $\mu P = \frac{\partial u}{\partial x}, \quad \mu Q = \frac{\partial u}{\partial y}, \quad \mu R = \frac{\partial u}{\partial z}$

and therefore

$$\frac{\partial}{\partial y} \mu P = \frac{\partial}{\partial x} \mu Q \quad \text{or} \quad P \frac{\partial \mu}{\partial y} + \mu \frac{\partial P}{\partial y} = Q \frac{\partial \mu}{\partial x} + \mu \frac{\partial Q}{\partial x}$$

$$\frac{\partial}{\partial z} \mu Q = \frac{\partial}{\partial y} \mu R \quad \text{or} \quad Q \frac{\partial \mu}{\partial z} + \mu \frac{\partial Q}{\partial z} = R \frac{\partial \mu}{\partial y} + \mu \frac{\partial R}{\partial y}$$

$$\frac{\partial}{\partial x} \mu R = \frac{\partial}{\partial z} \mu P \quad \text{or} \quad R \frac{\partial \mu}{\partial x} + \mu \frac{\partial R}{\partial x} = P \frac{\partial \mu}{\partial z} + \mu \frac{\partial P}{\partial z}$$

Multiplying these equations by  $R, P, Q$ , and adding, we get (2). Hence (2) is the necessary condition that (1) be integrable. In the next section it will be shown to be the sufficient condition also.

**EXAMPLE.** Show that if  $P, Q, R$  satisfy (2), and  $\lambda$  be a function of  $x, y, z$ , then also

$$\begin{aligned} \lambda P \left( \frac{\partial}{\partial z} \lambda Q - \frac{\partial}{\partial y} \lambda R \right) + \lambda Q \left( \frac{\partial}{\partial x} \lambda R - \frac{\partial}{\partial z} \lambda P \right) \\ + \lambda R \left( \frac{\partial}{\partial y} \lambda P - \frac{\partial}{\partial x} \lambda Q \right) \equiv 0 \end{aligned} \quad (3)$$

<sup>1</sup> This may be expressed in the more easily remembered symbolic form

$$\begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \equiv 0$$

**275. The solution of integrable equations.** When the condition (2) is satisfied, the solution of the equation (1) can be made to depend on solving equations in two variables.

EXAMPLE. Solve the equation

$$2xz dx + 2y^2z dy + (x^2 + 2y^2z - 1) dz = 0 \quad (1)$$

It will be found that the equation satisfies the condition § 274 (2).

Were  $z$  a constant, (1) would reduce to

$$2x dx + 2yz dy = 0 \quad (2) \quad \text{whose solution is} \quad x^2 + y^2z = C \quad (3)$$

Let us inquire whether if we regard  $C$  as a function of  $z$  instead of a constant, we can so determine it that (3) will represent a solution of (1).

Differentiating (3) with  $z$  variable,

$$2x dx + 2yz dy + y^2 dz = dC \quad (4)$$

Multiplying (4) by  $z$  and subtracting from (1)

$$(x^2 + y^2z - 1) dz + z dC = 0 \quad (5)$$

which by aid of (3) reduces to

$$(C - 1) dz + z dC = 0 \quad (6)$$

and this involves  $C$  and  $z$  only and gives

$$(C - 1)z = c \quad \text{or} \quad C = c/z + 1 \quad (7)$$

Substituting (7) in (3), we obtain the solution of (1), namely

$$x^2 + y^2z = c/z + 1 \quad \text{or} \quad x^2z + y^2z^2 - z = c$$

The success of the method is due to the fact that by (3) we can eliminate both  $x$  and  $y$  from (5). The reason for this, as we shall see, is that (1) satisfies the condition § 274 (2).

In general, to solve any integrable equation

$$P dx + Q dy + R dz = 0 \quad (1)$$

first regard one variable, say  $z$ , as constant and solve

$$P dx + Q dy = 0 \quad (2)$$

$$\text{Let the solution}^1 \text{ be} \quad v(x, y, z) = C \quad (3)$$

<sup>1</sup> We assume that for the values of  $x, y, z, C$  under consideration the solution of (2) can be reduced to the form (3), by aid of the theorem of § 226.

Differentiate (3) completely regarding  $C$  as a function of  $z$ :

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \left( \frac{\partial v}{\partial z} - \frac{\partial C}{\partial z} \right) dz = 0 \quad (4)$$

Since (3) satisfies (2), a function  $\lambda(x, y, z)$  exists such that

$$\frac{\partial v}{\partial x} = \lambda P \quad \text{and} \quad \frac{\partial v}{\partial y} = \lambda Q \quad (5)$$

Hence (4) is equivalent to (1) if  $C$  be such that

$$\frac{\partial v}{\partial z} - \frac{\partial C}{\partial z} = \lambda R \quad \text{or} \quad \frac{\partial v}{\partial z} - \lambda R = \frac{\partial C}{\partial z} \quad (6)$$

But, as will be shown immediately, when the condition of § 274 (2) is satisfied, (6) can by aid of (3) be reduced to an equation in  $C, z$  only. Let the solution of this reduced equation be  $C = \phi(z) + c$ . Then the solution of (1) is

$$v(x, y, z) = \phi(z) + c \quad (7)$$

It being supposed that the condition of § 274 (2) is satisfied, we are to prove that if (3) be solved for  $y$ , say, in terms of  $x, z, C$ , and the solution be substituted in (6), the result will involve only  $z$  and  $C$ , not  $x$ . But this is equivalent to showing that when  $x, z, C$  are the independent variables, and  $y$  the function of  $x, z, C$  defined by (3), the  $x$ -derivative of  $\partial v / \partial z - \lambda R$  is 0, in other words, that

$$\frac{\partial}{\partial x} \left[ \frac{\partial v}{\partial z} - \lambda R \right] + \frac{\partial}{\partial y} \left[ \frac{\partial v}{\partial z} - \lambda R \right] \frac{\partial y}{\partial x} = 0 \quad \text{where} \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial x} = 0$$

or that

$$\frac{\partial}{\partial x} \left[ \frac{\partial v}{\partial z} - \lambda R \right] \frac{\partial v}{\partial y} - \frac{\partial}{\partial y} \left[ \frac{\partial v}{\partial z} - \lambda R \right] \frac{\partial v}{\partial x} = 0 \quad (8)$$

But, using (5), the first member of (8) may be written

$$\left[ \frac{\partial}{\partial z} \lambda P - \frac{\partial}{\partial x} \lambda R \right] \lambda Q - \left[ \frac{\partial}{\partial z} \lambda Q - \frac{\partial}{\partial y} \lambda R \right] \lambda P$$

and by § 274 (3), this is identically equal to

$$\lambda R \left[ \frac{\partial}{\partial y} \lambda P - \frac{\partial}{\partial x} \lambda Q \right] = \lambda R \left[ \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial y} \right] = 0$$

**276. Geometrical interpretation. Non-integrable equations.** The equation

$$P dx + Q dy + R dz = 0 \quad (1)$$

is the condition that the directions whose direction ratios are  $dx : dy : dz$  and  $P : Q : R$  be perpendicular, § 212.



Hence if a point  $T$  be supposed to move in any manner such that at every position  $A$  the direction of the motion, given by  $dx : dy : dz$ , is perpendicular to the line through  $A$  whose direction ratios are  $P_A : Q_A : R_A$ , it will trace a curve  $K$  which represents a particular solution of (1). And the set of all such curves  $K$  will represent the general solution.

When (1) satisfies the condition § 274 (2), that is, is integrable, the curves  $K$  can be grouped so as to form the surfaces of some family of surfaces  $u(x, y, z) = C$ . On the other hand, when (1) is not integrable, no family of surfaces  $u = c$  exists which contains all the curves  $K$ . In this case, however, as in the integrable case, on an arbitrarily chosen surface  $S$  there are infinitely many of the curves  $K$ , one through each point of  $S$ .

**EXAMPLE.** Show that the equation  $y dx - x dy - x dz = 0$  is not integrable, and find the integral curves  $K$  which are on the plane  $z = x + y$ .

Substituting  $dz = dx + dy$  in  $y dx - x dy - x dz = 0$ , we get  $(x - y) dx + 2x dy = 0$  whose solution is  $(y + x)^2 = Cx$ . The cylindrical surfaces represented by this equation intersect the plane  $z = x + y$  in the required curves.

### EXERCISE LIV

Solve the following equations:

1.  $yz dx - xz dy - 2xy dz = 0$
2.  $xz dx + yz dy = (z + 1) dz$
3.  $x dx + z dz = (a^2 - x^2 - z^2) dy$
4.  $dx + dy = (x + y + e^{-x}) dz$
5.  $(2x + 2xy + 2xz^2) dx = dy + 2z dz$
6.  $x dx - (1 - x^2 - z^2)^{1/2} dy + z dz = 0$
7.  $(x^2 + 2y^2 + 2z^2) dx + xy dy + xz dz = 0$
8.  $(y + z) dx + (z + x) dy + (x + y) dz = 0$
9.  $(ay - bz) dx + (cz - ax) dy + (bx - cy) dz = 0$
10.  $2(y - z) dx + (x + 3y - 2z) dy - (x + y) dz = 0$

11. Show that the set of curves on the sphere  $x^2 + y^2 + z^2 = a^2$  which satisfy the equation  $3x dx + y(1 - x) dy + z dz = 0$  project into parabolas in the  $xy$ -plane.

## SIMULTANEOUS EQUATIONS

**277. Equations of the first order.** The pair of equations  
 $P dx + Q dy + R dz = 0 \quad P_1 dx + Q_1 dy + R_1 dz = 0 \quad (1)$   
 gives

$$dx : dy : dz = (QR_1 - Q_1R) : (RP_1 - R_1P) : (PQ_1 - P_1Q) \quad (2)$$

and therefore, by the reasoning in § 276, defines an infinite set of curves  $K$ , of which one will pass through any assigned point <sup>1</sup>  $A$  and will there have the direction given by (2) at  $A$ . The pair of equations (1) will be solved if two families of surfaces  $u = c_1, v = c_2$  can be found whose curves of intersection are the curves  $K$ .

1. If one of the equations (1) is integrable, we may find  $u = c_1, v = c_2$  as follows:

EXAMPLE 1. Solve the pair of equations

$$dz = dx + dy \quad (1), \quad z dy - x dz = 0 \quad (2)$$

The solution of (1) is  $z = x + y + c_1$  (3). Substituting (3) in (2) gives the  $x, y$  equation  $(y + c_1) dy - x dx = 0$ , whose solution is  $y^2 - x^2 + 2c_1y = c_2$  (4). In (4) replace  $c_1$  by its value as given by (3); we obtain  $2yz - x^2 - y^2 - 2xy = c_2$  (5). Hence the required solution is  $z - x - y = c_1, 2yz - x^2 - y^2 - 2xy = c_2$ .

2. Let  $L, M, N$  denote the right members of (2), and  $\lambda, \mu, \nu$  any functions of  $x, y, z$ . Then

$$\frac{dx}{L} = \frac{dy}{M} = \frac{dz}{N} = \frac{\lambda dx + \mu dy + \nu dz}{\lambda L + \mu M + \nu N} \quad (3)$$

Any pair of the equations (3) is equivalent to the given pair (1) or (2), and it may be easier to solve. It is sometimes possible to choose  $\lambda, \mu, \nu$  such that  $\lambda L + \mu M + \nu N \equiv 0$ . In that case, we must have  $\lambda dx + \mu dy + \nu dz = 0$ , and if

<sup>1</sup> Points are excepted where  $QR_1 - Q_1R, RP_1 - R_1P, PQ_1 - P_1Q$  vanish simultaneously.

this equation be integrable its solution is one of the required equations  $u = c_1, v = c_2$ .

EXAMPLE 2. Solve  $dx : dy : dz = x(y - z) : y(z - x) : z(x - y)$

$$\begin{aligned} x(y - z) + y(z - x) + z(x - y) &\equiv 0 \\ \therefore dx + dy + dz &= 0 \quad \therefore x + y + z = c_1 \end{aligned} \quad (1)$$

$$\begin{aligned} yzx(y - z) + zxy(z - x) + xyz(x - y) &\equiv 0 \\ \therefore yz dx + zx dy + xy dz &= 0 \quad \therefore xyz = c_2 \end{aligned} \quad (2)$$

The integral curves  $K$  are those in which the planes (1) cut the surfaces (2).

**278. Equations with constant coefficients.** The following example illustrates the method of solving a pair of equations of the type

$$f_1(D)y + \phi_1(D)z = X_1 \quad f_2(D)y + \phi_2(D)z = X_2$$

where  $D$  denotes  $d/dx$ ;  $f_1(D)$  and so on, polynomials in  $D$  with constant coefficients; and  $X_1, X_2$  functions of  $x$ .

EXAMPLE. Solve

$$\frac{d^2y}{dx^2} + \frac{dz}{dx} + 6z = x, \quad \frac{dy}{dx} - y + 2z = x^2$$

The equations may be written :

$$D^2y + (D + 6)z = x \quad (1) \quad (D - 1)y + 2z = x^2 \quad (2)$$

To eliminate  $y$ , we apply  $D - 1$  to (1) and  $D^2$  to (2), and subtract, the result being

$$(D^2 - 5D + 6)z = x + 1 \quad (3)$$

whose solution is  $z = c_1 e^{2x} + c_2 e^{3x} + \frac{x}{6} + \frac{11}{36}$  (4)

Similarly, by eliminating  $z$ ,

$$y = c'_1 e^{2x} + c'_2 e^{3x} - x^2 - \frac{5}{3}x - \frac{19}{18} \quad (5)$$

The constants  $c_1, c_2, c'_1, c'_2$  are not independent. For substituting (4), (5) in (2), we get

$$(2c_1 + c'_1)e^{2x} + (2c_2 + 2c'_2)e^{3x} \equiv 0$$

$$\therefore 2c_1 + c'_1 = 0, \quad c_2 + c'_2 = 0 \quad \therefore c'_1 = -2c_1, \quad c'_2 = -c_2.$$

## EXERCISE LV

Solve the following pairs of equations :

$$1. \frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z}$$

$$2. \frac{dx}{x} = \frac{dy}{z} = -\frac{dz}{y}$$

$$3. \frac{dx}{x^2} = -\frac{dy}{xy} = -\frac{dz}{z^2}$$

$$4. \frac{dx}{bz - cy} = \frac{dy}{cx - az} = \frac{dz}{ay - bx}$$

$$5. \frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

$$6. \frac{dx}{y + z} = \frac{dy}{z + x} = \frac{dz}{x + y}$$

$$7. \frac{dy}{dx} + 3y + 4z = 0, \quad \frac{dz}{dx} - 2y - 5z = 0$$

$$8. \frac{d^2x}{dt^2} = m^2y, \quad \frac{d^2y}{dt^2} = m^2x$$

$$9. \frac{dy}{dx} - \frac{dz}{dx} + y = \cos 2x, \quad \frac{d^2y}{dx^2} - \frac{dz}{dx} + 3y - z = e^{2x}$$

10. Show that the differential equations of the curves orthogonal to the surfaces  $F(x, y, z) = C$  are  $dx : dy : dz = F_x : F_y : F_z$ ; and find the curves orthogonal to

$$1. 2x^2 + 4y^2 + z^2 = C$$

$$2. xy + z = C$$

## PARTIAL DIFFERENTIAL EQUATIONS

**279. Linear equations of the first order.** Let  $x, y$  be independent variables, and  $z$  a variable dependent on  $x, y$ , and represent  $\partial z / \partial x$  and  $\partial z / \partial y$  by  $p$  and  $q$ . The equation

$$Pp + Qq = R \quad (1)$$

where  $P, Q, R$  denote functions of  $x, y, z$ , is said to be *linear*.

$$\text{Suppose} \quad z = f(x, y) \quad (2)$$

to be a particular solution of (1) and call the surface which it represents an *integral surface* of (1). The values of  $p : q : -1$  at any point  $A$  of this surface (2) are the direction ratios of its normal at  $A$ , § 233 (5). Therefore since, by (1), we have  $Pp + Qq + R(-1) = 0$ , the normal to (2) at  $A$  is perpendicular to the direction whose ratios are the values of  $P : Q : R$  at  $A$ . This consideration suggests the following method of finding the integral surfaces.

By solving the simultaneous equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad (3)$$

as in § 277, we obtain a doubly infinite set of curves  $u = c_1$ ,  $v = c_2$  of which one, and generally but one, will pass through a given point  $A$  and have there for its direction ratios the values of  $P : Q : R$  at  $A$ . From this set, select the singly infinite set  $S$  for which  $c_1$  and  $c_2$  satisfy any given relation  $\phi(c_1, c_2) = 0$ . Then the equation

$$\phi(u, v) = \phi(c_1, c_2) = 0 \quad (4)$$

will represent a surface on which all the curves of the set  $S$  lie; and the normal to this surface at any point  $A$  will be perpendicular to the curve of the set  $S$  through  $A$ , that is, to the direction whose ratios are the values of  $P : Q : R$  at  $A$ . Hence the surface is an integral surface of (1).

We have therefore proved that every equation of the form  $\phi(u, v) = 0$  is a solution of the given equation (1), and it can be shown conversely that, save in exceptional cases, every solution of (1) can be reduced to this form. Hence

*The general solution of the linear equation (1) is*

$$\phi(u, v) = 0 \quad (5)$$

where  $u = c_1$ ,  $v = c_2$  denote any two independent integrals of the equations (3), and  $\phi$  an arbitrary function subject only to the restriction that  $\phi_u, \phi_v$  exist.

The curves  $u = c_1$ ,  $v = c_2$  are called the *characteristics* of the equation (1). Each particular solution of (1) represents a surface made up of these characteristics, and the general solution represents the set of all such surfaces.

One may obtain the integral surface of (1) which contains a given curve  $g(x, y, z) = 0$ ,  $h(x, y, z) = 0$ , not a characteristic, by eliminating  $x, y, z$  between  $g = 0$ ,  $h = 0$ ,  $u = c_1$ ,  $v = c_2$ . Let the result of the elimination be  $f(c_1, c_2) = 0$ ; the required surface is  $f(u, v) = 0$ .

EXAMPLE 1. Find the general solution of  $yp - xq = 0$ .

The equations  $\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{0}$  give  $x^2 + y^2 = c_1, z = c_2$ .

Hence the general solution is  $\phi(x^2 + y^2, z) = 0$  or  $x^2 + y^2 = \phi(z)$  which represents all surfaces of revolution about  $Oz$  as axis.

The characteristics are the  $\infty^2$  set of circles  $x^2 + y^2 = c_1, z = c_2$ .

EXAMPLE 2. Find the integral surface of  $2p - q = 1$  which contains the curve  $y = x^2, z = x$  (1).

The equations  $\frac{dx}{2} = \frac{dy}{-1} = \frac{dz}{1}$  give  $x - 2z = c_1, y + z = c_2$  (2)

Eliminating  $x, y, z$  between (1) and (2) gives  $c_2 + c_1 = c_1^2$

Hence the required surface is  $(y + z) + (x - 2z) = (x - 2z)^2$ .

### EXERCISE LVI

Solve the following equations and discuss their characteristics :

1.  $(y + z)p + (z + x)q = x + y$
2.  $xp + z = 0$
3.  $(bz - cy)p + (cx - az)q = ay - bx$
4.  $yp + xq = z$
5.  $xzp + yzq = xy$
6.  $zp + yq - x = 0$
7.  $xp + yq + xy - z = 0$
8.  $x^2p - xyq + y^2 = 0$
9.  $z = px + qy + x^2 + y^2$
10.  $(x^2 - y^2)p + 2xyq = 0$

11. Show that  $ap + bq = 1$  represents all cylinders whose elements have the direction ratios  $a : b : 1$ .

12. Show that  $(x - a)p + (y - b)q = z - c$  represents all cones with vertex at  $(a, b, c)$ .

13. Show that  $xp + yq = 0$  represents all surfaces formed by lines which meet the  $z$ -axis and are parallel to the  $xy$ -plane.

14. Show that all surfaces for which the  $z$ -intercept of any tangent plane is twice the  $z$  of the point of contact satisfy the equation  $px + qy + z = 0$ . Find the surface of this type which cuts the plane  $z = 1$  in the parabola  $y = x^2$ .

15. Find the differential equation of which  $\phi(x - lz, y - mz) = 0$  is the general solution.

16. Show that the characteristics of  $Pp + Qq = R$  are perpendicular to the surfaces, if any, represented by  $Pdx + Qdy + Rdz = 0$ .

17. Show that the integrating factors  $\mu$  of  $Mdx + Ndy = 0$  are integrals of

$$\frac{\partial \mu}{\partial x} N - \frac{\partial \mu}{\partial y} M + \mu \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = 0$$

**280. Non-linear equations of the first order.** Let

$$F(x, y, z, p, q) = 0 \quad (1)$$

denote an equation of a degree higher than one in  $p, q$ . Sometimes by inspection, or by methods too complicated to be considered here but which depend on the solution of certain linear equations, there can be obtained an equation of the form

$$f(x, y, z, c_1, c_2) = 0 \quad (2)$$

which satisfies (1). Here  $f$  denotes a definite function, and  $c_1, c_2$  are arbitrary constants. It is customary to call (2) a *complete integral* of (1).

The equation (2) represents a doubly infinite set of surfaces. From this set select the singly infinite set for which  $c_2$  is some given function of  $c_1$ , say  $c_2 = \phi(c_1)$ . The envelope of this subset, got, § 244, by eliminating  $c_1$  between the equations

$$f[x, y, z, c_1, \phi(c_1)] = 0 \quad (3) \quad \text{and} \quad f_{c_1} = 0 \quad (4)$$

is a surface whose equation will also satisfy (1); for it will touch each of the surfaces (3) along the curve or curves in which that surface is cut by the corresponding surface (4). These curves are called *characteristics* of the equation (1).

When we interpret  $\phi$  as an arbitrary function, that is, as any function for which the set of surfaces (3) has an envelope, then the equations (3), (4) represent the set of all surfaces which touch the surfaces (2) along their characteristics, and the pair (3), (4) is called the *general integral* of (1).

Finally a surface may exist each of whose points is an isolated point of contact with one of the surfaces (2). In that case its equation also will satisfy (1), and it is called the *singular integral* of (1). It may be found by eliminating  $c_1, c_2$  between the equations

$$f = 0 \quad f_{c_1} = 0 \quad f_{c_2} = 0 \quad (5)$$



It can be shown (compare § 258) that the singular integral, if it exists, may also be found by eliminating  $p$ ,  $q$  between the equations

$$F = 0 \qquad F_p = 0 \qquad F_q = 0 \qquad (6)$$

There are infinitely many complete integrals, but generally speaking any complete integral with the corresponding general integral will include all the particular integrals of (1).

EXAMPLE. Solve the equation  $z = pq$ .

By inspection, we find the complete integral

$$z = (x + a)(y + b) \qquad (1)$$

where  $a$ ,  $b$  are arbitrary constants. The corresponding general integral is

$$z = (x + a)[y + \phi(a)] \qquad y + \phi(a) + (x + a)\phi'(a) = 0 \qquad (2)$$

The singular integral is  $z = 0$  (3)

The equation also has the complete integral

$$4z = (cx + y/c + d)^2 \qquad (4)$$

But the surfaces (4) are included in those represented by (2).

Thus (4) with  $c = 1$ ,  $d = 0$  and (2) with  $\phi(a) = -a$  both give

$$4z = (x + y)^2 \qquad (5)$$

**281. Equations of higher order.** There is a group of partial differential equations of the second order which play an important rôle in mathematical physics. The physicist is not concerned with general solutions of these equations but with those best adapted to the discussion of the problem at hand.

EXAMPLE. Physical considerations show that in the steady flow of heat in a thin plate lying in the  $xy$ -plane the temperature  $u$  at any point  $(x, y)$  satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

Let the plate be a rectangle  $R$  of breadth  $x$  and infinite length with its base on  $Ox$  and one of its sides on  $Oy$ . Suppose that the sides are kept at the temperature 0, and the base at the temperature 1; also that for interior points of  $R$  the temperature decreases as the distance from  $Ox$  increases, and approaches 0 as the distance approaches infinity.

We are required to find the solution of (1) which will satisfy these conditions and will give the temperature  $u$  at every interior point of  $R$ .

It is readily seen that (1) is satisfied by

$$u = e^{-ky} \sin kx \quad (2)$$

also, if  $k$  be any positive integer, that this  $u$  is 0 when  $x = 0$  or  $\pi$  and that it  $\rightarrow 0$  when  $y \rightarrow \infty$ . But it does not satisfy the condition  $u = 1$  when  $y = 0$  and  $0 < x < \pi$ .

To find a solution which will satisfy this last condition also, let us set

$$u = \sum_{k=0}^{\infty} A_k e^{-ky} \sin kx \quad (3)$$

where the  $A_k$ 's denote undetermined constants. Since every term of this series satisfies (1), the same is true of the series itself if, as we shall assume, it converges. It also satisfies the conditions  $u = 0$ , when  $x = 0$  or  $\pi$ , and  $u \rightarrow 0$  when  $y \rightarrow \infty$ . To satisfy the remaining condition,  $u = 1$  when  $y = 0$  and  $0 < x < \pi$ , the  $A_k$ 's must be such that

$$1 = \sum_{k=0}^{\infty} A_k \sin kx \quad (4)$$

for all values of  $x$  between 0 and  $\pi$ . As will be shown later in the chapter on Fourier series, we can in fact determine the  $A_k$ 's so as to satisfy this requirement. When the values so determined are substituted in (3) it becomes the required solution.

### EXERCISE LVII

1. Show that the general solution of  $\frac{\partial^2 z}{\partial x \partial y} = 0$  is  $z = \phi(x) + \psi(y)$  where  $\phi$  and  $\psi$  denote arbitrary functions.

2. Find the general solutions of the following equations

$$\frac{\partial^2 z}{\partial x \partial y} = x + y + \frac{y}{x} \qquad \frac{\partial^2 z}{\partial x^2} = x^2 y$$

3. Show that, if  $x, y$  are the independent variables, the general solution of  $\frac{\partial^2 z}{\partial x^2} = a^2 z$  is  $z = \phi(y)e^{ax} + \psi(y)e^{-ax}$ , where  $\phi$  and  $\psi$  denote arbitrary functions.

4. Show that the equation with constant coefficients

$$a \frac{\partial^2 z}{\partial x^2} + b \frac{\partial^2 z}{\partial x \partial y} + c \frac{\partial^2 z}{\partial y^2} = 0$$

is satisfied by  $z = e^{kx+ly}$  if  $k, l$  satisfy the equation  $ak^2 + bkl + cl^2 = 0$ . Show that the like is true of any partial differential equation with constant coefficients.

5. Show that  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$  has the solution  $u = \phi(x + iy) + \psi(x - iy)$  where  $\phi$  and  $\psi$  denote arbitrary functions, and  $i = \sqrt{-1}$ .

6. Show that  $\frac{\partial^2 z}{\partial t^2} - a^2 \frac{\partial^2 z}{\partial x^2} = 0$ , the equation of a vibrating cord, has the solution  $z = \phi(x + at) + \psi(x - at)$ , where  $\phi$  and  $\psi$  are arbitrary functions.

7. Show that  $z = px + qy + f(p, q)$  has the complete integral  $z = ax + by + f(a, b)$ . What is the corresponding general integral and what the singular integral when  $f(p, q) = pq$ ?

8. Show that any equation of the form  $f(p, q) = 0$  has the complete integral  $z = ax + by + c$ , where  $f(a, b) = 0$ . Show that the surfaces represented by the corresponding general integral are developable surfaces, § 244.

9. Show that every developable surface satisfies some equation of the form  $f(p, q) = 0$  and therefore that all developable surfaces satisfy the equation

$$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left( \frac{\partial^2 z}{\partial x \partial y} \right)^2 = 0$$

10. Find developable surfaces for which  $p, q$  satisfy each of the following conditions:

$$p^2 + q^2 = 1$$

$$pq = p + q$$

## XXVI. LINE AND SURFACE INTEGRALS

### THEOREMS RESPECTING DEFINITE INTEGRALS

**282. Mean value theorem for double and triple integrals.** As in § 149, let  $S$  denote a given region in the  $xy$ -plane and also its area.

1. If  $f(x, y)$  is continuous in  $S$ , there is a point  $(\bar{x}, \bar{y})$  in  $S$  such that

$$\int_S f(x, y) dS = f(\bar{x}, \bar{y}) \cdot S \quad (1)$$

For by the definition of  $\int_S f(x, y) dS$ , § 149, its value is between the least and greatest values of  $f(x, y) \cdot S$  in  $S$ . Therefore, § 18, 2, since  $f(x, y) \cdot S$  is continuous in  $S$ , there is a point  $(\bar{x}, \bar{y})$  in  $S$  such that  $f(\bar{x}, \bar{y}) \cdot S = \int_S f(x, y) dS$ .

The like is true of integrals of the type  $\int_S f(x, y, z) dS$ .

2. If  $f$  is continuous, and  $\phi$  is both continuous and positive in  $S$ , there is a point  $P$  in  $S$  such that,  $f_P$  denoting the value of  $f$  at  $P$ ,

$$\int_S f\phi dS = f_P \int_S \phi dS \quad (2)$$

For if  $M$  and  $m$  denote the greatest and least values of  $f$  in  $S$ , then

$$\int_S (M - f)\phi dS > 0 \quad \therefore M \int_S \phi dS > \int_S f\phi dS \quad (3)$$

$$\int_S (f - m)\phi dS > 0 \quad \therefore \int_S f\phi dS > m \int_S \phi dS \quad (4)$$

and (2) follows from (3) and (4), as in the proof of 1.

**EXAMPLE.** Extend the theorem of § 145 to double and triple integrals.

**283. Differentiation under the integral sign.** Let  $f(x, \alpha)$  be a continuous function of  $x$  and the parameter  $\alpha$  in the region  $R$  in which  $x$  is between  $a$  and  $b$  and  $\alpha$  between  $A$  and  $B$ . Suppose also that  $f_\alpha(x, \alpha)$  exists and is continuous in  $R$ .

Evidently  $\int_a^b f(x, \alpha) dx$  is a function of  $\alpha$ . We are to prove that

$$\text{If } F(\alpha) = \int_a^b f(x, \alpha) dx \quad \text{then} \quad F'(\alpha) = \int_a^b f_\alpha(x, \alpha) dx$$

$$\text{For} \quad F(\alpha + \Delta\alpha) - F(\alpha) = \int_a^b [f(x, \alpha + \Delta\alpha) - f(x, \alpha)] dx$$

Hence, applying the mean value theorem, and then dividing by  $\Delta\alpha$ ,

$$\frac{F(\alpha + \Delta\alpha) - F(\alpha)}{\Delta\alpha} = \int_a^b f_\alpha(x, \alpha + \theta \Delta\alpha) dx \quad (1)$$

$$\text{Set} \quad f_\alpha(x, \alpha + \theta \Delta\alpha) = f_\alpha(x, \alpha) + \eta$$

Since  $f_\alpha$  is uniformly continuous in  $R$ , § 148 (5), if any positive number  $\epsilon$  be assigned we can find another positive number  $\delta$  such that everywhere in  $R$ , when  $|\Delta\alpha| < \delta$  we shall have  $|\eta| < \epsilon$ , and therefore  $|\int_a^b \eta dx| < \epsilon |b - a|$ . Hence it follows from (1), when  $\Delta\alpha \rightarrow 0$ , that

$$F'(\alpha) = \int_a^b f_\alpha(x, \alpha) dx \quad (2)$$

If the limits  $a$  and  $b$  are themselves functions of  $\alpha$ , then, §§ 126, 217,

$$F'(\alpha) = \int_a^b f_\alpha(x, \alpha) dx + f(b, \alpha) \frac{db}{d\alpha} - f(a, \alpha) \frac{da}{d\alpha} \quad (3)$$

EXAMPLE 1. Show that  $v = \int_0^{2\pi} e^{k(x \cos \theta + y \sin \theta)} \phi(\theta) d\theta$  is a solution of

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = k^2 v$$

EXAMPLE 2. Verify (3) for the integral  $\int_a^{2\alpha} (3x^2 + 2\alpha x + \alpha^2) dx$ .

**284. The Gamma function.** By the tests indicated in Examples 26 and 27 on p. 152, it is readily shown that the integral  $\int_0^\infty x^{n-1} e^{-x} dx$  exists when  $n > 0$ . Its value is a function of the parameter  $n$ . This function is called the *gamma function* and is represented by  $\Gamma(n)$ . Hence

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \quad n > 0 \quad (1)$$

If we integrate  $\int_{x_1}^{x_2} x^n e^{-x} dx$  by parts, and in the result let  $x_1 \rightarrow 0$  and  $x_2 \rightarrow \infty$ , we obtain  $\int_0^\infty x^n e^{-x} dx = n \int_0^\infty x^{n-1} e^{-x} dx$ . Hence

$$\Gamma(n+1) = n\Gamma(n) \quad (2)$$

By (1), we have  $\Gamma(1) = 1$ . Hence (2) gives successively  $\Gamma(2) = 1$ ,  $\Gamma(3) = 2 \cdot 1$ , and in general, if  $n$  be any positive integer greater than one,  $\Gamma(n) = (n - 1)!$ .

By a similar procedure, we may express the value of  $\Gamma(n)$  for any positive non-integral  $n$  in terms of its value for an  $n$  between 1 and 2; and the values of  $\Gamma(n)$  for  $n$  between 1 and 2 have been tabulated.

The definition (1) applies only when  $n$  is positive. But we may write (2) in the form

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \quad (3)$$

and then use it to define  $\Gamma(n)$  for negative values of  $n$ .

For when  $n$  is between  $-1$  and  $0$ , then  $n+1$  is between  $0$  and  $1$ , and therefore  $\Gamma(n+1)$  is known by (1); hence (3) gives us  $\Gamma(n)$ . In like manner the values of  $\Gamma(n)$  for  $n$  between  $-2$  and  $-1$  are determined by its values for  $n$  between  $-1$  and  $0$ ; and so on. Setting  $n = 0$ ,  $-1$ ,  $-2$ ,  $\dots$  successively, we find that the definition (3) makes  $\Gamma(n)$  infinite for  $0$  and negative integral values of  $n$ .

## JACOBIANS

### 285. Properties of functional determinants or Jacobians.

Let  $u$ ,  $v$  be independent variables referred to rectangular axes  $Ou$ ,  $Ov$ , and let  $x$ ,  $y$  be given functions of  $u$ ,  $v$  in some region  $D$  of the  $uv$ -plane :

$$x = \phi(u, v) \quad y = \psi(u, v) \quad (1)$$

$\phi$ ,  $\psi$  and their first partial derivatives being continuous.

Represent the functional determinant, § 229, or Jacobian,  $J$ , of  $\phi$  and  $\psi$ , or  $x$  and  $y$ , with respect to  $u$  and  $v$ , namely

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \text{by} \quad \frac{\partial(x, y)}{\partial(u, v)} \quad (2)$$

The Jacobian has properties analogous to those of the derivative.

1. Suppose that  $u, v$  are themselves functions of independent variables  $r, s$ , these functions possessing continuous partial derivatives. Then

$$\frac{\partial(x, y)}{\partial(r, s)} = \frac{\partial(x, y)}{\partial(u, v)} \frac{\partial(u, v)}{\partial(r, s)} \quad (3)$$

$$\text{For} \quad \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} \frac{\partial u}{\partial r} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial r} & \frac{\partial x}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial s} \\ \frac{\partial y}{\partial u} \frac{\partial u}{\partial r} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial r} & \frac{\partial y}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial s} \end{vmatrix}$$

which is the product of the two determinants  $\partial(x, y)/\partial(u, v)$  and  $\partial(u, v)/\partial(r, s)$ , § 203.

2. Let  $(u_1, v_1)$  be any point of  $D$  where the Jacobian (2) is not 0, and let  $(x_1, y_1)$  be the corresponding point in the  $xy$ -plane given by (1) and § 230. Then  $u, v$  are one-valued continuous functions of  $x, y$  :

$$u = \Phi(x, y) \quad v = \Psi(x, y) \quad (4)$$

in the neighborhood of  $(x_1, y_1)$  and take the values  $u_1, v_1$  at  $(x_1, y_1)$ , § 230.

The pair of functions (4) is called the *inverse* of the pair (1). By setting  $r = x, s = y$  in (3), we obtain

$$1 = \frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)} \quad \therefore \quad \frac{\partial(x, y)}{\partial(u, v)} = 1 \div \frac{\partial(u, v)}{\partial(x, y)} \quad (5)$$

3. Let  $x, y$  be functions of the three variables  $r, s, t$  :

$$x = \phi(r, s, t) \quad y = \psi(r, s, t) \quad (6)$$

and let  $r, s, t$  in turn be functions of the independent variables  $u, v$ .

The Jacobian  $J$  of  $x$  and  $y$  with respect to  $u$  and  $v$  is given by the formula

$$J = \frac{\partial(x, y)}{\partial(r, s)} \frac{\partial(r, s)}{\partial(u, v)} + \frac{\partial(x, y)}{\partial(s, t)} \frac{\partial(s, t)}{\partial(u, v)} + \frac{\partial(x, y)}{\partial(t, r)} \frac{\partial(t, r)}{\partial(u, v)} \quad (7)$$



$$\text{For} \quad \begin{array}{ccc} 1 & 2 & 3 \\ \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial r} \frac{\partial r}{\partial u} + \frac{\partial x}{\partial s} \frac{\partial s}{\partial u} + \frac{\partial x}{\partial t} \frac{\partial t}{\partial u} & \frac{\partial x}{\partial r} \frac{\partial r}{\partial v} + \frac{\partial x}{\partial s} \frac{\partial s}{\partial v} + \frac{\partial x}{\partial t} \frac{\partial t}{\partial v} \\ \frac{\partial y}{\partial r} \frac{\partial r}{\partial u} + \frac{\partial y}{\partial s} \frac{\partial s}{\partial u} + \frac{\partial y}{\partial t} \frac{\partial t}{\partial u} & \frac{\partial y}{\partial r} \frac{\partial r}{\partial v} + \frac{\partial y}{\partial s} \frac{\partial s}{\partial v} + \frac{\partial y}{\partial t} \frac{\partial t}{\partial v} \end{vmatrix} \end{array}$$

Resolve this determinant into a sum of determinants with simple columns by the theorem of § 199, 7. and in this sum represent the determinant with the columns marked 1 and 1' by the symbol (1, 1'), and so on. It is easily shown that the determinants (1, 1'), (2, 2'), (3, 3') are 0, and that the sums

$$(1, 2') + (2, 1') \quad (2, 3') + (3, 2') \quad (3, 1') + (1, 3')$$

are equal to the first, second and third terms of (7) respectively.

4. The functional determinant or Jacobian of any number of functions  $x_1, x_2, \dots, x_n$ , of the same number of variables  $u_1, u_2, \dots, u_n$ , is represented by the symbol

$$\frac{\partial(x_1, x_2, \dots, x_n)}{\partial(u_1, u_2, \dots, u_n)}$$

The theorems in 1., 2., 3. hold good for any such Jacobian.

## LINE AND SURFACE INTEGRALS

**286. Line integrals.** Let  $AB$  be a curve arc which is met by parallels to  $Ox$  or  $Oy$  in single points, and on which  $y$  is a continuous function of  $x$ , and  $x$  of  $y$ ;  $AB$  having an equation of the form  $y = \phi(x)$ , or  $x = \psi(y)$ , where  $\phi$  and  $\psi$  denote one-valued, continuous functions.

And let  $f(x, y)$  be a function of  $x, y$  which is continuous on the arc  $AB$ .

It being also supposed that  $\phi(x)$  and  $\psi(y)$  have continuous derivatives, think of  $AB$  as traced by a point  $P$  which starts at  $A$ , and regard  $s$ , the length of the arc  $AP$ , as the independent variable.

Divide the arc  $AB$  into  $n$  parts all of which  $\rightarrow 0$  when  $n \rightarrow \infty$ . Let  $\delta s$  denote any one of these parts;  $(x, y)$ , any point of  $\delta s$ ; and  $\delta x, \delta y$ , the projections of  $\delta s$  on  $Ox, Oy$ .

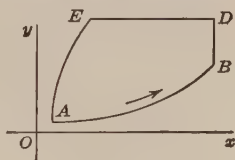


FIG. 127.

When  $n \rightarrow \infty$  the sum  $\Sigma f(x, y) \delta x$  approaches a limit. For

$$\Sigma f(x, y) \delta x = \Sigma f[x, \phi(x)] \delta x \rightarrow \int_a^b f[x, \phi(x)] dx$$

where  $a$  and  $b$  are the abscissas of  $A$  and  $B$ . We call  $\lim \Sigma f(x, y) \delta x$  the  $x$ -integral of  $f(x, y)$  along  $AB$ , and represent it by  $\int_{AB} f(x, y) dx$ .

In like manner, by definition,

$$\int_{AB} f(x, y) dy = \lim \Sigma f(x, y) \delta y$$

Evidently

$$\int_{BA} f dx = - \int_{AB} f dx \quad \text{and} \quad \int_{BA} f dy = - \int_{AB} f dy$$

We extend these definitions to line segments, like  $BD$  or  $DE$ , parallel to  $Oy$  or  $Ox$ . Since the projection of any part  $\delta s$  of  $BD$  on  $Ox$  is 0,  $\int_{BD} f dx = 0$ . Similarly  $\int_{DE} f dy = 0$ .

Finally let the point  $P$  trace any path  $C$  which can be separated into a finite number of non-intersecting pieces of the types  $AB, BD, DE$ . We define  $\int_C f dx$  and  $\int_C f dy$  as the algebraic sums of the corresponding integrals along the several pieces. If we reverse the sense in which  $C$  is traced, we merely change the signs of  $\int_C f dx$  and  $\int_C f dy$ . When  $C$ , like  $ABDEA$  in Fig. 127, is the contour of a closed region  $S$ , we take as the positive sense of motion along  $C$  that for which the tracing point has  $S$  at its left.

Let  $P$  and  $Q$  be any two functions of  $x, y$ , both continuous on  $C$ . The sum of  $\int_C P dx$  and  $\int_C Q dy$  is written

$$\int_C P dx + Q dy,$$

and this is taken as the general expression for a line integral in  $x, y$ . Observe that if  $v_T$  denote the projection of the vector whose  $x$ - and  $y$ -components are  $P$  and  $Q$ , on the tangent to the curve  $C$ , we have  $\int_C P dx + Q dy = \int_C v_T ds$ .

We have analogous definitions of integrals along space curves.

EXAMPLE 1. If  $AB$  is the arc of the parabola  $y = x^2$  between  $A(0, 0)$  and  $B(1, 1)$ , then

$$\int_{AB} (x + y) dx + x^2 dy = \int_0^1 (x + x^2) dx + \int_0^1 y dy = 4/3$$

**EXAMPLE 2.** If  $C$  consists of the line segments  $AB$  and  $BD$  joining  $A(a, b)$ ,  $B(x, b)$  and  $D(x, y)$ , then

$$\begin{aligned}\int_C y \, dx + x \, dy &= \int_{AB} y \, dx + \int_{BD} x \, dy = \int_a^x b \, dx + \int_b^y x \, dy \\ &= b(x - a) + x(y - b) = xy - ab\end{aligned}$$

**287. Green's formula.** 1. Let  $S$  denote any region, bounded by a single contour  $C$ , which can be divided into pieces like  $S_1 = KLMN$  in Fig. 128,  $KL$  and  $MN$  being non-intersecting curve arcs of the type described in § 286. And let  $P$  and  $Q$  be any functions of  $x, y$  which, together with  $\partial P/\partial y$  and  $\partial Q/\partial x$  are continuous in  $S$  and on  $C$ . We have

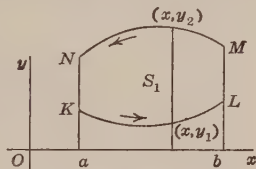


FIG. 128.

$$\begin{aligned}\iint_{S_1} \frac{\partial P}{\partial y} \, dy \, dx &= \int_a^b \int_{y_1}^{y_2} \frac{\partial P}{\partial y} \, dy \, dx \\ &= \int_a^b P(x, y_2) \, dx - \int_a^b P(x, y_1) \, dx \\ &= - \int_{MN} P(x, y) \, dx - \int_{KL} P(x, y) \, dx\end{aligned}\quad (1)$$

the line integrals in the last member being taken in the positive sense along the parts of the contour  $C$  of  $S$  which belong to  $S_1$ . Hence by adding the equations (1) for all the parts  $S_1$  of  $S$ , we have

$$\iint_S \frac{\partial P}{\partial y} \, dy \, dx = - \int_C P \, dx \quad (2)$$

We can show in like manner that

$$\iint_S \frac{\partial Q}{\partial x} \, dx \, dy = \int_C Q \, dy \quad (3)$$

and subtracting (2) from (3) we obtain

$$\iint_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \int_C P \, dx + Q \, dy \quad (4)$$

This identity is called *Green's formula*. It has many important applications. Observe that (2) and (3) are particular cases of it.

If in (4) we set  $P = y, Q = 0$ ;  $P = 0, Q = x$ ;  $P = -y, Q = x$ , we obtain the formulas:

$$\text{Area } S = - \int_C y \, dx = \int_C x \, dy = \frac{1}{2} \int_C x \, dy - y \, dx \quad (5)$$

2. If  $\partial Q/\partial x = \partial P/\partial y$  throughout  $S$ , then

$$\int_C P \, dx + Q \, dy = 0 \quad (6)$$

Conversely, if along every closed curve in a certain region  $R$  the integral  $\int_C$  in (4) is 0, then  $\partial Q/\partial x - \partial P/\partial y$  is 0 throughout  $R$ . For otherwise,  $\partial Q/\partial x$  and  $\partial P/\partial y$  being continuous, there would be a region in  $R$  in which the sign of  $\partial Q/\partial x - \partial P/\partial y$  was constant, and, by (4), the integral  $\int_C$  along the contour of this region would not be 0.

3. The formula (4) may be extended to regions bounded by more than one contour.

Thus consider the region  $S$  between the contours  $C_1$  and  $C_2$  in Fig. 129. Make the cut  $ad$  and so transform  $S$  into a region  $S'$  bounded by the single contour  $C = abcdefda$ . By (4) we have  $\iint_S = \iint_{S'} = \int_C$ . But the parts of  $\int_C$  along  $ad$  and  $da$  cancel each other. Hence  $\iint_S = \int_{C_1} + \int_{C_2}$ , where both  $C_1$  and  $C_2$  are supposed traced by a point having  $S$  at its left.

If  $\partial Q/\partial x = \partial P/\partial y$  throughout  $S$ , and both  $C_1$  and  $C_2$  be supposed traced clockwise, then  $\int_{C_1} = \int_{C_2}$ .

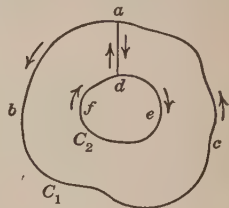


FIG. 129.

4. A region in which no closed curve can be drawn that encloses a boundary point of the region is said to be *simply connected*; one in which such a curve can be drawn, *multiply connected*. Thus in Fig. 129 the region bounded by  $C_1$  and  $C_2$  is multiply connected; that bounded by  $C_1$ ,  $C_2$  and  $ad$  is simply connected.

**288.** The function  $F(x, y) = \int_{a,b}^{x,y} P \, dx + Q \, dy$ . 1. Suppose that throughout a region  $R$ , of the kind described in § 214, 2, we have  $\partial Q/\partial x = \partial P/\partial y$ . Then if  $A$  and  $B$  are any two points of  $R$ , the value of  $\int_{AB} P \, dx + Q \, dy$  is the

same for all paths in  $R$  from  $A$  to  $B$  and may be represented by  $\int_A^B P dx + Q dy$ .

For if  $ADB$  and  $AEB$  denote any two such paths, and  $(ADB)$ ,  $(AEB)$  the values of  $\int_{AB} P dx + Q dy$  along them, then, by § 287 (6),

$$(ADB) + (BEA) = 0$$

$$\therefore (ADB) = -(BEA) = (AEB).$$

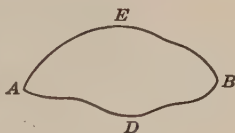


FIG. 130.

This proof does not apply to all paths in a region like  $S$  in Fig. 129. Let  $C'$  be a curve in  $S$  that encloses  $C_2$ . It cannot be inferred that  $\int_{C'} P dx + Q dy$  is 0 unless it be known that  $P$ ,  $Q$ ,  $\partial Q/\partial x$ ,  $\partial P/\partial y$  are continuous and  $\partial Q/\partial x = \partial P/\partial y$ , not only in  $S$ , but also in the entire region interior to  $C_2$ .

2. Let  $(a, b)$  denote a fixed point, and  $(x, y)$  a variable point, both in  $R$ . By 1.,  $\int_{a,b}^{x,y} P dx + Q dy$  has the same value for all paths in  $R$  from  $(a, b)$  to  $(x, y)$ ; it is therefore a function of  $(x, y)$  only.

$$\text{If } F(x, y) = \int_{a,b}^{x,y} P dx + Q dy, \text{ then } \frac{\partial F}{\partial x} = P \text{ and } \frac{\partial F}{\partial y} = Q.$$

For give  $x$  the increment  $\Delta x$ . It then follows from 1. that

$$F(x + \Delta x, y) - F(x, y) = \int_{x,y}^{x+\Delta x,y} P dx + Q dy$$

$$\text{But } \int_{x,y}^{x+\Delta x,y} P dx = P(x + \theta \Delta x, y) \Delta x, \text{ and } \int_{x,y}^{x+\Delta x,y} Q dy = 0$$

$$\text{Hence } \frac{\partial F}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{F(x + \Delta x, y) - F(x, y)}{\Delta x} = P(x, y)$$

$$\text{Similarly } \frac{\partial F}{\partial y} = Q(x, y)$$

Hence  $P dx + Q dy$  is the differential of  $F(x, y)$ , and we have another proof that  $P dx + Q dy$  is an exact differential if  $\partial Q/\partial x = \partial P/\partial y$ .

If  $u(x, y)$  be any known integral of  $P dx + Q dy$ , then, by § 223, 2.,

$$F(x, y) = \int_{a,b}^{x,y} P dx + Q dy = u(x, y) + C$$

Let  $x, y \rightarrow a, b$ . We get

$$0 = u(a, b) + C \quad \therefore C = -u(a, b)$$

Hence  $F(x, y) = u(x, y) - u(a, b)$

We can find such an integral  $u(x, y)$  by the formula

$$\int_{x_0, y_0}^{x, y} P dx + Q dy = \int_{x_0}^x P(x, y_0) dx + \int_{y_0}^y Q(x, y) dy \quad (1)$$

**289. Transformation of double integrals.** Into what integral is  $\iint_S f(x, y) dx dy$  transformed by the substitution

$$x = \phi(u, v) \quad y = \psi(u, v) \quad (1)$$

where  $u, v$  denote new independent variables?

Regarding  $u, v$  as variables referred to rectangular axes,  $O_1u, O_1v$ , we assume that there is a region  $S'$  in the  $uv$ -plane to which the equations (1) make the given  $xy$ -region  $S$  correspond in such a manner that to each interior point  $(u, v)$  of  $S'$  there corresponds an interior point  $(x, y)$  of  $S$ , and but one, and that when  $(u, v)$  traces the contour  $C'$  of  $S'$ , then  $(x, y)$  traces the contour  $C$  of  $S$  in the same or

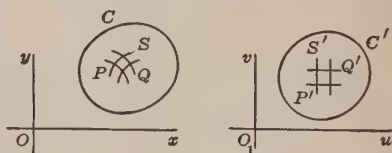


FIG. 131.

the opposite sense. We also suppose that in  $S'$  and on  $C'$  the functions  $\phi$  and  $\psi$  and their first and second partial derivatives are continuous, and that

$$J = \frac{\partial \phi}{\partial u} \frac{\partial \psi}{\partial v} - \frac{\partial \phi}{\partial v} \frac{\partial \psi}{\partial u} \quad (2)$$

does not vanish, and is therefore of constant sign. We shall prove that

$$\iint_S f(x, y) dx dy = \iint_{S'} f(\phi, \psi) |J| du dv \quad (3)$$

To a division of  $S'$  into small rectangles  $\delta S'(P'Q')$  by sets of lines  $u = a$  and  $v = b$  parallel to  $O_1v$  and  $O_1u$  corresponds the division of  $S$  into small curvilinear quadrilaterals  $\delta S(PQ)$  by the sets of curves

$x = \phi(a, v)$ ,  $y = \psi(a, v)$  and  $x = \phi(u, b)$ ,  $y = \psi(u, b)$ . By § 149, the given integral  $\iint_S f(x, y) dx dy$  equals  $\lim \Sigma f(x, y) \delta S$ . Hence to prove our theorem we must find the expression for  $\delta S$  in terms of  $u, v$ .

1. We begin by finding the expression for the area  $S$  in terms of  $u, v$ . By § 287 (5), we have  $S = \int_C x dy$ , the integral being taken in the positive sense along  $C$ . But, by hypothesis,  $(x, y)$  will trace  $C$  in this sense when  $(u, v)$  traces  $C'$  in one sense or the other. Hence if  $ds'$  denote the differential of arc of  $C'$ , we have, by §§ 132, 107,

$$S = \int_C x dy = \pm \int_{C'} \phi \frac{d\psi}{ds'} ds' = \pm \int_{C'} \phi \frac{\partial \psi}{\partial u} du + \phi \frac{\partial \psi}{\partial v} dv \quad (4)$$

the sign being  $+$  or  $-$  according as  $C'$  is traced in the positive or negative sense. But by Green's formula, § 287 (4), with  $x, y, P, Q$  replaced by  $u, v, \phi \partial \psi / \partial u, \phi \partial \psi / \partial v$ , the last member of (4) reduces to

$$\pm \iint_{S'} \left[ \frac{\partial}{\partial u} \left( \phi \frac{\partial \psi}{\partial v} \right) - \frac{\partial}{\partial v} \left( \phi \frac{\partial \psi}{\partial u} \right) \right] du dv = \pm \iint_{S'} J du dv \quad (5)$$

Since the area  $S$  is positive and by hypothesis the sign of  $J$  is constant in  $S'$ , it follows from (4) and (5) that  $\pm J = |J|$  and therefore that

$$S = \iint_{S'} |J| du dv \quad (6)$$

2. It follows from (6) and § 282 (1), that the area of the part  $\delta S$  of  $S$  corresponding to the part  $\delta S'$  of  $S'$  can be expressed in the form

$$\delta S = |J(\bar{u}, \bar{v})| \delta S' \quad (7)$$

where  $(\bar{u}, \bar{v})$  denotes some point of  $\delta S'$ . Hence, if  $(\bar{x}, \bar{y})$  be the corresponding point of  $\delta S$ , we have

$$f(\bar{x}, \bar{y}) \delta S = f[\phi(\bar{u}, \bar{v}), \psi(\bar{u}, \bar{v})] |J(\bar{u}, \bar{v})| \delta S'$$

and since  $\iint_S f(x, y) dx dy = \lim \Sigma f(x, y) \delta S$ ,

we have finally  $\iint_S f(x, y) dx dy = \iint_{S'} f(\phi, \psi) |J| du dv$

EXAMPLE. If  $x = r \cos \theta$ ,  $y = r \sin \theta$ , show that  $J = r$ , therefore that

$$\iint_S f(x, y) dx dy = \iint_{S'} f(r \cos \theta, r \sin \theta) r dr d\theta$$

**290. Surface area.** Let  $S$  denote a piece of a surface bounded by a single contour  $L$ , and let  $D$  denote the projection of  $S$  on some plane  $E$ ,  $S$  and  $E$  being such that no normal to  $E$  meets  $S$  in more than one point and that no normal to  $S$  is parallel to  $E$ .

Referred to rectangular axes  $Oxyz$  in which  $E$  is the  $xy$ -plane,  $S$  has an equation of the form  $z = f(x, y)$ . We



suppose  $S$  to be such that  $f(x, y)$ ,  $p = f_x$ ,  $q = f_y$  are continuous in  $D$ , — which means that  $S$  has a definite tangent plane at every point and that this plane moves continuously when the point of tangency moves continuously on  $S$ .

Divide  $D$  into parts of the type described in § 149, and by cylindrical surfaces through their boundaries and parallel to  $Oz$  make the corresponding division of  $S$ . Let  $\delta D$  denote any one of these parts of  $D$  and also its area; and let  $\delta S$  denote the corresponding part of  $S$ , and  $\delta T$  the area of that part of any tangent plane to  $\delta S$  which like  $\delta S$  projects into  $\delta D$ . The angle which the tangent plane makes with the  $xy$ -plane  $E$  is the same as the acute angle  $\gamma$  which the corresponding normal to  $\delta S$  makes with  $Oz$ . Hence  $\delta D = \delta T \cos \gamma$ , and therefore, § 233 (5),

$$\delta T = \delta D / \cos \gamma = (p^2 + q^2 + 1)^{1/2} \delta D$$

By § 149, since  $(p^2 + q^2 + 1)^{1/2}$  is continuous in  $D$ , the sum

$$\Sigma \delta T = \Sigma \delta D / \cos \gamma = \Sigma (p^2 + q^2 + 1)^{1/2} \delta D$$

will approach a limit when all the parts  $\delta D \rightarrow 0$ . This limit,  $\int_D \sec \gamma \, dD$ , is called the *area of  $S$* . Hence if we make the division of  $D$  into parts by parallels to  $Ox$ ,  $Oy$ , so that  $\delta D = \delta x \, \delta y$ , we have, by definition,

$$\text{area } S = \iint_D \frac{dx \, dy}{\cos \gamma} = \iint_D (p^2 + q^2 + 1)^{1/2} \, dx \, dy \quad (1)$$

The value of area  $S$  given by (1) is evidently independent of the choice of the  $xy$ -axes in the plane  $E$ . It is also independent of the choice of the plane  $E$  itself.<sup>1</sup> For let  $E'$  denote a second plane,  $l$  its line of intersection with  $E$ , and  $\theta$  the angle which  $E'$  makes with  $E$ . Take any point  $O$  of  $l$  as origin and  $l$  as  $y$ -axis in both planes, the axes  $Oxyz$  and  $O'x'y'z'$  then lying as in Fig. 132. Finally let  $D'$  denote the pro-

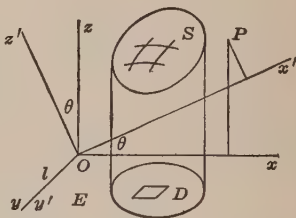


FIG. 132.

<sup>1</sup> E. B. Wilson, *Advanced Calculus*, p. 339.

jection of  $S$  on  $E'$ , and  $\gamma'$  the acute angle which a normal to  $S$  makes with  $Oz'$ . Then

$$\iint_D \frac{dx dy}{\cos \gamma} = \iint_{D'} \frac{dx' dy'}{\cos \gamma (\cos \theta + p \sin \theta)} = \iint_{D'} \frac{dx' dy'}{\cos \gamma'} \quad (2)$$

For, as the figure shows, the  $x, y$  and  $x', y'$  coordinates of any point of  $S$  are connected by the relations

$$y' = y \quad x' = x \cos \theta + z \sin \theta, \text{ where } z = f(x, y) \quad (3)$$

$$\text{Hence} \quad dx = [1/(\cos \theta + p \sin \theta)] dx' \quad \text{and} \quad dy = dy' \quad (4)$$

The substitution (3), (4) transforms the first integral in (2) into the second, the two integrals being equal by § 132 since only one variable,  $x$ , is changed. But the denominator in the second integral equals  $\cos \gamma'$ . For the direction ratios of a normal to  $S$ , referred to the axes  $Oxyz$ , are  $-p : -q : 1$ , and those of  $Oz'$  are  $-\sin \theta : 0 : \cos \theta$ . Hence, § 211 (1),

$$\cos \gamma' = \frac{p \sin \theta + \cos \theta}{(p^2 + q^2 + 1)^{1/2}} = \cos \gamma (p \sin \theta + \cos \theta)$$

### EXERCISE LVIII

1. Find  $\int_C x^2 ds$  when  $C$  is the circle  $x = a \cos \theta, y = a \sin \theta$ .
2. Find  $\int_C (x^2 + y) dx + (2x + y^2) dy$  when  $C$  is the square whose angular points are  $(1, 1), (2, 1), (2, 2), (1, 2)$ . Verify by aid of Green's formula.
3. Given the points  $A(2, 1), B(6, 1), C(6, 3)$ , compute the integral  $\int_{2,1}^{6,3} y^2 dx + x^2 dy$  along the straight paths  $ABC$  and  $AC$ .
4. Compute  $\int_{0,0}^{1,-1} (x + 2y) dx + yx dy$  along the paths  $y = -x^2$ , and  $x = t^2, y = t^3$ .
5. Show that the value of  $\int_{1,2}^{4,5} 2xy dx + (x^2 + 2y) dy$  is independent of the path, and compute it.
6. Show that  $\frac{x^2 - y^2}{x^2 y} dx + \frac{y^2 - x^2}{xy^2} dy$  is an exact differential, and find its integral by aid of the formula § 288 (1).
7. Show that  $\int_C \frac{x dy - y dx}{x^2 + y^2}$  is 0 if  $C$  is any circle which does not meet or enclose the origin; is  $2\pi$  if  $C$  is the circle  $x = a \cos \theta, y = a \sin \theta$ ; is  $\pi$  if  $C$  is the circle  $x - a = a \cos \theta, y = a \sin \theta$ .
8. Find the area of the surface  $z^2 = x^3$  included by the planes  $x = 0, y = 0, x + y = 1$ .

9. Find the area of the surface  $z = x^{3/2} + y^{3/2}$  bounded by the planes  $y = x$ ,  $y = 0$ ,  $x = a$ .

10. Show that the area of the portion of the surface of the sphere  $x^2 + y^2 + z^2 = a^2$  within the cylinder  $x^2 + y^2 - ax = 0$  is given by

$$4a \int_0^a \int_0^{\sqrt{ax-x^2}} \frac{dy dx}{(a^2 - x^2 - y^2)^{1/2}} = 4a \int_0^{\pi/2} \int_0^{a \cos \theta} \frac{r dr d\theta}{(a^2 - r^2)^{1/2}} = 4a^2 \left( \frac{\pi}{2} - 1 \right)$$

11. Find the area of the portion of the surface  $x^2 + y^2 = z^2$  within the surface  $x^2 + y^2 = 2x$ .

12. Show that the portion of the surface  $x^2 + y^2 + z^2 = 2z$  within the surface  $z = x^2 + y^2$  has the area  $2\pi$ .

13. Derive the formula for the area of a surface of revolution, § 144 (1), from the formula § 290 (1).

14. Fill in the details of the following proof that  $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$ .

$$\text{Set } K = \int_0^a e^{-x^2} dx = \int_0^a e^{-v^2} dy \quad \therefore K^2 = \int_0^a \int_0^a e^{-(x^2+y^2)} dx dy$$

$$\text{Then} \quad \int_0^{\pi/2} \int_0^a e^{-r^2} r dr d\theta < K^2 < \int_0^{\pi/2} \int_0^{a\sqrt{2}} e^{-r^2} r dr d\theta$$

$$\text{Hence } \lim_{a \rightarrow \infty} K^2 = \pi/4, \text{ and therefore } \int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2.$$

**291. Surface integrals.** Let  $F(x, y, z)$  denote a function which is continuous on the piece of surface  $S$  described in § 290. Divide  $S$  into parts each of which can be inscribed in a sphere of radius  $\rho$ . Let  $\delta S$  denote any one of these parts and also its area, and let  $x, y, z$  denote any point of  $\delta S$ . When  $\rho \rightarrow 0$ , the sum  $\Sigma F(x, y, z) \delta S$  will approach a limit. This limit is called the integral of  $F(x, y, z)$  over  $S$ , and is denoted by  $\iint_S F(x, y, z) dS$ .

If the equation of  $S$  be  $z = f(x, y)$ , and  $D$  be the projection of  $S$  on the  $xy$ -plane, and  $\gamma$  the acute angle which the normal at a point of  $S$  makes with  $Oz$ , then

$$\iint_S F(x, y, z) dS = \iint_D F(x, y, z) \sec \gamma dx dy \quad (1)$$

For, by § 282 (1), we can express  $\delta S$  in the form

$$\delta S = [f_x^2(\bar{x}, \bar{y}) + f_y^2(\bar{x}, \bar{y}) + 1]^{1/2} \delta D$$

where  $(\bar{x}, \bar{y})$  denotes some point in  $\delta D$ ; and, by §§ 282, Example,

$$\lim \Sigma F(x, y, z) [f_x^2(\bar{x}, \bar{y}) + f_y^2(\bar{x}, \bar{y}) + 1]^{1/2} \delta D = \iint_D F(x, y, z) \sec \gamma dx dy$$

**292. Integrals over a designated side of a surface.** 1. We may regard the piece of surface  $S$  of §§ 290, 291 as having two distinct sides.<sup>1</sup> At any point  $P$  of  $S$  there are two half normals  $PQ$  and  $PQ'$  one directed upward the other downward; we call them the normals to the upper and lower sides of  $S$ .

Let  $X, Y, Z$  denote the direction angles of normals to one side of  $S$ , and  $R(x, y, z)$  a function which is continuous on  $S$ , and consider the integral

$$\iint_S R \cos Z \, dS \quad (1)$$

For the normals to the upper side of  $S$ ,  $Z$  is the acute angle  $\gamma$  in § 291 (1), and if in that formula we replace  $F$  by  $R \cos Z$ , we find

$$\iint_S R \cos Z \, dS = \iint_D R \, dx \, dy \quad (2)$$

For normals to the lower side of  $S$ ,  $Z$  is  $\pi - \gamma$  and

$$\iint_S R \cos Z \, dS = - \iint_D R \, dx \, dy \quad (3)$$

In the first case the integral (1) is said to be taken over the upper side of  $S$ , in the second, over the lower side. In both cases it is customary to represent the integral (1) by the symbol  $\iint_S R \, dx \, dy$ . Hence, according as the integration is over the upper or the lower side of  $S$ , we have

$$\iint_S R \, dx \, dy = \iint_D R \, dx \, dy \quad \text{or} \quad - \iint_D R \, dx \, dy \quad (4)$$

The formulas (2) and (3) also hold good when  $\cos Z$  becomes 0 on the contour of  $S$ . For if  $S'$  denote a variable interior portion of  $S$ , and  $D'$  its projection on the  $xy$ -plane,  $\iint_{S'} R \cos Z \, dS = \pm \iint_{D'} R \, dx \, dy$ , and this gives (2) or (3) when  $S' \rightarrow S$ .

Evidently  $\iint_S R \cos Z \, dS$  is 0 when taken over a cylindrical surface parallel to  $Oz$ .

<sup>1</sup> By this it is meant that were  $S$  to be covered by a thin layer of some impenetrable substance, a point moving on one side of this layer could not pass to the other side without crossing the contour of  $S$ . One can form a surface which does not possess this property, and is therefore called one-sided, by twisting a rectangular piece of paper  $ABCD$  so as to be able to fasten the edges  $BC$  and  $AD$  together with  $B$  on  $D$  and  $C$  on  $A$ .

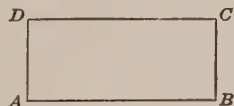


FIG. 133.

2. Let  $T$  denote any two-sided surface which can be divided into pieces  $S$  of the type just considered, or into such pieces and portions of cylindrical surfaces parallel to  $Oz$ . We define

$$\iint_T R \, dx \, dy = \iint_T R \cos Z \, dS \quad (5)$$

over one side of  $T$  as the sum of the integrals over the corresponding sides of the parts  $S$ .

We have analogous definitions of the surface integrals

$$\begin{aligned} \iint_T P \, dy \, dz &= \iint_T P \cos X \, dS \\ \iint_T Q \, dz \, dx &= \iint_T Q \cos Y \, dS \end{aligned} \quad (6)$$

where  $P$  and  $Q$  are continuous functions of  $x, y, z$  on  $T$ ; and with these three integrals, all taken over the same side of  $T$ , we form the general surface integral

$$\int_T P \, dy \, dz + Q \, dz \, dx + R \, dx \, dy \quad (7)$$

Observe that in (5), (6), (7),  $dS$  is positive and the signs of  $dx \, dy$ ,  $dy \, dz$ , and  $dz \, dx$  are those of  $\cos Z$ ,  $\cos X$ ,  $\cos Y$ .

**293. Green's formula for space.** 1. Let  $V$  denote a closed region bounded below and above by pieces of surface  $S_1$  and  $S_2$  of the type described in § 290, and laterally by a cylindrical surface  $S_3$  parallel to  $Oz$ . Call the entire surface  $S$ , and let  $X, Y, Z$  denote the direction angles of normals to its *outer* side.

Let  $R(x, y, z)$  be any function which, together with  $\partial R / \partial z$ , is continuous in  $V$  and on  $S$ . If  $z_2$  and  $z_1$  are the ordinates of the points,

$P_2$  and  $P_1$ , where a parallel to  $Oz$  meets  $S_2$  and  $S_1$ , we have

$$\begin{aligned} \iiint_V \frac{\partial R}{\partial z} \, dz \, dx \, dy &= \iint_D R(x, y, z_2) \, dx \, dy \\ &\quad - \iint_D R(x, y, z_1) \, dx \, dy \end{aligned} \quad (1)$$

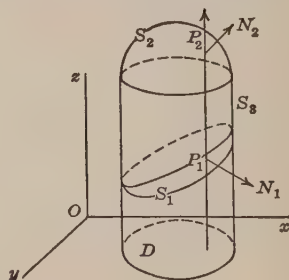


FIG. 134.

where  $D$  is the projection of  $S_2$  and  $S_1$  on the  $xy$ -plane. But since the outer normal to  $S$  is the upper normal along  $S_2$  and the lower normal along  $S_1$ , we have, § 292 (2), (3),

$$\begin{aligned}\iint_D R(x, y, z_2) dx dy &= \iint_{S_2} R \cos Z dS \\ \iint_D R(x, y, z_1) dx dy &= - \iint_{S_1} R \cos Z dS\end{aligned}$$

Hence the second member of (1) equals  $\iint_{S_1} R \cos Z dS + \iint_{S_2} R \cos Z dS$ , and, since  $\iint_{S_3} R \cos Z dS = 0$ , we have

$$\iiint_V \frac{\partial R}{\partial z} dx dy dz = \iint_S R \cos Z dS \quad (2)$$

2. The formula (2) is also true for any region  $V$  bounded by a closed surface  $S$  which can be divided into parts of the type just considered, — as can be shown by applying (2) to each of the parts and adding the resulting equations.

We establish analogous formulas for  $\iiint_V (\partial P / \partial x) dx dy dz$ ,  $\iiint_V (\partial Q / \partial y) dx dy dz$ , and adding them and (2), obtain

$$\begin{aligned}\iiint_V \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx dy dz \\ = \iint_S (P \cos X + Q \cos Y + R \cos Z) dS\end{aligned} \quad (3)$$

the surface integral being taken over the outer side of  $S$ . This is known as *Green's formula* for space.

By making the appropriate choices of  $P$ ,  $Q$ ,  $R$  in (3) we obtain the following formulas for the volume  $V$ :

$$V = \iint_S z dx dy = \iint_S x dy dz = \iint_S y dz dx \quad (4)$$

$$V = \frac{1}{3} \iint_S x dy dz + y dz dx + z dx dy \quad (5)$$

**294. Stokes's theorem.** Let  $S$  be a piece of surface of the type described in § 292 and let  $L$  be its contour; also let  $P$ ,  $Q$ ,  $R$  denote functions of  $x$ ,  $y$ ,  $z$  which, together with their

first partial derivatives, are continuous on  $S$  and  $L$ . We are to prove that

$$\int_L P dx + Q dy + R dz = \iint_S \left[ \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \right] \quad (1)$$

provided the following relation exists between the side of  $S$  and the sense along  $L$  for which the two integrals are taken: Take the  $x$ - and  $y$ -axes as in Fig. 135, so that the sense of rotation about  $Oz$  from  $Ox$  to  $Oy$  may be counter-clockwise, or positive. Suppose an observer to be standing on the side of  $S$  over which the integral  $\iint_S$  is taken; then  $\int_L$  is to be taken in the sense in which he must move along  $L$  to have  $S$  to his left.

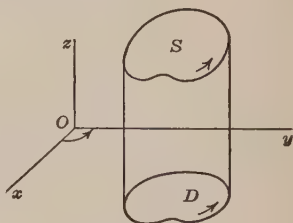


FIG. 135.

First consider the integral  $\int_L P dx$ .

Let  $D$  and  $C$  be the projections of  $S$  and  $L$  on the  $xy$ -plane and give the equation of  $S$  the form

$$z = f(x, y), \text{ also setting } p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

To fix the ideas, suppose  $\iint_S$  to be taken over the upper side of  $S$ . The corresponding sense along  $L$  and  $C$  is counter clockwise as in the figure.

Finally, represent  $P[x, y, f(x, y)]$  by  $F(x, y)$ .

We then have, by Green's theorem for the plane, § 287 (4),

$$\int_L P dx = \int_C F dx = - \iint_D \frac{\partial F}{\partial y} dy dx = - \iint_S \left[ \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} q \right] dx dy$$

But, if  $X, Y, Z$  be the direction angles of any normal to  $S$ , then

$$\cos X : \cos Y : \cos Z = -p : -q : 1, \text{ and } dx dy = \cos Z dS$$

$$\therefore q dx dy = - \frac{\cos Y}{\cos Z} \cos Z dS = - \cos Y dS = - dz dx$$

Hence

$$\int_L P dx = \iint_S \frac{\partial P}{\partial z} dz dx - \frac{\partial P}{\partial y} dx dy$$



Similarly, from the projections of  $S$  and  $L$  on the  $yz$ - and  $zx$ -planes, we get

$$\begin{aligned}\int_L Q \, dy &= \iint_S \frac{\partial Q}{\partial x} \, dx \, dy - \frac{\partial Q}{\partial z} \, dy \, dz \\ \int_L R \, dz &= \iint_S \frac{\partial R}{\partial y} \, dy \, dz - \frac{\partial R}{\partial x} \, dz \, dx\end{aligned}$$

Adding these three equations, we obtain (1). We suppose the projections of  $L$  on the  $xy$ -,  $yz$ -,  $zx$ -plane to be curves of the kind in § 286.

It is obvious from (1) that the value of the integral in the second member depends on  $P$ ,  $Q$ ,  $R$  and  $L$  only, not on  $S$ .

Suppose that throughout some region  $T$ , § 214, 2, we have

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \quad (2)$$

Then, by (1), the integral of

$$P \, dx + Q \, dy + R \, dz \quad (3)$$

along any closed curve  $L$  in  $T$  is 0. But from this it follows, as in § 288, that if  $A(x_0, y_0, z_0)$  and  $B(x, y, z)$  are points of  $T$ , the value of the integral of (3) along any path in  $T$  from  $A$  to  $B$  is independent of the path, and therefore, when  $A$  is fixed and  $B$  varies, is a one-valued function of the coordinates  $x, y, z$  of  $B$ . We call this function

$$F(x, y, z) = \int_{x_0, y_0, z_0}^{x, y, z} P \, dx + Q \, dy + R \, dz \quad (4)$$

It can be shown, as in § 288, that  $\partial F / \partial x = P$ ,  $\partial F / \partial y = Q$ ,  $\partial F / \partial z = R$ . Hence (3) is the differential of  $F$  and we have proved that (3) is an exact differential when the equations (2) are satisfied.

Furthermore if  $u(x, y, z)$  be any known function whose differential is (3), then, as in § 288,

$$F(x, y, z) = u(x, y, z) - u(x_0, y_0, z_0) \quad (5)$$

We can find such a function  $u(x, y, z)$  by the formula

$$u(x, y, z) = \int_{x_0}^x P(x, y_0, z_0) \, dx + \int_{y_0}^y Q(x, y, z_0) \, dy + \int_{z_0}^z R(x, y, z) \, dz \quad (6)$$

**295. Work.** Suppose that the point  $P(x, y, z)$  moves from  $A$  to  $B$  under the action of a force  $F$  which is a continuous function of  $x, y, z$  and therefore of  $s$ , the length of the path  $AP$ . If  $\theta$  denote the angle between the direction of  $F$  and the direction of the motion at any instant, then  $F \cos \theta$  is the component of  $F$  in the direction of the motion at that instant. Therefore, by an obvious extension of the reasoning in § 147, we are led to define the work done by  $F$  as

$$W = \int_{AB} F \cos \theta \, ds \quad (1)$$

The direction cosines of the motion at  $P$  are  $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ ; let those of  $F$  be  $l, m, n$ . Then

$$\cos \theta \, ds = \left[ l \frac{dx}{ds} + m \frac{dy}{ds} + n \frac{dz}{ds} \right] ds = l \, dx + m \, dy + n \, dz$$

and therefore 
$$W = \int_{AB} Fl \, dx + Fm \, dy + Fn \, dz \quad (2)$$

Observe that  $Fl, Fm, Fn$  are the  $x$ -,  $y$ -,  $z$ -components of  $F$  at  $P$ . If  $Fl, Fm, Fn$  satisfy the conditions of § 294 (2), the value of  $W$  is independent of the path from  $A$  to  $B$ ;  $Fl \, dx + Fm \, dy + Fn \, dz$  is the differential of some function  $\phi(x, y, z)$ ; and

$$W = \int_A^B d\phi = \phi(x, y, z)_B - \phi(x, y, z)_A \quad (3)$$

In this case the force  $F$  is called a *conservative force*.

In general,  $Fl \, dx + Fm \, dy + Fn \, dz$  is not an exact differential, and  $W$  depends on the path as well as on the positions of  $A$  and  $B$ .

### EXERCISE LIX

1. A particle  $P$  moves from  $A$  to  $B$  under the action of a force directed toward  $O$  and of the magnitude  $k/OP^2$ ; show that the work done is independent of the path.

2. Find the  $x$ -,  $y$ -,  $z$ -components of a force which would cause a particle  $P$  to move along the path  $x = t, y = t^2, z = t^3, t$  denoting time; also the work done between the times  $t = 1$  and  $t = 4$ .

3. Show that  $(yz\,dx - zx\,dy + xy\,dz)/y^2$  is an exact differential and find an integral by the formula § 294 (6).

4. Respecting the function  $u$  defined by § 294 (6), prove by aid of the theorem of § 283 that  $\partial u/\partial x = P$ ,  $\partial u/\partial y = Q$ ,  $\partial u/\partial z = R$ , if the conditions of § 294 (2) are satisfied.

5. Find the following surface integrals by means of the equivalent volume integrals:

1.  $\iiint xz\,dy\,dz + zy\,dz\,dx + z^2\,dx\,dy$  over the surface of one of the solids bounded by the sphere  $x^2 + y^2 + z^2 = a^2$  and the planes  $x = 0$ ,  $y = 0$ ,  $z = 0$ .

2.  $\iiint xy\,dy\,dz + yz\,dz\,dx + xz\,dx\,dy$  over the surface of one of the solids bounded by the cylindrical surface  $x^2 + y^2 = a^2$  and the planes  $x = 0$ ,  $y = 0$ ,  $z = 0$ ,  $z = b$ .

6. Let  $T$  be a region bounded by two closed surfaces  $S_1$  and  $S_2$ ,  $S_2$  lying entirely within  $S_1$ . Show that Green's formula, § 293 (3), holds good for  $T$  on the understanding that the integral  $\iint_S$  is to be taken over the sides of  $S_1$  and  $S_2$  which are outward with respect to  $T$ .

7. Let  $T$  be a region, § 214, 2., in which  $P$ ,  $Q$ ,  $R$ , and their first partial derivatives are continuous. Show that the sufficient and necessary condition that

$$\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy \quad (1)$$

be 0 for every closed surface  $S$  in  $T$  is that throughout  $T$

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0 \quad (2)$$

8. Let  $P$  be any point of a closed surface  $S$ , and  $(r, n)$  the angle which  $OP$  produced makes with the outward normal to  $S$  at  $P$ . Show that the volume  $V$  bounded by  $S$  is given by the formula

$$V = (1/3) \iint_S r \cos(r, n) dS.$$

9. Let  $S$  be a closed surface,  $M$  a given point, and  $(r, n)$  the angle which the outward normal at any point  $P$  of  $S$  makes with  $MP$  ( $= r$ ) produced; and let

$$I = \iint_S \frac{\cos(r, n)}{r^2} dS$$

Show that when  $S$  is a sphere and  $M$  its center,  $I = 4\pi$ ; when  $S$  is a sphere and  $M$  a point on  $S$ , then  $I = 2\pi$ ; when  $S$  is any closed surface,  $I$  is  $4\pi$ ,  $2\pi$ , or 0 according as  $M$  is within, on, or outside of  $S$ .

**10.** Show that if  $S$  denote any piece of surface having a given contour  $L$ , as in § 294, the equation (2) of Ex. 7 is the condition that the value of (1) of Ex. 7 shall depend on  $L$  only, not on  $S$ .

To prove that (2) is the sufficient condition, show that if it be satisfied we can find functions  $A, B, C$  such that

$$P = \frac{\partial C}{\partial y} - \frac{\partial B}{\partial z} \quad Q = \frac{\partial A}{\partial z} - \frac{\partial C}{\partial x} \quad R = \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \quad (3)$$

and so reduce (1) to the line integral  $\int_L A dx + B dy + C dz$  by Stokes's formula. The equations (3) imply (2); but if (2) be satisfied we can assign  $C$ , say, arbitrarily and then obtain  $A$  and  $B$  by simple integration.

To prove that (2) is the necessary condition, take any closed surface  $S'$  made up of parts  $S_1, S_2$  of which  $L$  is the common contour. The sum of the integrals (1) taken over the outer and inner sides of the part  $S_1$  is 0, and if the value of (1) is the same for every  $S$  having the contour  $L$ , it is the same for the inner side of  $S_1$  as for the outer side of  $S_2$ ; hence the value of (1) taken over the outer side of  $S'$  is 0, and this implies (2).

**11.** Show that if  $S$  be any piece of surface such as in Ex. 10, then

$$\iint_S x dy dz + y dz dx - 2z dx dy = \int_L yz dx + 3xz dy + 2xy dz$$

**296. Parametric representation of surfaces.** 1. A surface  $S$  may be represented by a set of equations of the form

$$x = \phi_1(u, v) \quad y = \phi_2(u, v) \quad z = \phi_3(u, v) \quad (1)$$

where  $u, v$  denote independent variables or parameters. It is supposed that the functions  $\phi_1, \phi_2, \phi_3$  and their partial derivatives of the first order are continuous, and that none of the functional determinants

$$A = \frac{\partial(y, z)}{\partial(u, v)} \quad B = \frac{\partial(z, x)}{\partial(u, v)} \quad C = \frac{\partial(x, y)}{\partial(u, v)} \quad (2)$$

vanishes identically. At an ordinary point  $P$  of  $S$  at least one of  $A, B, C$  is not 0. If  $C \neq 0$ , the first and second of the equations (1) can be solved at  $P$  for  $u, v$  in terms of  $x, y$ , § 230, and when the results are substituted in the third equation it takes the form  $z = f(x, y)$ . Points, if any, where  $A, B, C$  are all 0 are called *singular points* of  $S$ .

2. Any given pair of values  $u, v$ , as  $u = a, v = b$ , determines a single point  $P$  of  $S$ . Hence  $u, v$  are called the *curvilinear coordinates* of the points of  $S$ .

Any equation of the form  $F(u, v) = 0$  represents a curve on  $S$ . In particular,  $u = a$  represents a curve along which  $v$  alone varies and whose parametric equations are  $x = \phi_1(a, v), y = \phi_2(a, v), z = \phi_3(a, v)$ , the direction ratios of its tangent therefore being  $\partial x / \partial v : \partial y / \partial v : \partial z / \partial v$ . The like is true of the equation  $v = b$ .

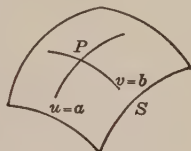


FIG. 136.

3. Let  $P(x_1, y_1, z_1)$  be the point whose curvilinear coordinates are  $u = a, v = b$ , and let  $A_1, B_1, C_1$  denote the values of  $A, B, C$  at  $P$ . Then if  $P$  is not a singular point of  $S$ , the equation of the tangent plane to  $S$  at  $P$  is

$$A_1(x - x_1) + B_1(y - y_1) + C_1(z - z_1) = 0 \quad (3)$$

For the normal at  $P$  is perpendicular to both of the curves  $u = a$  and  $v = b$  at  $P$ ; hence if its direction cosines are  $l, m, n$ , we have at  $P$

$$l \frac{\partial x}{\partial v} + m \frac{\partial y}{\partial v} + n \frac{\partial z}{\partial v} = 0, \quad l \frac{\partial x}{\partial u} + m \frac{\partial y}{\partial u} + n \frac{\partial z}{\partial u} = 0$$

and therefore  $l : m : n = A_1 : B_1 : C_1$  (4)

4. If in the formula for the differential of curve arc, namely  $ds^2 = dx^2 + dy^2 + dz^2$ , we substitute the values of  $dx, dy, dz$  got from (1), we obtain for a curve on  $S$ ,

$$ds^2 = E du^2 + 2F du dv + G dv^2 \quad (5)$$

where  $E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$

$$F = \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right) + \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial y}{\partial v}\right) + \left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial z}{\partial v}\right)$$

$$G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$$

When the curves  $u = \text{const.}, v = \text{const.}$  are orthogonal, then  $F = 0$ , § 212, and (5) becomes  $ds^2 = E du^2 + G dv^2$ .

5. Let  $S'$  denote a portion of  $S$  on which the normal is nowhere perpendicular to  $Oz$ . By § 290, we have

$$\text{area } S' = \iint_{D'} \frac{dx dy}{\cos \gamma} \quad (6)$$

where  $D'$  denotes the projection of  $S'$  on the  $xy$ -plane, and  $\gamma$  the acute angle which a normal to  $S'$  makes with  $Oz$ . We are to prove that the substitution  $x = \phi_1(u, v)$ ,  $y = \phi_2(u, v)$  of (1) transforms (6) into

$$\begin{aligned} \text{area } S' &= \iint_D (A^2 + B^2 + C^2)^{1/2} du dv \\ &= \iint_D (EG - F^2)^{1/2} du dv \end{aligned} \quad (7)$$

where  $D$  denotes the region (supposed to exist) in the  $uv$ -plane to which the region  $D'$  corresponds in the sense explained in § 289.

For by (3), and because  $\cos \gamma$  is positive throughout  $D$ , we have

$$\cos \gamma = |C| \div (A^2 + B^2 + C^2)^{1/2}$$

Again  $C$  is the functional determinant  $J$  in the formula § 289 (3), and by hypothesis  $\cos \gamma$  and therefore  $C$  does not vanish in  $D$ . Hence by this formula § 289 (3) we have immediately

$$\iint_{D'} \frac{dx dy}{\cos \gamma} = \iint_D (A^2 + B^2 + C^2)^{1/2} du dv$$

And, by using the expanded forms of  $A$ ,  $B$ ,  $C$  and the expressions for  $E$ ,  $F$ ,  $G$  given above, it is easily shown that

$$A^2 + B^2 + C^2 \equiv EG - F^2$$

When the curves  $u = \text{const.}$ ,  $v = \text{const.}$  are orthogonal, then

$$\text{area } S' = \iint_D \sqrt{EG} du dv$$

### EXERCISE LX

1. Let  $P(x, y, z)$  be any point on the sphere  $x^2 + y^2 + z^2 = a^2$ . And let  $CPD$  be the great circle through  $P$  and the pole  $C$ . If  $u = POC$  and  $v = AOD$ , show that

$$\begin{aligned} x &= a \sin u \cos v & y &= a \sin u \sin v \\ z &= a \cos u \end{aligned} \quad (1)$$

The coordinate curves  $u = c_1$  and  $v = c_2$  through  $P$ , namely  $FPG$  and  $CPD$ , are orthogonal.

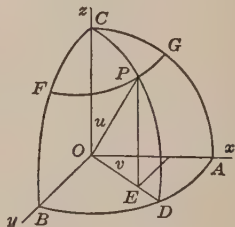


FIG. 137.

2. Show that for the sphere, Ex. 1., the formulas § 296 (5), (7) are

$$ds^2 = a^2(du^2 + \sin^2 u dv^2) \quad (2) \quad dS = a^2 \sin u du dv \quad (3)$$

3. Show, by Ex. 2 (2), that the angle  $\theta$  which a curve on the sphere through  $P$  makes with the curve  $u = c$  through  $P$  is given by  $\tan \theta = du/\sin u dv$ ; and therefore that the equation of a curve which cuts all the curves  $u = c$  at the angle whose tangent is  $k$  is of the form

$$kv = \log \tan (u/2) + C \quad (\S 119, \text{Example})$$

4. Find the area of the portion of the sphere bounded by the curves  $u = \pi/4$ ,  $u = \pi/3$ ,  $v = 0$ ,  $v = \pi/6$ ; also that of the portion bounded by  $v = 0$ ,  $v = u$ ,  $u = 0$ ,  $u = \frac{\pi}{2}$ .

5. Show that  $x = u \cos v$ ,  $y = u \sin v$ ,  $z = ku$  are the parametric equations of a cone. Interpret  $u$ ,  $v$ ,  $k$  and find  $ds$  and  $dS$  for this cone.

6. Show that the equations  $x = u$ ,  $y = f(u) \cos v$ ,  $z = f(u) \sin v$  represent a surface of revolution. Find the formula for its area between  $x = a$  and  $x = b$ , and reduce this formula to the form § 144 (1).

7. Find the equation of the tangent plane at any point of the surface  $x = u$ ,  $y = v$ ,  $z = 1/uv$ .

8. Using the notation of § 296 (2), show that if  $x = \phi_1(u, v)$ ,  $y = \phi_2(u, v)$ ,  $z = \phi_3(u, v)$  represent the surface  $f(x, y, z) = 0$ , then

$$f_x : f_y : f_z = A : B : C$$

9. Let  $\theta$  be the angle between two curves on the surface  $x = \phi_1$ ,  $y = \phi_2$ ,  $z = \phi_3$  which meet at the point  $(u, v)$  and there have the directions given by  $du_1 : dv_1$  and  $du_2 : dv_2$ . Show that

$$\cos \theta = \frac{E du_1 du_2 + F(du_1 dv_2 + du_2 dv_1) + G dv_1 dv_2}{ds_1 \cdot ds_2}$$

10. Show that if  $\omega$  be the angle between the coordinate curves  $u = c_1$ ,  $v = c_2$ , then

$$\cos \omega = F/\sqrt{EG} \quad \therefore \sin \omega = \sqrt{EG - F^2}/\sqrt{EG} \quad \therefore dS = ds_u ds_v \sin \omega$$

which is the area of the parallelogram with sides of lengths  $ds_u$ ,  $ds_v$  tangent to the curves  $u = c_1$ ,  $v = c_2$  at the point  $(c_1, c_2)$ .

**297. Transformation of surface integrals.** Suppose that the formulas

$$x = \phi_1(x', y', z') \quad y = \phi_2(x', y', z') \quad z = \phi_3(x', y', z')$$

make the points of a given piece of surface  $S$  in the  $xyz$ -space correspond one to one to the points of a piece of surface  $S'$



in the  $x'y'z'$ -space. And let  $P(x, y, z)$  and its partial derivatives of the first order be continuous on  $S$ . What integral  $\iint_{S'}$  corresponds to the integral  $\iint_S P \, dx \, dy$ ?

We are to prove that

$$\iint_S P \, dx \, dy = \iint_{S'} P \left[ \frac{\partial(x, y)}{\partial(y', z')} dy' \, dz' + \frac{\partial(x, y)}{\partial(z', x')} dz' \, dx' + \frac{\partial(x, y)}{\partial(x', y')} dx' \, dy' \right] \quad (1)$$

We shall assume that we may regard the coordinates  $x, y, z$  and  $x', y', z'$  of corresponding points of  $S$  and  $S'$  as functions of two independent variables or parameters  $u, v$  in a region  $D$  of the  $uv$  plane, one pair of these points of  $S$  and  $S'$  corresponding to each point of  $D$  and vice versa. Let

$$A = \frac{\partial(y, z)}{\partial(u, v)}, \quad B = \frac{\partial(z, x)}{\partial(u, v)}, \quad C = \frac{\partial(x, y)}{\partial(u, v)}, \quad K = (A^2 + B^2 + C^2)^{1/2}$$

$$A' = \frac{\partial(y', z')}{\partial(u, v)}, \quad B' = \frac{\partial(z', x')}{\partial(u, v)}, \quad C' = \frac{\partial(x', y')}{\partial(u, v)}, \quad K' = (A'^2 + B'^2 + C'^2)^{1/2}$$

Then if  $X, Y, Z$  denote the direction angles of the normals to one side of  $S$ , and  $X', Y', Z'$  those of the normals to one side of  $S'$ , we have

$$A = \pm K \cos X, \quad B = \pm K \cos Y, \quad C = \pm K \cos Z$$

$$A' = \pm K' \cos X', \quad B' = \pm K' \cos Y', \quad C' = \pm K' \cos Z' \quad (2)$$

the  $\pm$  signs in each row being the same. By the formula § 285 (7), we have

$$C = \frac{\partial(x, y)}{\partial(y', z')} A' + \frac{\partial(x, y)}{\partial(z', x')} B' + \frac{\partial(x, y)}{\partial(x', y')} C'$$

Therefore, by (2),

$$K \cos Z = \pm K' \left[ \frac{\partial(x, y)}{\partial(y', z')} \cos X' + \frac{\partial(x, y)}{\partial(z', x')} \cos Y' + \frac{\partial(x, y)}{\partial(x', y')} \cos Z' \right]$$

and we can make the  $\pm$  sign  $+$  by properly choosing the side of  $S'$  to which the normals are to be taken.

Multiply both members of this equation by  $P \, du \, dv$  and integrate over  $D$ . Then since  $K \, du \, dv = dS$  and  $K' \, du \, dv = dS'$  by § 296 (7), we have

$$\iint_S P \cos Z \, dS$$

$$= \iint_{S'} P \left[ \frac{\partial(x, y)}{\partial(y', z')} \cos X' + \frac{\partial(x, y)}{\partial(z', x')} \cos Y' + \frac{\partial(x, y)}{\partial(x', y')} \cos Z' \right] dS'$$

and, by § 292, this equation can be written in the form (1).

**298. Transformation of triple integrals.** Suppose that the formulas

$$x = \phi_1(x', y', z') \quad y = \phi_2(x', y', z') \quad z = \phi_3(x', y', z') \quad (1)$$

make the points of the region  $V$  bounded by the closed surface  $S$  in the  $xyz$ -space correspond one to one to the points of the region  $V'$  bounded by the closed surface  $S'$  in the  $x'y'z'$ -space; also that in  $V'$  and on  $S'$ , the functions  $\phi_1, \phi_2, \phi_3$  and their partial derivatives of the first and second orders are continuous, and the functional determinant

$$J = \frac{\partial(x', y', z')}{\partial(x, y, z)}$$

different from 0, and therefore of constant sign. Then, if  $f(x, y, z)$  denote any function which is continuous in  $V$  and on  $S$ ,

$$\iiint_V f(x, y, z) dx dy dz = \iiint_{V'} f(\phi_1, \phi_2, \phi_3) |J| dx' dy' dz' \quad (2)$$

For by § 293 (4) we have  $V = \iiint_S z dx dy$ , and by § 297 (1) we can reduce  $\iiint_S z dx dy$  to the form

$$\pm \iint_{S'} z \left[ \frac{\partial(x, y)}{\partial(y', z')} dy' dz' + \frac{\partial(x, y)}{\partial(z', x')} dz' dx' + \frac{\partial(x, y)}{\partial(x', y')} dx' dy' \right]$$

the two integrals being taken over the outer sides of  $S$  and  $S'$  respectively.

Let us transform the second integral into a triple integral by aid of Green's formula. If in the formula § 293 (3) we set

$$P = z \frac{\partial(x, y)}{\partial(y', z')} \quad Q = z \frac{\partial(x, y)}{\partial(z', x')} \quad R = z \frac{\partial(x, y)}{\partial(x', y')}$$

we obtain, on carrying out the indicated reckoning,

$$\frac{\partial P}{\partial x'} + \frac{\partial Q}{\partial y'} + \frac{\partial R}{\partial z'} = \frac{\partial(x, y, z)}{\partial(x', y', z')} = J$$

We therefore have

$$V = \pm \iiint_{V'} J dx' dy' dz' = \iiint_{V'} |J| dx' dy' dz' \quad (3)$$

the third member following from the second because  $V$  is positive.

The formula (2) follows from (3) by an obvious extension of the argument given in § 289, 2.

EXAMPLE. The formulas for transformation to polar coordinates in space are

$$x = r \sin \phi \cos \theta \qquad y = r \sin \phi \sin \theta \qquad z = r \cos \phi$$

where, in Fig. 137,  $r = OP$ ,  $\phi = POC$ , and  $\theta = AOD$ . Show that

$$\iiint_V f(x, y, z) \, dx \, dy \, dz = \iiint_{V'} f(x, y, z) r^2 \sin \phi \, dr \, d\theta \, d\phi$$

## XXVII. DOUBLE SERIES. IMPLICIT FUNCTIONS

**299. Double series.** It can be proved of any finite number of infinite series,  $\Sigma a_n, \Sigma b_n, \dots$ , as was done for two such series in § 191, 1, that if they converge to the sums  $A, B, \dots$ , the series got by adding their corresponding terms, namely  $\Sigma(a_n + b_n + \dots)$ , will converge to the sum  $A + B + \dots$ . The following theorem relates to the case in which the number of the given series is infinite, the resulting series  $\Sigma(a_n + b_n + \dots)$  then being called a *double series*.

**Theorem.** *Let  $U_1 + U_2 + \dots$  be a convergent series whose terms are themselves sums of absolutely convergent series:*

$$U_1 = u_1^{(1)} + u_2^{(1)} + \dots, \quad U_2 = u_1^{(2)} + u_2^{(2)} + \dots, \quad \dots \quad (1)$$

Also let

$$U'_1 = |u_1^{(1)}| + |u_2^{(1)}| + \dots, \quad U'_2 = |u_1^{(2)}| + |u_2^{(2)}| + \dots, \quad \dots$$

*If the series  $U'_1 + U'_2 + \dots$  converges, then the several series obtained by adding the corresponding terms of the series (1), namely*

$$u_1^{(1)} + u_1^{(2)} + u_1^{(3)} + \dots, \quad u_2^{(1)} + u_2^{(2)} + u_2^{(3)} + \dots, \quad \dots \quad (2)$$

*will also converge, and if their sums be  $V_1, V_2, \dots$ , we shall have*

$$U_1 + U_2 + U_3 + \dots = V_1 + V_2 + V_3 + \dots \quad (3)$$

For let the remainders after  $n$  terms in the series (1) be  $R_n^{(1)}, R_n^{(2)}, \dots$ ; then

$$U_1 = u_1^{(1)} + u_2^{(1)} + \dots + u_n^{(1)} + R_n^{(1)}$$

$$U_2 = u_1^{(2)} + u_2^{(2)} + \dots + u_n^{(2)} + R_n^{(2)}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$U_k = u_1^{(k)} + u_2^{(k)} + \dots + u_n^{(k)} + R_n^{(k)}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Each of the column series, as  $u_1^{(1)} + u_1^{(2)} + \dots$ , is convergent since each of its terms is numerically less than the corresponding term of the convergent positive series  $U'_1 + U'_2 + \dots$ , §§ 168, 176. Let the sums of these series be  $V_1, V_2, \dots, V_n, R_n$ .

If we add the corresponding terms of these  $n + 1$  column series, we obtain the original series  $U_1 + U_2 + \dots$ . Therefore

$$U_1 + U_2 + \dots + U_k + \dots = V_1 + V_2 + \dots + V_n + R_n$$

Hence the theorem will be proved if it can be shown that  $\lim_{n \rightarrow \infty} R_n = 0$ .

Let the remainder after the  $k$ th term in  $R_n^{(1)} + R_n^{(2)} + \dots$  be  $T_n^{(k)}$ ; then

$$R_n = R_n^{(1)} + R_n^{(2)} + \dots + R_n^{(k)} + T_n^{(k)}$$

But if any positive number  $\epsilon$  be assigned, we can first find a value  $k'$  of  $k$  such that  $|T_n^{(k)}| < \epsilon/2$  when  $k \geq k'$ , whatever the value of  $n$  may be, and then find a value  $n'$  of  $n$  such that  $|R_n^{(1)}|, |R_n^{(2)}|, \dots, |R_n^{(k)}|$  are each  $< \epsilon/2$  when  $n \geq n'$ . We shall then have  $|R_n| < \epsilon$ . And since  $\epsilon$  may be taken as small as we please,  $\lim_{n \rightarrow \infty} R_n = 0$ .

**300. Operations with power series.** The theorem of § 299 gives us a means of proving the theorems respecting power series stated in § 191.

**1. Multiplication.** If  $f(x) = \Sigma a_n x^n$  and  $\phi(x) = \Sigma b_n x^n$ , for  $|x| < l$ , then also, for  $|x| < l$ , the product  $f(x) \phi(x)$  is the sum of the series whose terms are

$$U_1 = f(x)b_0 = a_0b_0 + a_1b_0x + a_2b_0x^2 + \dots \quad (1)$$

$$U_2 = f(x)b_1x = 0 + a_0b_1x + a_1b_1x^2 + \dots \quad (2)$$

$$U_3 = f(x)b_2x^2 = 0 + 0 + a_0b_2x^2 + \dots \quad (3)$$

. . . . .

Since  $\Sigma |a_n x^n|$  converges, the series obtained by replacing the terms of (1), (2),  $\dots$  by their absolute values converge to sums  $U'_1, U'_2, \dots$ ; and  $U'_1 + U'_2 + \dots$  converges since  $\Sigma |b_n x^n|$  converges. Hence, § 299, the product  $f(x) \phi(x)$  is also the sum of the series got by adding the terms of (1), (2),  $\dots$  by columns: that is

$$f(x) \phi(x) = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \dots, |x| < l$$

**2. Substitution.** Let  $z = \Sigma a_n y^n$ , and let  $y = \Sigma b_n x^n$ , and suppose that the limits of convergence of the two series are  $l$  and  $r$  respectively.

If  $|b_0| < l$ , we can find<sup>1</sup> a positive number  $r' (\leq r)$ , such that  $\Sigma |b_n x^n| < l$  when  $|x| < r'$ .

It being supposed that  $|x| < r'$ , replace  $y, y^2, \dots$  in  $\Sigma a_n y^n$  by the series  $\Sigma b_n x^n$  and the series derived from it by repeated applications of the rule in 1. The resulting double series satisfies the conditions of § 299, and if in it we group and add terms involving like powers of  $x$ , we obtain a power series in  $x$  which converges and represents  $z$ , at least when  $|x| < r'$ .

In case  $l = \infty$  we have  $\Sigma |b_n x^n| < l$  when  $|x| < r$ . Hence the substitution under consideration is valid for all finite values of  $x$  when  $l = \infty$  and  $r = \infty$ .

**3. Division.** Consider  $(a_0 + a_1 x + \dots)/(b_0 + b_1 x + \dots)$ , a quotient of two power series which have limits of convergence  $> 0$  and in which  $b_0 \neq 0$ . We are to prove that a power series  $C_0 + C_1 x + \dots$  exists which has a limit of convergence  $> 0$  and which when convergent equals this quotient.

For a positive number  $c$  can be found such that when  $|x| < c$  the series  $a_0 + a_1 x + \dots$  converges and also  $|b_1 x| + |b_2 x^2| + \dots < |b_0|$ .

$$\text{Let} \quad y = b_1 x + b_2 x^2 + \dots \quad (1)$$

and suppose  $|x| < c$ , and therefore  $|y| < |b_0|$ . We then have

$$\frac{1}{b_0 + b_1 x + \dots} = \frac{1}{b_0 + y} = \frac{1}{b_0} \frac{1}{1 + y/b_0} = \frac{1}{b_0} - \frac{y}{(b_0)^2} + \frac{y^2}{(b_0)^3} - \dots \quad (2)$$

Transform the series (2) into a power series in  $x$  by the substitution (1) and then multiply this  $x$ -series by the series  $a_0 + a_1 x + \dots$ . The resulting  $x$ -series will converge and represent the given quotient, at least when  $|x| < c$ .

**301. Implicit functions.** 1. By § 225 an algebraic equation of the form

$$0 = ax + by + cx^2 + exy + fy^2 + \dots \quad b \neq 0 \quad (1)$$

<sup>1</sup> For let  $c$  be any positive number  $< r$ . Then since  $\Sigma |b_n c^n|$  converges, each of its terms is less than some finite positive number  $M < l$ . Hence if  $|x| < c$ , we have

$$|b_n x^n| = |b_n c^n| \left| \frac{x}{c} \right|^n < M \left| \frac{x}{c} \right|^n$$

$$\therefore \quad \Sigma |b_n x^n| < M \left[ 1 + \left| \frac{x}{c} \right| + \left| \frac{x}{c} \right|^2 + \dots \right] = \frac{M}{1 - |x|/c}$$

$$\text{and} \quad \frac{M}{1 - |x|/c} < l \text{ when } |x| < c \left( 1 - \frac{M}{l} \right) = r'$$

has a solution  $y = \phi(x)$  at the origin. We are to prove that  $\phi(x)$  can be expressed as a power series in  $x$  with a limit of convergence  $l > 0$ .

By transposing the  $y$  term we can reduce (1) to the form:

$$y = (a_{10}x + a_{20}x^2 + \cdots) + y(a_{11}x + a_{21}x^2 + \cdots) + y^2(a_{02} + a_{12}x + a_{22}x^2 + \cdots) + \cdots \quad (2)$$

$$\text{In (2) set } y = C_1x + C_2x^2 + C_3x^3 + \cdots \quad (3)$$

and, after reducing the second member to a power series in  $x$ , equate coefficients of like powers of  $x$  in the two members. We thus get

$$C_1 = a_{10}, \quad C_2 = a_{20} + C_1a_{11} + C_1^2a_{02}, \quad \cdots$$

Each of these equations after the first expresses one of the  $C$ 's as a polynomial in certain of the  $a_{ik}$ 's and the preceding  $C$ 's. Therefore when solved they give expressions for the  $C$ 's as polynomials in the  $a_{ik}$ 's. Thus

$$C_1 = a_{10}, \quad C_2 = a_{20} + a_{10}a_{11} + a_{10}^2a_{02}, \quad \cdots \quad (4)$$

The series (3) with the coefficients (4) satisfies (2) formally. It is an *actual* solution of (2) if, as has been assumed in deriving it, it has a limit of convergence  $l > 0$ . To prove that it has, let  $\lambda$  denote the value of the greatest  $|a_{ik}|$ , and form the equation

$$y = \lambda(x + x^2 + \cdots) + \lambda y(x + x^2 + \cdots) + \lambda y^2(1 + x + x^2 + \cdots) + \cdots \quad (5)$$

the right member being an infinite series in  $y$  with coefficients which are infinite series in  $x$  of the forms indicated.

Proceeding as in the case of (2), we can find a series

$$y = C'_1x + C'_2x^2 + C'_3x^3 + \cdots \quad (6)$$

which satisfies (5) formally. Moreover, since  $\lambda \geq |a_{ik}|$ , we shall have, by (4),  $C'_1 \geq |C_1|$ ,  $C'_2 \geq |C_2|$ , and so on. Hence if (6) has a limit of convergence  $l > 0$ , the same must be true of (3).



But if we add  $\lambda + \lambda y$  to both members of (5) and suppose  $|x|, |y| < 1$ , we have

$$\begin{aligned} (\lambda + 1)y + \lambda &= \lambda/(1 - x)(1 - y) \\ \therefore (\lambda + 1)y^2 - y + \lambda x/(1 - x) &= 0 \end{aligned} \quad (7)$$

Solving (7) for the root  $y$  which vanishes with  $x$ , we get

$$y = \{1 - [1 - (2\lambda + 1)^2 x]^{1/2}(1 - x)^{-1/2}\} \div 2(\lambda + 1) \quad (8)$$

By the binomial theorem and § 300, 1., we can expand the right member of (8) in a power series in  $x$  which converges when  $(2\lambda + 1)^2 |x| < 1$ , or

$$|x| < 1/(2\lambda + 1)^2 \quad (9)$$

And (9) and (8) imply that  $|x|, |y| < 1$ . The series thus obtained is identical with (6), by § 186. Hence (6) and therefore (3) has a limit of convergence  $l > 0$ , as was to be proved.

**EXAMPLE.** Solve  $y^2 - y + x = 0$  for  $y$  in terms of  $x$  at 0.

In  $y = x + y^2$  substitute  $y = C_1x + C_2x^2 + C_3x^3 + \dots$

We get  $C_1x + C_2x^2 + C_3x^3 + \dots = x + x^2(C_1 + C_2x + C_3x^2 + \dots)^2$

Hence

$C_1 = 1, C_2 = C_1^2 = 1, C_3 = 2C_1C_2 = 2, C_4 = C_2^2 + 2C_1C_3 = 5, \dots$

Therefore  $y = x + x^2 + 2x^3 + 5x^4 + \dots, |x| < 1/(2 + 1)^2$ .

**302. Method of successive approximations.** We may write the equation § 301 (2) in the form

$$y = cx + u_2 + u_3 + u_4 + \dots \quad (1)$$

where  $c = a_{10}$ , and  $u_2, u_3, \dots$  denote the groups of terms of the second, third,  $\dots$  degrees in  $x, y$ .

It has been proved that the equation (1) has a solution of the form :

$$y = cx + C_2x^2 + C_3x^3 + C_4x^4 + \dots \quad (2)$$

We call  $y = cx, y = cx + C_2x^2, \dots$  the *first, second,  $\dots$  approximations* to (2).

Since (2) satisfies (1) identically, the series  $cx + C_2x^2 + C_3x^3 + \dots$  must be identical with the  $x$ -series  $S$  got by substituting (2) in the second member of (1), expanding, and collecting terms in like powers of  $x$ .

But the only  $x^2$  terms in  $S$  are those got by setting  $y = cx$  in  $u_2$ . Hence we can get the second approximation  $y = cx + C_2x^2$  directly from (1) by discarding  $u_3, u_4, \dots$  and substituting the first approximation  $y = cx$  in  $u_2$ .

In like manner, the second approximation having been found, we can get the third by the process: In (1) discard  $u_4, \dots$ , substitute the first approximation in  $u_3$ , and the second in  $u_2$ , ignoring the  $x^4$  term in the result. And so on.

This is the most expeditious method of finding the solution (2).

EXAMPLE 1. Solve  $y = 2x + y^2 + y^3$  for  $y$  in terms of  $x$  at  $O$ .

1st approx'n.  $y = 2x$

2nd approx'n.  $y = 2x + (2x)^2 = 2x + 4x^2$

3rd approx'n.  $y = 2x + (2x + 4x^2)^2 + (2x)^3 = 2x + 4x^2 + 24x^3$

4th approx'n.  $y = 2x + (2x + 4x^2 + 24x^3)^2 + (2x + 4x^2)^3$   
 $= 2x + 4x^2 + 24x^3 + 160x^4$

EXAMPLE 2. Find the shape of the graph  $K$  of  $y = x + 2y^2 + x^2$  near  $O$ .

The first approximation  $y = x$  represents the tangent  $OT$ .

The second approximation is  $y = x + 3x^2$ . When  $x = OD$  is very small the term  $3x^2$  is approximately equal to  $QP$ . Therefore, since  $3x^2$  is always  $+$ ,  $K$  lies above  $OT$  near  $O$ .

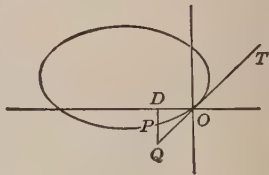


FIG. 138.

EXAMPLE 3. Find the fourth approximation to the solution of the equation  $0 = x - 2y + xy + y^3 + x^4$  for  $y$  at  $O$ .

EXAMPLE 4. Find the shape of the graph near  $O$  of the equation  $x + 2y - xy + y^2 - x^3 = 0$ .

**303. Newton's Diagram.** Let  $f(x, y) = 0$  denote any irreducible algebraic equation in  $x, y$  which lacks the constant term. Let us inquire whether in every case, that is, even when the  $y$  term is missing,  $f(x, y) = 0$  has a solution of the

form  $y = Cx^\mu + \dots$ , the series being one in increasing positive powers of  $x$ , integral or fractional.

Evidently if  $y = Cx^\mu + \dots$  is to satisfy  $f(x, y) = 0$  identically, it is necessary first of all that  $\mu$  be such as to make the degrees of two or more of the terms of  $f(x, Cx^\mu)$  equal and less than the degrees of the remaining terms, and then that  $C$  be such as to make the sum of these terms of lowest degree 0.

It is therefore required to find all substitutions  $y = x^\mu$  which will make the degrees in  $x$  of two or more of the terms of  $f(x, y)$  equal and less than the degrees of the remaining terms. We proceed as follows:

Apart from numerical coefficients, the terms of  $f(x, y)$  are of the type  $x^k y^l$ , and when  $y = x^\mu$  the degree of  $x^k y^l$  in  $x$  is  $k + l\mu$ .

1. Take rectangular axes  $OX, OY$ , and for any term  $x^k y^l$  plot the point  $P$  whose  $X, Y$  coordinates are the exponents  $k, l$ ; we call  $P(k, l)$  the *degree point* of the term  $x^k y^l$ . Next, supposing  $y = x^\mu$  and that  $\mu$  is known, take  $OM = \mu$  and  $OI = 1$ , and through  $P$  take  $AB$  parallel to  $MI$ .

The equation of  $AB$  is

$$(X - k)/\mu + (Y - l)/1 = 0$$

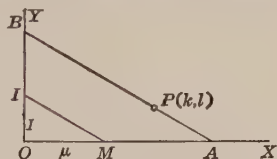


FIG. 139.

Hence the  $X$ -intercept  $OA$  is  $k + l\mu$ , the degree of  $x^k y^l$  in  $x$ , and the  $Y$ -intercept  $OB$  is  $k/\mu + l$ , the degree of  $x^k y^l$  in  $y$ .

2. Plot the degree points  $P, Q, \dots$  of all the terms of  $f(x, y)$ . Then find every line  $L$  which will pass through two or more of these points and have the rest to the side remote from  $O$ . Let  $x^k y^l$  and  $x^{k'} y^{l'}$  be two of the terms whose points are on such a line  $L$ , and determine  $\mu$  so that their degrees in  $x$  when  $y = x^\mu$ , are equal: which makes  $k + l\mu = k' + l'\mu$ , and therefore  $\mu = (k' - k)/(l - l')$ . Then, as follows from 1., all the terms whose points are on  $L$  will be of the same degree  $k + l\mu$  in  $x$ , and all the other terms will be of a higher

degree, their degrees increasing with the distances of their points from  $L$ .

We call the groups of terms determined by the lines  $L$  the *possible groups of terms of lowest order* in  $f(x, y) = 0$ . By equating each such group to 0, discarding factors in  $x$  or  $y$  only, and then solving for  $y$ , we find the initial term, or first approximation, of every solution  $y = Cx^\mu + \dots$  that  $f(x, y) = 0$  can have at  $O$ ,  $\mu$  being a positive rational number, integral or fractional.

**EXAMPLE.** For  $f(x, y) = xy^3 + 3x^2y^2 + 2x^3y - 3x^5 - y^6 + x^7 = 0$ , the degree points are those indicated in Fig. 140, and the lines  $L$  are  $BC$ ,  $CDE$ ,  $EF$ .

The corresponding possible groups of terms of lowest order are

$$\begin{aligned} & -3x^5 + 2x^3y \\ & 2x^3y + 3x^2y^2 + xy^3 \\ & xy^3 - y^6 \end{aligned}$$

Equating each group to 0 and solving for  $y$ , we obtain

$$y = (3/2)x^2; \quad y = -x, \quad y = -2x; \quad y = x^{1/3}, \quad y = \alpha x^{1/3}, \quad y = \alpha^2 x^{1/3},$$

where

$$\alpha = (-1 + i\sqrt{3})/2$$

These are the initial terms of the possible solutions  $y = Cx^\mu + \dots$  of  $f(x, y) = 0$  at  $O$ .

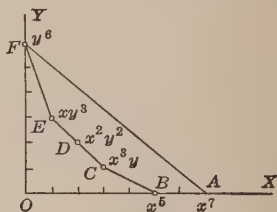


FIG. 140.

### 304. Solutions of algebraic equations at the origin.

1. Taking any one of the first approximations just found,  $y = Cx^\mu$  ( $\mu = p/q$ ), substitute  $y = x^\mu(C + v)$  in the given equation  $f(x, y) = 0$  of § 303. We obtain an algebraic equation in  $v$  and  $x^{1/q}$ , call it  $\phi(v, x^{1/q}) = 0$ , which lacks the constant term but ordinarily has the  $v$  term, and which may therefore be solved for  $v$  in powers of  $x^{1/q}$  by the method of § 301. When this value of  $v$  is substituted in  $y = x^\mu(C + v)$ , we obtain a solution  $y = \phi(x)$  of  $f(x, y) = 0$  in the form of a series in ascending powers of  $x^{1/q}$ , with a limit of convergence  $l > 0$ .

In case the  $v$  term is missing in  $\phi(v, x^{1/q}) = 0$ , we may deal with this equation as we have just dealt with  $f(x, y) = 0$ .

It can be shown that it is always possible by a limited number of steps of this kind to obtain equations in which the first power of the dependent variable is present and which therefore will lead back to solutions of  $f(x, y) = 0$ .

If in  $f(x, y)$  the term of lowest degree among those that involve  $y$  only is the  $y^q$  term, the number of the solutions of  $f(x, y) = 0$  for  $y$  at  $O$  is  $q$ . See § 303, Example.

2. The most expeditious method of finding the solutions whose existence has thus been proved is that of successive approximations described in § 302.

Let  $y^q - Cx^p$  occur once or  $r > 1$  times as a factor of one of the groups of terms of lowest order in  $f(x, y) = 0$ , so that  $y = C^{1/q}x^{p/q}$  is the first approximation to one or more of the required solutions. The second, third,  $\dots$  approximations may be obtained by first expressing  $f(x, y) = 0$  in the form

$$(y^q - Cx^p)^r = u_1 + u_2 + u_3 + \dots \quad (1)$$

where  $u_1, u_2, \dots$  denote terms or groups of terms arranged in the ascending order of their degrees in  $x$  when  $y = C^{1/q}x^{p/q}$ , and then proceeding as in § 302.

EXAMPLE 1. In  $y^2 - 4x^2 - x^3 - y^4 = 0$  (a) the group of lowest order is  $y^2 - 4x^2 = (y - 2x)(y + 2x)$ .

To find the solution whose first approximation is  $y = 2x$ , write (a) in the form

$$y - 2x = \frac{x^3}{y + 2x} + \frac{y^4}{y + 2x} \quad (b)$$

Substituting  $y = 2x$  in  $x^3/(y + 2x)$ , we get the second approximation

$$y = 2x + \frac{x^3}{4x} = 2x + \frac{x^2}{4} \quad (c)$$

Substituting (c) in  $x^3/(y + 2x)$ , and  $y = 2x$  in  $y^4/(y + 2x)$ , we get the third approximation

$$\begin{aligned} y &= 2x + \frac{x^3}{4x + x^2/4} + \frac{16x^4}{4x} = 2x + \frac{x^2}{4} \left( 1 - \frac{x}{16} + \dots \right) + 4x^3 \\ &= 2x + \frac{x^2}{4} + \frac{255}{64}x^3 \end{aligned} \quad (d)$$

If  $y^q - Cx^p$  is a factor of  $u_1$ , we retain  $u_2$  in computing the second approximation.

Thus in the case of  $(y - x)^2 = 2(y - x)x^2 + 3y^4$ , the first approximation is  $y = x$ . But  $y - x$  is a factor of  $u_1$ . We therefore substitute  $y = x$  in  $u_2 = 3y^4$  and then solve the equation as a quadratic in  $y - x$ , thus obtaining the second approximations  $y = x + 3x^2$ ,  $y = x - x^2$ .

**EXAMPLE 3.** Find the fourth approximation to one of the solutions of  $(y - x)^2 - y^3 = 0$  at  $O$ .

**EXAMPLE 4.** Find the second approximation to each of the solutions of  $y^4 - xy^2 + 4x^3 = 0$  at  $O$ .

**305. Singular points of algebraic curves.** 1. Such of the solutions of  $f(x, y) = 0$  considered in § 304 as have real coefficients represent curves which are called the "branches" of the graph  $K$  of  $f(x, y) = 0$  at and near  $O$ . These branches differ very little near  $O$  from the graphs of the approximations to the solutions.

The first approximations are given by irreducible equations of one of the two types  $y^q - Cx^p = 0$  ( $q \neq p$ ), and  $y - Cx = 0$ . To each equation  $y^q - Cx^p = 0$  corresponds a branch of  $K$  which near  $O$  almost coincides with the curve  $y^q - Cx^p = 0$  and, like it, touches  $Ox$  or  $Oy$  at  $O$  according as  $q \leq p$ . To each equation  $y - Cx = 0$  corresponds a branch of  $K$  which touches the line  $y - Cx = 0$ . Near  $O$  this branch almost coincides with the graph of the second approximation to the solution of  $f(x, y) = 0$  corresponding to  $y - Cx = 0$ .

**EXAMPLE 1.** Find the graph  $K$  of  $x^4 + y^4 - xy(x + y) = 0$  (a)

The possible groups of terms of lowest order are

$$x^4 - x^2y, \quad y^4 - xy^2, \quad -xy(x + y)$$

Setting  $x^4 - x^2y = 0$  gives  $y = x^2$

Hence there is a branch  $AOB$  which close to  $O$  nearly coincides with the parabola  $y = x^2$ .

Setting  $y^4 - xy^2 = 0$  gives  $y^2 = x$ . Hence the branch  $DOE$ .

Setting  $-xy(x + y) = 0$  gives  $y = -x$ . The second approximation of the corresponding solution of (a) is

$$y + x = \left[ \frac{x^4 + y^4}{xy} \right]_{y=-x} \quad \text{or} \quad y = -x - \frac{x^2}{2}$$



which represents a parabola touching the line  $y = -x$  at  $O$  and lying below it. Hence  $FOG$ .

The substitution  $y = tx$ , § 93, in (a)

gives  $x = t(t+1)/(t^4+1)$

$$y = t^2(t+1)/(t^4+1)$$

and these equations show that as  $t$  increases from  $-\infty$  to  $\infty$ ,  $P(t)$  traces successively  $OEFO$ ,  $OGAO$ ,  $OBDO$ .

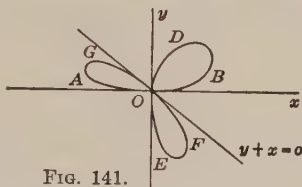


FIG. 141.

EXAMPLE 2. Find the graph of  $x^4 + 2y^4 + (x-y)^2(2x+y) = 0$

2. A point of the graph of an equation  $F(x, y) = 0$  at which both  $F_x$  and  $F_y$  are 0 is called a singular point, § 42, 4. Hence in the case of an equation  $f(x, y) = 0$  which lacks the constant term,  $O$  is a singular point when  $f(x, y)$  also lacks the  $x$  term and the  $y$  term (Exs. 1 and 2). In general the three equations  $F = 0$ ,  $F_x = 0$ ,  $F_y = 0$  have no common solution, and therefore the curve  $F = 0$  has no singular point. But if such a point  $(a, b)$  exists (found by solving  $F_x = 0$ ,  $F_y = 0$ ), its character can be determined by transforming  $F(x, y) = 0$  into an equation  $f(x', y') = 0$  which lacks the constant term — the method is explained in the next section — and then proceeding as in 1.

**306. Continuity of algebraic functions.** 1. Since all of the solutions at  $O$  of the equation  $f(x, y) = 0$  of § 304 are of the type  $y = x^\mu(C + v)$ , where  $\mu$  is positive and  $\lim_{x \rightarrow 0} v = 0$ , all of them are continuous functions of  $x$  at  $O$ .

2. If  $(a, b)$  be any point at which a given irreducible algebraic equation  $F(x, y) = 0$  is satisfied, and we make the substitution  $x = a + x'$ ,  $y = b + y'$ , and then apply Taylor's theorem, we transform  $F(x, y) = 0$  into an equation  $f(x', y') = 0$  which is satisfied at  $x', y' = 0, 0$ . We can get the solutions  $y' = \phi(x')$  of  $f(x', y') = 0$  at  $x', y' = 0, 0$  by § 304. If in each of them we substitute  $x' = x - a$ ,  $y' = y - b$ , we get the solutions  $y - b = \phi(x - a)$  of  $F(x, y) = 0$  at  $(a, b)$ . Hence *all of these solutions are continuous functions of  $x$  at  $(a, b)$ .*



**307. Infinite values of  $x$  and  $y$ .** 1. An irreducible algebraic equation  $f(x, y) = 0$  being given, let us inquire how  $y$  varies when  $x \rightarrow \infty$ .

In Newton's diagram, with the constant term of  $f(x, y)$ , if any, represented by  $O$ , each line  $L$ , with a positive  $X$ -intercept, which passes through two or more of the degree points of  $f(x, y)$  and has the rest to the *origin* side, determines a possible group of terms of *highest* order with respect to  $x$ . Equate this group to 0, discard factors in  $x$  or  $y$  only, and solve for  $y$  in terms of  $x$ . We thus obtain one or more equations of the type  $y = Cx^\mu$ , where  $\mu$  is rational and  $\mu \geq 0$  according as the  $Y$ -intercept of  $L$  is  $+$ ,  $\infty$ , or  $-$ .

Corresponding to each such equation  $y = Cx^\mu$ , substitute  $y = x^\mu(C + v)$  in  $f(x, y) = 0$ , and then divide throughout by the highest power of  $x$  that appears in any of the terms. We thus obtain an equation in  $v$  and negative powers of  $x$  which lacks the constant term but ordinarily has the  $v$  term. By the reasoning of § 304, this equation has one or more solutions of the form

$$v = a/x^\alpha + b/x^\beta + \dots \quad (\alpha > 0, \beta > \alpha, \dots) \quad (1)$$

there being a number  $l > 0$  such that the series on the right converges when  $1/|x| < l$ , and therefore when  $|x| > l$ .

Substituting (1) in  $y = x^\mu(C + v)$ , we get

$$y = x^\mu[C + a/x^\alpha + b/x^\beta + \dots] \quad |x| > l \quad (2)$$

When  $\mu > 0$ , then  $\lim_{x \rightarrow \infty} y/x^\mu = C$ , and we say that  $y$  becomes infinite to the order  $\mu$  with respect to  $x$ .

When  $\mu = 0$ ,  $\lim_{x \rightarrow \infty} y = C$ ; when  $\mu < 0$ ,  $\lim_{x \rightarrow \infty} y = 0$ .

We may obtain (2) by the method of successive approximations. We first arrange the terms of  $f(x, y) = 0$  in the descending order of their degrees with respect to  $x$  when  $y = Cx^\mu$ , and then proceed as in § 304, 2.

2. In like manner, each line  $L$  with a positive  $Y$ -intercept determines a possible group of terms of highest order with respect to  $y$  and leads to one or more solutions of  $f(x, y) = 0$  of the type

$$x = y^\mu [C' + a'/y^{a'} + b'/y^{b'} + \dots] \quad |y| > l' \quad (3)$$

EXAMPLE 1. The possible groups of terms of highest order with respect to  $x$  in

$y^3 - x^2y + x = 0$  (a) are  $y^3 - x^2y$  (b) and  $-x^2y + x$  (c)  
 $y^3 - x^2y = 0$  gives  $y = x$ ,  $y = -x$ ;  $-x^2y + x = 0$  gives  $y = 1/x$ .

Substituting  $y = x(1 + v)$  in (a), we get

$$\begin{aligned} x^3(1 + v)^3 - x^3(1 + v) + x &= 0 \\ \therefore 2v + 1/x^2 + 3v^2 + v^3 &= 0 \end{aligned} \quad (d)$$

Solving (d) for  $v$ ,

$$\begin{aligned} v &= -1/2 x^2 - 3/8 x^4 \dots \\ \therefore y &= x(1 + v) = x - 1/2 x^3 - 3/8 x^5 \dots \end{aligned} \quad (e)$$

Similarly, corresponding to  $y = -x$  and  $y = 1/x$ , we find

$$y = -x - 1/2 x^3 + 3/8 x^5 \dots \quad \text{and} \quad y = 1/x + 1/x^3 + 3/x^5 \dots$$

EXAMPLE 2. For the equation  $y^2x - 3xy - y^2 + 2x = 0$  (a), the line  $AC$  determines a group of highest order with respect to  $x$ , namely  $y^2x - 3xy + 2x$  (b), and  $BC$  a group of highest order with respect to  $y$ , namely  $y^2x - y^2$  (c).

Setting (b) = 0 gives  $y = 1$ ,  $y = 2$ . To find the solution corresponding to  $y = 1$ , we write (a) in the form

$$y - 1 = \frac{y^2}{x(y - 2)} \quad (d)$$

Hence the 2nd approximation, given by  $y - 1 = \frac{1}{x(-1)}$ , is  $y = 1 - \frac{1}{x}$  and the 3rd approximation, given by  $y - 1 = \frac{(1 - 1/x)^2}{x(-1 - 1/x)}$ , is

$$y = 1 - \frac{1}{x} + \frac{3}{x^2}$$

Similarly corresponding to  $y = 2$ , we find  $y = 2 + 4/x + 16/x^3 + \dots$

Equating (c) to 0 gives  $x = 1$ . Hence we express (a) in the form  $x - 1 = 3x/y - 2x/y^2$  and by successive approximations find

$$x = 1 + 3/y + 7/y^2 + \dots$$

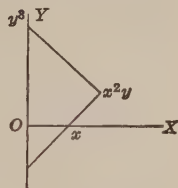


FIG. 142.

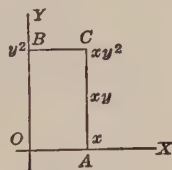


FIG. 143.

3. When  $f(x, y)$  has a constant term, so that  $O$  is one of the degree points, and  $P$  is another degree point (not on  $OX$  or  $OY$ ) such that all the degree points not on the line  $OP$  lie to one side of it; then if these remaining points are above  $OP$ ,  $OP$  is a line of terms of highest (zero) degree which leads to solutions of the type  $y = Cx^\mu + \dots$ , if below, to solutions of the type  $x = Cy^\nu + \dots$ .

Thus for the equation  $xy - y - 1 = 0$  the line joining the degree points of  $xy$  and  $-1$  gives the solution  $y = 1/x + 1/x^2 + \dots$ ; the other solution  $x = 1 + 1/y$  being given by the line joining the degree points of  $xy$  and  $-y$ .

**308. Infinite branches of algebraic curves.** Each of the solutions § 307 (2) which has real coefficients represents, for  $x > l$ , or  $x < -l$ , or both, an infinite branch of the curve  $f(x, y) = 0$ . By § 51, 2., a branch represented by (2) when  $\mu = 1$  and  $\alpha = 1$ , that is, by  $y = Cx + a + b/x^{\beta-1} + \dots$ , has the asymptote  $y = Cx + a$ . It is therefore called a *hyperbolic branch*. The branch is also hyperbolic when  $\mu = 1$  and  $\alpha > 1$ , when  $\mu = 0$ , and when  $\mu < 0$ , the corresponding asymptotes being  $y = Cx$ ,  $y = C$ , and  $y = 0$ .

In the remaining cases a branch represented by (2) has no asymptote, but it approaches parallelism with some given line, as does the half parabola  $y = x^{1/2}$  with the  $x$ -axis; it is therefore called a *parabolic branch*.

The like is to be said regarding the infinite branches represented by § 307 (3) when real. The solutions considered in § 307, 3. all represent hyperbolic branches.

A knowledge of the infinite branches of the curve, when coupled with information which may be obtained directly from its equation  $f(x, y) = 0$  — as the points where the curve cuts  $Ox$  and  $Oy$ , the finite points where it cuts its asymptotes, and its singular points, if any — often enable one to determine the general course of the curve without serious difficulty. Observe in particular that an asymptote, since it plays the

role of a tangent, cannot meet a curve whose equation is of degree  $n$  in more than  $n - 2$  finite points.

EXAMPLE 1. Find the asymptotes and graph of

$$x^2y + xy^2 - x^2 - y^2 = 0 \quad (a)$$

The possible groups of terms of highest order are

$$xy(y + x) \quad x^2(y - 1) \quad y^2(x - 1)$$

Solving (a), as in § 307, Ex. 2, for  $y + x$ ,  $y - 1$ ,  $x - 1$ , we get

$$y = -x - 2 - \frac{4}{x^2} \dots \quad (b)$$

$$y = 1 - \frac{1}{x} \dots \quad (c)$$

$$x = 1 - \frac{1}{y} \dots \quad (d)$$

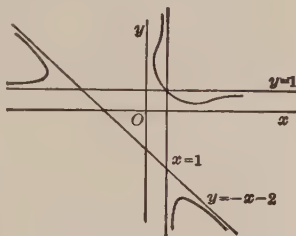


FIG. 144.

Hence the asymptotes are  $y = -x - 2$ ,  $y = 1$ ,  $x = 1$ . Moreover the term  $-4/x^2$  in (b) shows that the infinite branches represented by (b) toward the  $x = \infty$  and  $x = -\infty$  "ends" of  $y = -x - 2$  both lie below this asymptote. Similarly, the terms  $-1/x$  in (c) and  $-1/y$  in (d) show that the end parts of the corresponding infinite branches are to the sides of the asymptotes indicated in Fig. 144. It will be found that  $y = 1$  and  $x = 1$  both meet the curve at  $(1, 1)$ , but  $y = -x - 2$  at no finite point. The group of terms of lowest order in (a) is  $x^2 + y^2$ ; hence  $O$  is a conjugate point.

EXAMPLE 2. In  $x^2y^2 - x^3 - y^3 = 0$  (a) the possible groups of terms of highest order are  $x^2y^2 - x^3$  and  $x^2y^2 - y^3$ . When equated to 0, they give  $y^2 - x = 0$  (b) and  $x^2 - y = 0$  (c)

Hence there are no asymptotes. Parts of the infinite branches of (a) tend two to parallelism with  $Ox$ , like those of (b), and two to parallelism with  $Oy$ , like those of (c). They are parabolic branches.

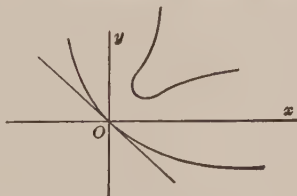


FIG. 145.

The curve is symmetric with respect to the line  $y = x$ , which cuts it at  $(1, 1)$ .

EXAMPLE 3. Find the graph of  $(y^2 - x^2)x - y(y - 3x) = 0$ .

The asymptotes are  $y = x - 1$ ,

$$y = -x - 2, x = 1$$

The tangents at  $O$  are  $y = 3x$ ,

$$y = 0$$

The substitution  $y = tx$  gives

$$x = \frac{t(t-3)}{t^2-1} \quad y = \frac{t^2(t-3)}{t^2-1}$$

These equations show that, as  $t$  increases from  $-\infty$  to  $\infty$ ,  $P(t)$  traces successively the arcs marked (1) ... (5) in Fig. 146.

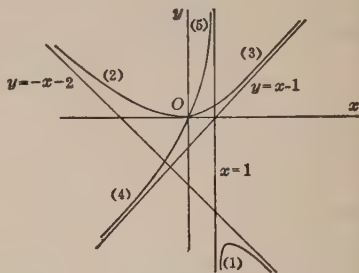


FIG. 146.

### EXERCISE LXI

1. Let  $f(x, y) = 0$  denote an irreducible algebraic equation of the  $n$ th degree,  $u_n$  its group of terms of the  $n$ th degree, and  $C$  its graph; prove the following, by aid of §§ 305, 307 :

1. If  $u_n$  has no real factor, then  $C$  has no infinite branch.
2. The asymptotes of  $C$ , if any, are parallel to the real lines through  $O$  represented by  $u_n = 0$ .
3. An asymptote  $l$  cannot meet  $C$  in more than  $n - 2$  finite points.
4. Ordinarily there are portions of  $C$  near both "ends" of an asymptote  $l$  and on opposite sides of  $l$ . If they are on the same side, or at the same end,  $l$  meets  $C$  in less than  $n - 2$  finite points.
5. In the case  $n = 3$ , an asymptote  $l$  ordinarily meets  $C$  in one finite point.
6. If  $u_k$  ( $k > 0$ ) denote the group of terms of lowest degree in  $f(x, y)$ , then  $C$  passes through  $O$ , and the tangents to  $C$  at  $O$  are the real lines represented by  $u_k = 0$ .

2. Trace the following curves :

1.  $y^2 + 2xy - 3x^2 - 2y = 0$
2.  $y^2 - 4x^2 + x - y + 3 = 0$
3.  $x^2y + y^2x + y - 2x = 0$
4.  $x^2y - xy^2 + y^2 - 4x^2 = 0$
5.  $x^3 - y^3 + y^2 = 0$
6.  $(y^2 - x^2)(y - 2x) - 2x^2 = 0$
7.  $x^4 - y^4 + x^2 = 0$
8.  $xy^2 + x - y = 0$
9.  $x^3 - x^2y - 2xy^2 - 5xy + 2y^2 = 0$
10.  $y^4 - xy^2 + 3x^2y - 2x^3 = 0$
11.  $y^2x + x^2 - y = 0$
12.  $(x - 1)y^2 + y - 2x = 0$
13.  $(x - 1)^2y^2 = x - y$
14.  $(y - x)^2(y + x) + y^2 - 3x^2 = 0$
15.  $(y - 2x)^2y - x = 0$
16.  $(x^2 - 1)y^2 = y - x$
17.  $x^2y^2 + x^3 - y^3 + x^2 - y^2 = 0$
18.  $(y - x)^2(y + x) + 2(y - x)x - y = 0$

**309. Uniform convergence.** Consider a series  $\Sigma u_n$  whose terms are functions of  $x$ , and which converges for all values of  $x$  in a certain interval  $(a, b)$ . Let  $S$  denote the sum of  $\Sigma u_n$  for values of  $x$  in  $(a, b)$ , also  $S_n$  the sum of  $n$  terms and  $R_n$  the remainder after  $n$  terms. Then  $S$  is a function of  $x$  in  $(a, b)$ , and  $S = S_n + R_n$ .

Suppose that, any positive number  $\epsilon$  having been assigned, it is always possible to find a value  $n'$  of  $n$ , which is independent of  $x$ , and is such that when  $n \geq n'$  then  $|R_n| < \epsilon$  for all values of  $x$  in  $(a, b)$ . We then say that the series is *uniformly convergent* in  $(a, b)$ .

The proof in § 181 shows that any power series  $\Sigma a_n x^n$  is uniformly convergent in any interval  $(a, b)$  which lies *within* its interval of convergence  $(-l, l)$ , and by an obvious extension of the reasoning of that section, it can be proved that

*If for all values of  $x$  in  $(a, b)$  the terms of  $\Sigma u_n$  are continuous, and are numerically less than the corresponding terms of a convergent positive series  $\Sigma M_n$ , then  $\Sigma u_n$  is uniformly convergent, and its sum  $S$  is a continuous function of  $x$  in  $(a, b)$ .*

As an example of non-uniform convergence, consider the series

$$x + x(1 - x) + x(1 - x)^2 + \dots$$

It converges in the interval  $(0, 1)$ , but not uniformly. For it will be found, when  $x \neq 0$ , that  $R_n = (1 - x)^n$ . Hence corresponding to any given  $x'$  between 0 and 1 we can find  $n'$  such that  $R_n < \epsilon$  when  $n \geq n'$  for every  $x$  in  $(x', 1)$ . But when  $x' \rightarrow 0$ , then  $n' \rightarrow \infty$ .

It can be proved, as in §§ 182, 183, that  $\Sigma u_n$ , when uniformly convergent, can be integrated term by term; and differentiated term by term, if the resulting series is uniformly convergent. The definition and theorems can be immediately extended to series whose terms are functions of more than one variable, or of the complex variable  $z = x + iy$ .

**310. Double power series.** Let  $a_{mn}x^m y^n$ ,  $m, n = 1, 2, \dots$ , denote an array of elements of the type  $a_{mn}x^m y^n$ ,  $m$  indicating the column and  $n$  the row in which the element stands.

Suppose that for  $x = r(> 0)$  and  $y = \rho(> 0)$  every element  $a_{mn}x^m y^n$  is numerically less than some positive number  $M$ . Then for  $|x| < r$ ,  $|y| < \rho$  the double series got by taking the elements of the array by rows and adding their sums will be an absolutely convergent series of the type discussed in § 299.

For when  $|x| < r$ ,  $|y| < \rho$ , then for every  $m, n$ ,

$$|a_{mn}x^m y^n| = |a_{mn}r^m \rho^n| \cdot \left|\frac{x}{r}\right|^m \cdot \left|\frac{y}{\rho}\right|^n < M \cdot \left|\frac{x}{r}\right|^m \left|\frac{y}{\rho}\right|^n$$

Hence the terms of the double series under consideration are numerically less than the corresponding terms of the similar double series got from the array  $M|x/r|^m \cdot |y/\rho|^n$ ; and it can be shown, as in § 301 (5), that when  $|x| < r$ ,  $|y| < \rho$  this latter series converges to the sum

$$M \div \left(1 - \left|\frac{x}{r}\right|\right) \left(1 - \left|\frac{y}{\rho}\right|\right)$$

**311. Differential equations.** Let  $dy/dx = f(x, y)$  be a given differential equation of the first order, and  $(x_0, y_0)$  a point in whose neighborhood  $f(x, y)$  can be expressed as a power series in  $x - x_0$ ,  $y - y_0$ . We are to prove that at  $(x_0, y_0)$  the equation has a solution of the form  $y = \Sigma a_n(x - x_0)^n$ , the series having a limit of convergence  $l > 0$ .

If in  $dy/dx = f(x, y)$  we substitute  $x - x_0 = x'$ ,  $y - y_0 = y'$  and then drop the accents in the result, we get an equation of the form

$$\frac{dy}{dx} = \Sigma a_{mn}x^m y^n \quad (1)$$

If in (1) we make the substitution

$$y = C_1x + C_2x^2 + C_3x^3 + \dots \quad (2)$$

and then proceed as in § 301 we obtain expressions for the  $C$ 's which are polynomials in the coefficients  $a_{mn}$  of (1). Thus

$$C_1 = a_{00}, \quad C_2 = (a_{10} + a_{00}a_{10})/2, \quad \text{and so on} \quad (3)$$

The series (2) with the coefficients (3) satisfies (1) formally. It is therefore a solution of (1) if it has a limit of convergence  $l > 0$ .

But assuming that  $\Sigma a_{mn}x^m y^n$  satisfies the conditions of § 310, and using the notation of that section, consider the equation

$$\frac{dy}{dx} = \Sigma M \left(\frac{x}{r}\right)^m \left(\frac{y}{\rho}\right)^n = \frac{M}{(1 - x/r)(1 - y/\rho)}, \quad |x| < r, |y| < \rho \quad (4)$$



Each coefficient  $M/r^m \rho^n$  in the second member of (4) is numerically greater than the corresponding coefficient  $a_{mn}$  in (1). It therefore follows from (3) that the series

$$y = C'_1 x + C'_2 x^2 + C'_3 x^3 + \dots \quad (5)$$

which satisfies (4) formally, as (2) satisfies (1), will have in each of its terms a numerically greater coefficient than (2) has in that term : so that (2) will converge when (5) converges. But the solution at  $x = 0$ ,  $y = 0$  of the differential equation formed of the first and third members of (4) is

$$y = \rho - \left[ \rho^2 + 2 M \rho r \log \left( 1 - \frac{x}{r} \right) \right]^{1/2}$$

and this can be expanded in a power series in  $x$  which will be found to converge when  $|x| < r(1 - e^{-k})$ , where  $k = \rho/2 Mr$ . Therefore, since, by § 186, this power series is identical with (5), the series (2) has a limit of convergence  $l > 0$ .

It may be added that, with a change in notation,  $\Sigma a_{mn} x^m y^n$  is the infinite series which the Taylor series of § 222 (6) becomes when  $f(x, y)$  and all its partial derivatives are continuous at the point  $(x_0, y_0)$  and in its neighborhood and the remainder term  $\rightarrow 0$ , as  $n \rightarrow \infty$ . It is not difficult to prove that the sum of this Taylor series is not changed when its terms  $a_{mn} x^m y^n$  are combined in the manner explained at the top of p. 376.

By an extension of this method it can be proved that a pair of equations  $dy/dx = f(x, y, z)$ ,  $dz/dx = \phi(x, y, z)$  has solutions. The equation  $d^2 y/dx^2 = F(x, y, dy/dx)$  becomes such a pair by setting  $dy/dx = z$ ; hence it has solutions. Similarly for sets of simultaneous equations of the first order in  $n$  variables, and equations of the  $n$ th order in two variables.

## XXVIII. FOURIER SERIES

**312. Fourier series.** 1. Assume that, for values of  $x$  between  $-\pi$  and  $\pi$ , a given function  $f(x)$  can be expressed by a series of the form

$$f(x) = a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx) \quad (1)$$

and that it is permissible to integrate this series term by term, as in § 182. We can then show that the coefficients  $a_0, a_1, b_1, \dots$  are given by the formulas

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx & a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx \\ b_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx \end{aligned} \quad (2)$$

The series (1) with the coefficients given by (2) is called a *Fourier series*.

For it is easy to prove, §§ 112, 115, that

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx dx &= 0 & \int_{-\pi}^{\pi} \sin mx dx &= 0 & \int_{-\pi}^{\pi} \cos mx \sin nx dx &= 0 \\ \int_{-\pi}^{\pi} \cos mx \cos nx dx &= \int_{-\pi}^{\pi} \sin mx \sin nx dx & &= 0 & (m \neq n) \\ \int_{-\pi}^{\pi} dx &= 2\pi & \int_{-\pi}^{\pi} \cos^2 mx dx &= \pi & \int_{-\pi}^{\pi} \sin^2 mx dx &= \pi \end{aligned}$$

Hence, if the equation (1) and the two equations got by multiplying it by  $\cos mx$  and by  $\sin mx$  be integrated between the limits  $x = -\pi$  and  $x = \pi$ , the results obtained will be

$$\int_{-\pi}^{\pi} f(x) dx = a_0 2\pi \quad \int_{-\pi}^{\pi} f(x) \cos mx dx = a_m \pi \quad \int_{-\pi}^{\pi} f(x) \sin mx dx = b_m \pi$$

and these equations give the formulas (2).

**EXAMPLE 1.** Let  $f(x)$  have the value 1 when  $-\pi < x < 0$ , and the value 2 when  $0 < x < \pi$ . On the assumption that this function can be expressed by a Fourier series, we have

$$\begin{aligned} a_0 2\pi &= \int_{-\pi}^0 dx + \int_0^{\pi} 2 dx = 3\pi \\ a_m \pi &= \int_{-\pi}^0 \cos mx dx + 2 \int_0^{\pi} \cos mx dx = 3 \int_0^{\pi} \cos mx dx = 0 \\ b_m \pi &= \int_{-\pi}^0 \sin mx dx + 2 \int_0^{\pi} \sin mx dx = \int_0^{\pi} \sin mx dx = -\frac{1}{m} \cos mx \Big|_0^{\pi} \end{aligned}$$

When  $m$  is odd,  $\cos mx \Big|_0^\pi = -2$ , and when  $m$  is even,  $\cos mx \Big|_0^\pi = 0$ . Hence

$$f(x) = \frac{3}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \frac{2}{5\pi} \sin 5x + \dots$$

Observe that the given  $f(x)$  is not defined at  $x = -\pi, 0, \pi$ , and that it is discontinuous at  $x = 0$ . The foregoing reckoning tacitly assumes the following definitions of  $\int_{-\pi}^\pi$ ,  $\int_{-\pi}^0$ ,  $\int_0^\pi$  for this  $f(x)$  (compare § 134). Using  $\alpha, \beta, \gamma, \delta$  to denote positive variables which  $\rightarrow 0$ , we have

$$\int_{-\pi}^\pi = \lim \left[ \int_{-\pi+\alpha}^{-\beta} + \int_{\gamma}^{\pi-\delta} \right] = \int_{-\pi}^0 + \int_0^\pi$$

EXAMPLE 2. Assuming that (1), (2) apply to  $f(x) = x$ , show that for any value of  $x$  between  $-\pi$  and  $\pi$

$$\frac{x}{2} = \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots$$

2. To prove that the series (1), (2) does actually represent  $f(x)$  within the interval  $(-\pi, \pi)$ , we must show that the sum of its first  $n$  terms  $\rightarrow f(x)$  when  $n \rightarrow \infty$ . This proof is given in § 315 under certain restrictions as to the character of  $f(x)$ . It is based on a theorem which is established in the following two sections.

**313. The integral**  $\int_0^{\pi/2} [\sin(2m+1)x / \sin x] dx$ . From this integral we shall derive an expression for  $\pi/2$  as the sum of an alternating series.

It can be proved by mathematical induction that

$$\frac{1}{2} + \cos \beta + \cos 2\beta + \dots + \cos m\beta = \sin \frac{2m+1}{2} \beta / 2 \sin \frac{\beta}{2} \quad (1)$$

Hence, setting  $\beta = 2x$ ,

$$\int_0^{\pi/2} \frac{\sin(2m+1)x}{\sin x} dx = 2 \int_0^{\pi/2} \left[ \frac{1}{2} + \cos 2x + \dots + \cos 2mx \right] dx = \frac{\pi}{2} \quad (2)$$

1. Setting  $2m+1 = n$ , and observing that  $\sin nx$  changes sign at the points  $x = \pi/n, 2(\pi/n), \dots$ , we divide the interval  $(0, \pi/2)$  into the subintervals whose end points are

$$0, \frac{\pi}{n}, 2\frac{\pi}{n}, 3\frac{\pi}{n}, \dots, r\frac{\pi}{n}, \frac{\pi}{2} \quad (3)$$

where  $r(\pi/n)$  denotes the greatest multiple of  $\pi/n$  which is less than  $\pi/2$ , and then express the integral (2) as the sum of the integrals correspond-

ing to these  $r + 1$  subintervals, so that if  $u_k$  denote the numerical value of the  $k$ th of these integrals we shall have (for  $k < r$ )

$$(-1)^k u_k = \int_{k\frac{\pi}{n}}^{(k+1)\frac{\pi}{n}} \frac{\sin nx}{\sin x} dx \quad (4)$$

and 
$$\frac{\pi}{2} = u_0 - u_1 + u_2 - \dots + (-1)^r u_r \quad (5)$$

2. Substituting  $x = k\frac{\pi}{n} + y$  in (4) and simplifying, we get

$$u_k = \int_0^{\frac{\pi}{n}} \frac{\sin ny}{\sin [y + k(\pi/n)]} dy \quad (6)$$

which shows that the value of  $u_k$  decreases as the subscript  $k$  increases.

3. We shall prove that, when  $n \rightarrow \infty$ ,  $u_k$  approaches a limit  $l_k$  such that

$$\frac{2}{(k+1)\pi} < l_k \leq \frac{2}{k\pi} \quad (7)$$

For substituting  $nx = z$  in (4), we get

$$(-1)^k u_k = \int_{k\pi}^{(k+1)\pi} \frac{\sin z}{n \sin (z/n)} dz \quad (8)$$

As  $n$  increases,  $n \sin \frac{z}{n} = z \frac{\sin(z/n)}{z/n}$  increases and  $\rightarrow z$ .

Hence, as  $n$  increases,  $u_k$  decreases but remains greater than

$$\left| \int_{k\pi}^{(k+1)\pi} \frac{\sin z}{z} dz \right| \therefore > \frac{\left| \int_{k\pi}^{(k+1)\pi} \sin z dz \right|}{(k+1)\pi} = \frac{2}{(k+1)\pi}$$

Therefore, § 5, as  $n \rightarrow \infty$ ,  $u_k$  approaches a limit  $l_k$  which is greater than  $2/(k+1)\pi$ .

Also 
$$u_k < \frac{\left| \int_{k\pi}^{(k+1)\pi} \sin z dz \right|}{n \sin (k\pi/n)} \therefore l_k \leq \lim_{n \rightarrow \infty} \frac{2}{n \sin (k\pi/n)} = \frac{2}{k\pi}.$$

4. But when  $n \rightarrow \infty$ , then  $r \rightarrow \infty$ . Therefore (5) becomes

$$\frac{\pi}{2} = l_0 - l_1 + l_2 - \dots + (-1)^k l_k + \dots \quad (9)$$

the right member being an alternating series in which  $l_0 > l_1 > l_2 > \dots$ , by (7); the sum of the terms after  $(-1)^k l_k$  is numerically  $< l_{k+1}$ ; and  $l_{k+1} \rightarrow 0$  when  $k \rightarrow \infty$ .

**314.** The integral  $\int_0^h \phi(x) [\sin nx / \sin x] dx$ . 1. Let  $n$  denote an odd integer,  $h$  any positive number  $\leq \pi/2$ , and  $\phi(x)$  a function which in the interval  $(0, h)$  is continuous,

positive and decreasing. And let us represent the integral  $\int_0^h \phi(x) [\sin nx / \sin x] dx$  by  $J[\phi(x)]$  or  $J$ . We are to prove that

$$\lim_{n \rightarrow \infty} J = \phi(0) \frac{\pi}{2} \quad (1)$$

As in § 313, 1., divide  $(0, h)$  into the subintervals whose end points are  $0, \pi/n, \dots, s(\pi/n), h$ , and let  $v_k$  be the numerical value of the part of  $J$  which belongs to the  $k$ th of these subintervals: so that (for  $k < s$ )

$$(-1)^k v_k = \int_{k\frac{\pi}{n}}^{(k+1)\frac{\pi}{n}} \phi(x) \frac{\sin nx}{\sin x} dx \quad (2)$$

and 
$$J = v_0 - v_1 + v_2 - \dots + (-1)^s v_s \quad (3)$$

By § 282, 2., there is a value  $x_k$  of  $x$  between  $k\frac{\pi}{n}$  and  $(k+1)\frac{\pi}{n}$  such that

$$\int_{k\frac{\pi}{n}}^{(k+1)\frac{\pi}{n}} \phi(x) \frac{\sin nx}{\sin x} dx = \phi(x_k) \int_{k\frac{\pi}{n}}^{(k+1)\frac{\pi}{n}} \frac{\sin nx}{\sin x} dx \quad (4)$$

Hence, § 313, (4),  $v_k = \phi(x_k) u_k \quad k\frac{\pi}{n} < x_k < (k+1)\frac{\pi}{n} \quad (5)$

Both  $\phi(x_k)$  and  $u_k$  decrease as  $k$  increases. Hence  $v_0 > v_1 > v_2 > \dots$ , and the sum of the terms in (3) after the term  $(-1)^k v_k$  is numerically less than the term  $v_{k+1}$ .

Hence if we first assign to  $k$  any odd value, as great as we please, and then take  $n$  great enough to make  $k(\pi/n) < h$ , we can express (3) in the form

$$J = [\phi(x_0)u_0 - \phi(x_1)u_1 + \dots - \phi(x_k)u_k] + \theta \phi(x_{k+1})u_{k+1} \quad (0 < \theta < 1) \quad (6)$$

But by § 313, (9), we also have

$$\phi(0) \frac{\pi}{2} = [\phi(0)l_0 - \phi(0)l_1 + \dots - \phi(0)l_k] + \theta' \phi(0)l_{k+1} \quad (0 < \theta' < 1) \quad (7)$$

Let  $n \rightarrow \infty$ . Then  $u_0, u_1, \dots \rightarrow l_0, l_1, \dots$ ; also  $\phi(x_0), \phi(x_1), \dots \rightarrow \phi(0)$  by (5); therefore the bracketed expression in (6)  $\rightarrow$  that in (7), and

$$J - \phi(0) \frac{\pi}{2} \rightarrow \theta'' \phi(0) l_{k+1} \quad (-1 < \theta'' < 1) \quad (8)$$

But we can at the outset take  $k$  as great as we please, great enough therefore, by § 313, (7), to make  $\phi(0)l_{k+1}$  less than any positive number  $\epsilon$  that can be assigned. Therefore

$$\lim_{n \rightarrow \infty} J = \phi(0) \frac{\pi}{2} \quad (9)$$

Observe that throughout this discussion  $\phi(0)$  has meant  $\lim_{x \rightarrow 0} \phi(x)$ .

2. The formula (9) also holds good when  $\phi(x)$  is a decreasing function which becomes negative in  $(0, h)$ . For let  $-C = \phi(h)$ , the least value of  $\phi(x)$  in  $(0, h)$ . Then  $\phi(x) + C$  is a positive decreasing function in  $(0, h)$ . We have

$$J[\phi(x)] = J[\phi(x) + C] - J[C]$$

$$\text{Hence } \lim_{n \rightarrow \infty} J[\phi(x)] = [\phi(0) + C] \frac{\pi}{2} - C \frac{\pi}{2} = \phi(0) \frac{\pi}{2}$$

3. Again (9) holds good for a positive or negative function  $\phi(x)$  which (when not constant) always increases in  $(0, h)$ . For  $-\phi(x)$  is then a decreasing function; hence

$$\lim_{n \rightarrow \infty} J[\phi(x)] = - \lim_{n \rightarrow \infty} J[-\phi(x)] = - \left[ -\phi(0) \frac{\pi}{2} \right] = \phi(0) \frac{\pi}{2}$$

4. Let  $g$  and  $h$  be any numbers such that  $0 < g < h \leq \pi/2$ , and let  $\phi(x)$  denote any continuous function which (when not constant) always decreases or always increases in  $(g, h)$ ; then

$$\lim_{n \rightarrow \infty} \int_g^h \phi(x) \frac{\sin nx}{\sin x} dx = 0 \quad (10)$$

For let  $\phi_1(x)$  denote a function which has the constant value  $\phi(g)$  in  $(0, g)$  and coincides with  $\phi(x)$  in  $(g, h)$ ; then

$$\int_g^h \phi(x) \frac{\sin nx}{\sin x} dx = \int_0^h \phi_1(x) \frac{\sin nx}{\sin x} dx - \int_0^g \phi_1(x) \frac{\sin nx}{\sin x} dx$$

When  $n \rightarrow \infty$  both the integrals on the right  $\rightarrow \phi_1(0)\pi/2$ , by (9); hence that on the left  $\rightarrow 0$ .

5. Thus far we have supposed  $h \leq \pi/2$ . Let us now suppose  $h > \pi/2$  but  $< \pi$ . Break up the integral  $J$  into the part belonging to  $(0, \pi/2)$  and the part belonging to  $(\pi/2, h)$ , and in the second part set  $x = \pi - y$ . We obtain

$$J = \int_0^{\pi/2} \phi(x) \frac{\sin nx}{\sin x} dx + \int_{\pi-h}^{\pi/2} \phi(\pi - y) \frac{\sin ny}{\sin y} dy$$

When  $n \rightarrow \infty$ , the first integral  $\rightarrow \phi(0)\pi/2$ , the second  $\rightarrow 0$ . Hence (9) holds for any  $\phi(x)$  which varies in but one sense in  $(0, h)$ ,  $h < \pi$ .

6. Suppose  $0 < h < \pi$ , and let  $\phi(x)$  be a function for which the interval  $(0, h)$  can be divided into a finite number of parts  $(0, a)$ ,  $(a, b)$ ,  $\dots$   $(g, h)$  within each of which  $\phi(x)$  is continuous and varies in but one sense; then

$$\lim_{n \rightarrow \infty} J = \lim_{n \rightarrow \infty} \int_0^h \phi(x) \frac{\sin nx}{\sin x} dx = \phi(0) \frac{\pi}{2} \quad (11)$$

For break  $J$  up into the parts belonging to  $(0, a)$ ,  $(a, b)$ ,  $\dots$ . By (9), (10), when  $n \rightarrow \infty$ , the first of these parts  $\rightarrow \phi(0)\pi/2$  and the others  $\rightarrow 0$ .

The proof holds good even when at any point of division, as  $a$ , we have  $\lim_{x \rightarrow a} \phi(x) \neq \lim_{x \rightarrow a} \phi(x)$ , so that  $\phi(x)$  is discontinuous at  $x = a$ .

**315. Proof of Fourier's theorem.** Let  $f(x)$  denote a function which is finite in the interval  $(-\pi, \pi)$ , and is such that this interval can be divided into a finite number of parts within each of which  $f(x)$  is continuous and varies in but one sense. If at any point of division,  $x = c$ , we have  $\lim_{x \rightarrow c} f(x) \neq \lim_{x \rightarrow c} f(x)$ , we represent these limits by  $f(c - 0)$  and  $f(c + 0)$  and then define  $f(c)$  by the formula

$$f(c) = \frac{f(c - 0) + f(c + 0)}{2}$$

Obviously this formula is true identically at any point where  $f(x)$  is continuous.

We are to prove that for all values of  $x$  between  $-\pi$  and  $\pi$  this function  $f(x)$  may be represented by the Fourier series of § 312, and that when  $x$  is  $-\pi$  or  $\pi$  the sum of the series is

$$\frac{f(-\pi + 0) + f(\pi - 0)}{2}$$

For taking  $a$  as the variable of integration, we may write the  $k$ th term under the  $\Sigma$  sign in § 312 (1), with the coefficients given by § 312 (2), in the form

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(\alpha) [\cos k\alpha \cos kx + \sin k\alpha \sin kx] d\alpha = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\alpha) \cos k(\alpha - x) d\alpha$$



Hence, if  $S_m$  denote the sum of the first  $m + 1$  terms of the series, we have

$$S_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\alpha) \left[ \frac{1}{2} + \cos(\alpha - x) + \cdots + \cos m(\alpha - x) \right] d\alpha \quad (1)$$

We shall prove that when  $x$  is between  $-\pi$  and  $\pi$ , then  $\lim S_m = f(x)$ .

Apply the formula § 313 (1); then set  $\alpha - x = 2y$ ,  $2m + 1 = n$ .

We thus obtain 
$$S_m = \frac{1}{\pi} \int_{-\frac{\pi+x}{2}}^{\frac{\pi-x}{2}} f(x + 2y) \frac{\sin ny}{\sin y} dy \quad (2)$$

Separate this integral into one between  $-(\pi + x)/2$  and  $0$ , and one between  $0$  and  $(\pi - x)/2$ , and in the first of these integrals set  $y = -z$ . We thus get

$$S_m = \frac{1}{\pi} \int_0^{\frac{\pi+x}{2}} f(x - 2z) \frac{\sin nz}{\sin z} dz + \frac{1}{\pi} \int_0^{\frac{\pi-x}{2}} f(x + 2y) \frac{\sin ny}{\sin y} dy \quad (3)$$

When  $-\pi < x < \pi$ , both  $\frac{\pi+x}{2}$  and  $\frac{\pi-x}{2}$  are between  $0$  and  $\pi$ . Hence, by § 314 (11),

$$\lim_{m \rightarrow \infty} S_m = \frac{1}{\pi} f(x - 0) \frac{\pi}{2} + \frac{1}{\pi} f(x + 0) \frac{\pi}{2} = \frac{f(x - 0) + f(x + 0)}{2} = f(x) \quad (4)$$

This conclusion does not follow when  $x = \pi$  or  $-\pi$ . But when  $x = -\pi$ , (2) becomes

$$S_m = \frac{1}{\pi} \int_0^{\pi} f(-\pi + 2y) \frac{\sin ny}{\sin y} dy \quad (5)$$

Separate this integral into one between  $0$  and  $\pi/2$ , and one between  $\pi/2$  and  $\pi$ , and in the second of these integrals set  $y = \pi - z$ . We thus obtain

$$S_m = \frac{1}{\pi} \int_0^{\pi/2} f(-\pi + 2y) \frac{\sin ny}{\sin y} dy + \frac{1}{\pi} \int_0^{\pi/2} f(\pi - 2z) \frac{\sin nz}{\sin z} dz \quad (6)$$

Hence, § 314 (11), 
$$\lim_{m \rightarrow \infty} S_m = \frac{f(-\pi + 0) + f(\pi - 0)}{2} \quad (7)$$

We obtain the same result for  $x = \pi$ .

**316. Convergence of Fourier series.** The Fourier series for  $f(x)$  has been shown to be convergent; but, generally speaking, it is only conditionally not absolutely convergent. In case  $f(x)$  has points of discontinuity in  $(-\pi, \pi)$ , the series cannot be uniformly convergent, § 309, but one can prove that it is uniformly convergent if  $f(x)$  is continuous in  $(-\pi, \pi)$ .

**317. Fourier series in cosines or sines only.** It is readily seen that if  $f(x)$  is an even function, that is, if  $f(-x) = f(x)$ , then

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) dx \quad a_m = \frac{2}{\pi} \int_0^{\pi} f(x) \cos mx dx \quad b_m = 0 \quad (1)$$

Similarly, if  $f(x)$  is an odd function, so that  $f(-x) = -f(x)$ ,

$$a_m = 0 \quad b_m = \frac{2}{\pi} \int_0^{\pi} f(x) \sin mx dx \quad (2)$$

It follows that any function  $f(x)$  which satisfies the conditions of § 314, 6. in the interval  $(0, \pi)$  may be expressed in that interval by a Fourier series in cosines only, or sines only, with the coefficients given by (1) or (2). For let  $F(x)$  denote the function which equals  $f(x)$  for any value of  $x$  in  $(0, \pi)$ , and also for the negative of that value. This  $F(x)$  is an even function; the formulas (1) give its series in  $(-\pi, \pi)$ ; and in  $(0, \pi)$ ,  $F(x)$  is  $f(x)$ . Similarly for the case of the sine series.

### EXERCISE LXII

1. Find the Fourier series for the following in the interval  $(-\pi, \pi)$ 
  1.  $x^2 + x$
  2.  $1 + \sin x$
  3.  $x \sin x$
2. Express  $\pi/4$  by a sine series and also by a cosine series in the interval  $(0, \pi)$ .
3. Express  $\sin x$  by a cosine series in the interval  $(0, \pi)$ .
4. If  $f(x)$  has the value  $-1$  when  $0 < x < \pi/2$ , and the value  $1$  when  $\pi/2 < x < \pi$ , find the sine series for  $f(x)$  in the interval  $(0, \pi)$ .
5. Find the cosine series in  $(0, \pi)$  for  $\cos ax$  when  $a$  is not an integer.

## XXIX. PROPERTIES OF CONTINUOUS FUNCTIONS

### THE SET OF REAL NUMBERS

**318. The rational numbers.** 1. *The infinite set of symbols called rational numbers is an ordered or "ordinal" set.*

We have first of all the *natural scale* 1, 2, 3, 4, 5, ..., consisting of the natural numbers arranged as they are used in counting. Next, to make subtraction of natural numbers always possible, this scale is extended backward. We invent successively the symbol 0 which we place before 1, the symbol  $-1$  which we place before 0, and so on without end. We thus obtain the *complete scale*

...,  $-5$ ,  $-4$ ,  $-3$ ,  $-2$ ,  $-1$ , 0, 1, 2, 3, 4, 5, ...

Also, to make division of natural numbers always possible, we invent *fractions*. Taking any pair of natural numbers,  $a$  and  $b$ , of which  $a$ , but not  $b$ , may also be 0, we form the symbol  $a/b$ , read " $a$  over  $b$ ," with the understanding that  $a/1 = a$ . We then think of all such symbols as inserted among the numbers of the scale  $0/1$ ,  $1/1$ ,  $2/1$ ,  $3/1$ , ... in such an order that if  $a/b$  and  $c/d$  denote any two of them,  $a/b$  shall precede, coincide with, or follow  $c/d$ , according as in the scale the number  $ad$  precedes, coincides with, or follows the number  $bc$ : or, employing  $<$ ,  $=$ ,  $>$  to mean "precedes," "coincides with," "follows," so that

$a/b <$ ,  $=$ , or  $> c/d$  according as  $ad <$ ,  $=$ , or  $> bc$

Finally we assign to  $-a/b$  a position relative to the numbers of the complete scale which is symmetric with respect to 0 to that of  $a/b$ .

2. *Between any two rationals there are infinitely many other rationals. The set of rational numbers is therefore said to be "dense."*

**319. Irrational numbers.** The rational numbers meet all the requirements of the four fundamental operations of arithmetic. But other requirements suggest the invention of additional numbers. There is, for example, no rational whose square is 2, none therefore that will satisfy the equation  $x^2 - 2 = 0$  or express the length of the diagonal of a square in terms of the side.

In relation to any particular rational, say  $1/2$ , all other rationals naturally fall into two classes, the one consisting of all that precede  $1/2$ , the other of all that follow  $1/2$ . Call these two classes of numbers  $C_1$  and  $C_2$ . The relations existing between them are these :

1. Each number in  $C_1$  precedes every number in  $C_2$ .
2. There is no last number in  $C_1$  and no first number in  $C_2$ .

As for  $1/2$  itself, it is the one number between all numbers in  $C_1$  and all numbers in  $C_2$ .

Let us now consider the problem of defining a number whose square shall be 2. Since there is no rational whose square is 2, every rational is either a number whose square is less than 2 or one whose square is greater than 2. Assign all positive rationals whose squares are greater than 2 to a class  $A_2$ , and all other rationals to a class  $A_1$ .

1. Each number in  $A_1$  precedes every number in  $A_2$ .
2. There is no last number in  $A_1$  and no first in  $A_2$ .

Hence the classes  $A_1$  and  $A_2$  stand in precisely the same relation to each other as did the classes  $C_1$  and  $C_2$  in the case just considered. But since all the rationals are in  $A_1$  and  $A_2$  themselves, no number as yet exists lying between  $A_1$  and  $A_2$ , as  $1/2$  lies between  $C_1$  and  $C_2$ . We invent such a number, defining it ordinally as the one number which lies between all the numbers in  $A_1$  on the one hand, and all in  $A_2$  on the other.

When multiplication has been defined for this new number and others of the same kind, we shall find that its square is 2, and that it may therefore be represented by the symbol  $\sqrt{2}$ . Hence, by definition, " $\sqrt{2}$  is that number which lies between all positive rationals whose squares are less than 2 and all whose squares are greater than 2."

We call the number  $\sqrt{2}$  thus defined an *irrational number*.

*An irrational number  $a$  is defined whenever a law is stated which assigns every given rational to one and but one of two classes  $A_1, A_2$  so related that (1) each number in  $A_1$  precedes every number in  $A_2$ , and (2) there is no last number in  $A_1$  and no first in  $A_2$ ; the definition of  $a$  then being:  $a$  is the one number between all numbers in  $A_1$  and all in  $A_2$ . The definition supposes that there are numbers in both  $A_1$  and  $A_2$ .*

Think of every such irrational  $a$  as inserted among the rationals in the place given it by its definition. The entire set of numbers, rational and irrational, thus obtained is called the set of *real numbers*.

It is easily proved that the set of real numbers is ordinal and dense. It also possesses the following property, — which the rational set does not possess.

**320. Theorem.** *The set of real numbers is "continuous": that is, if the entire set be separated into any two parts  $R_1$  and  $R_2$  so related that each number in  $R_1$  precedes every number in  $R_2$ , there is always either a last number in  $R_1$  or a first in  $R_2$ , but never both.*

For let  $A_1$  and  $A_2$  be the sets of rationals in  $R_1$  and  $R_2$ .

If there be a last number in  $A_1$ , it must also be the last in  $R_1$ ; for between it and any number of  $R_1$  supposed to follow it there would be rationals, in  $R_1$  but not in  $A_1$ , which is impossible.

Similarly, if there be a first number in  $A_2$ , it must also be the first in  $R_2$ .

If there be neither a last in  $A_1$  nor a first in  $A_2$ , there is an irrational  $a$  between the numbers  $A_1$  and the numbers  $A_2$ , § 319. By hypothesis,

$a$  belongs either to  $R_1$  or to  $R_2$ . If  $a$  belongs to  $R_1$ , it is the last number in  $R_1$ ; for between it and any number of  $R_1$  supposed to follow it there would be rationals, in  $R_1$ , but not in  $A_1$ , which is impossible. Similarly if  $a$  belongs to  $R_2$ , it is the first number in  $R_2$ .

Hence in every case there is either a last number in  $R_1$  or a first in  $R_2$ ; and there cannot be both since between them there would be rationals, in neither  $A_1$  nor  $A_2$ , which is impossible.

**321. Approximate values of irrationals.** Let  $a$  denote a given irrational. If any positive rational  $\epsilon$  be assigned, however small, it is always possible to find rationals  $a_1, a_2$ , such that

$$a_1 < a < a_2 \quad \text{and} \quad a_2 - a_1 \leq \epsilon$$

For if we take any rational  $< a$  and repeatedly add  $\epsilon$ , we shall ultimately reach such a pair of rationals  $a_1, a_2$ .

We call  $a_1$  and  $a_2$  *approximate values* of  $a$  to less than  $\epsilon$ .

**322. The number  $-a$ .** Let  $a$  denote any given irrational, and  $a_1, a_2$  any rationals whatsoever such that  $a_1 < a < a_2$ .

Consider the sets of numbers  $-a_2$  and  $-a_1$ . Every rational belongs to one or the other of these sets; we have always  $-a_2 < -a_1$ , and there is no last  $-a_2$  nor first  $-a_1$ . Hence there is a single number, irrational, between all numbers  $-a_2$  and all numbers  $-a_1$ . We call it  $-a$ .

**323. The number  $1/a$ .** Let  $a$  denote any given irrational, and  $a_1, a_2$  any rationals to the same side of 0 as  $a$  and such that  $a_1 < a < a_2$ . By the reasoning in § 322, there is a single number, irrational, between all numbers  $1/a_2$  and all numbers  $1/a_1$ . We call this number  $1/a$ .

**324. Addition and subtraction.** Let  $a$  and  $b$  denote any two real numbers, and  $a_1, a_2$  and  $b_1, b_2$  any rationals such that

$$a_1 < a < a_2 \quad b_1 < b < b_2$$

We have always  $a_1 + b_1 < a_2 + b_2$ ; and there is no last  $a_1 + b_1$  nor first  $a_2 + b_2$ . Hence there is at least one number

between all numbers  $a_1 + b_1$  and all numbers  $a_2 + b_2$ , and it can be shown that there is but one.<sup>1</sup> We call it  $a + b$ .

We define  $a - b$  by the formula  $a - b = a + (-b)$ .

**325. Multiplication and division.** Let  $a$  and  $b$  be any positive real numbers, and  $a_1, a_2, b_1, b_2$  any positive rationals such that  $a_1 < a < a_2$ , and  $b_1 < b < b_2$ . By the reasoning in § 324, there is a single number<sup>2</sup> between all numbers  $a_1 b_1$  and all numbers  $a_2 b_2$ . We call this number  $ab$ .

We define  $a/b$  by the formula  $a/b = a(1/b)$ .

Also, by definition, we make  $a \cdot 0 = 0$ ,  $a(-b) = -ab$ , and so on.

**326. Measurement.** Let  $s$  denote a unit line segment,  $S$  a segment incommensurable with  $s$ , and  $a_1$  and  $a_2$  the rationals which are the lengths, in terms of  $s$ , of any two segments,  $S_1$  and  $S_2$ , commensurable with  $s$ , and less, and greater than  $S$ . There is a single number, an irrational  $a$ , between all the numbers  $a_1$  and all the numbers  $a_2$ ; we call  $a$  the *length of  $S$  in terms of  $s$* . Thus the ratio of the diagonal to the side of a square is  $\sqrt{2}$ .

It is assumed conversely (*axiom of continuity*) that to any given positive irrational  $a$  corresponds an  $S$  whose length in terms of  $s$  is  $a$ .

It may be proved that if the lengths of  $S$  and  $T$  in terms of  $s$  are  $a$  and  $b$ , then the length of  $S$  in terms of  $T$  is  $a/b$ .

**327. Bounded sets of numbers.** Let  $A$  denote a given "bounded set of numbers," that is, a set all numbers of which are between two definite finite numbers  $c$  and  $c'$ .

<sup>1</sup> Were there more than one, there would be rationals between them, and if  $d$  were the difference of two of these rationals, we should have  $(a_2 + b_2) - (a_1 + b_1) > d$  for every  $a_1, a_2$  and  $b_1, b_2$ . But by § 321, we can take  $a_1, a_2$  and  $b_1, b_2$  such that  $a_2 - a_1 < d/2$ ,  $b_2 - b_1 < d/2$   $\therefore (a_2 + b_2) - (a_1 + b_1) < d$ .

<sup>2</sup> Were there more than one, we could find  $d > 0$  such that always  $a_2 b_2 - a_1 b_1 > d$ . But take any  $a_2$  and call it  $a'_2$ , then take  $b_1, b_2$  such that  $b_2 - b_1 < d/2 a'_2$ , and using this  $b_2$ , take  $a_1, a_2$  such that  $a_2 - a_1 < d/2 b_2$ . We then have  $a_2 b_2 - a_1 b_1 = (a_2 - a_1) b_2 + (b_2 - b_1) a_1 < d/2 + a_1 d/2 a'_2 < d$ .



1. *There is a number  $M$  which is either the greatest of the different numbers of  $A$ , or the least of all numbers which are greater than those of  $A$ . We call  $M$  the "least upper bound" of  $A$ .*

2. *There is a number  $m$  which is either the least of the different numbers of  $A$  or the greatest of all numbers which are less than those of  $A$ . We call  $m$  the "greatest lower bound" of  $A$ .*

For let  $R_2$  be the set of numbers greater than those of  $A$ , and  $R_1$  the rest of the real system. By § 320, there is either a greatest number in  $R_1$  or a least in  $R_2$ . This number is the least upper bound  $M$ .

The existence of  $m$  may be proved in a similar manner.

3. *Except when  $M$  is an isolated number of  $A$ , there are numbers of  $A$  between  $M$  and every number  $M - \epsilon$  less than  $M$ . Similarly for  $m$ .*

EXAMPLE. The set  $2, 2\frac{1}{2}, 2\frac{3}{4}, 2\frac{7}{8}, \dots$  has the least upper bound 3 and the greatest lower bound 2. There are numbers of the set between 3 and  $3 - \epsilon$ . But 2 is isolated.

**328. Limits.** 1. *If a variable  $x$  continually increases but remains less than some definite number  $c$ , it approaches a limit; and this limit is either  $c$  or some lesser number.*

2. *If  $x$  continually decreases but remains greater than some definite finite number  $c$ , it approaches a limit; and this limit is either  $c$  or some greater number.*

For in Case 1. the values taken by  $x$  have a least upper bound  $M$ ; and since  $x$  remains less than  $M$  but will ultimately become greater than every number  $M - \epsilon$  which is less than  $M$ , it approaches  $M$  as limit, § 4.

Similarly in Case 2.,  $x \rightarrow m$ , the greatest lower bound of the values of  $x$ .

3. *If  $x$  increases and  $y$  decreases,  $x$  remains less than  $y$ , and  $y - x \rightarrow 0$ , then  $x$  and  $y$  approach the same number as limit.*

For, by 1., 2.,  $x$  and  $y$  approach limits,  $a$  and  $b$ , such that  $x < a \leq b < y$ . But we cannot have  $a < b$ , since in that case  $y - x$  would not become less than  $b - a$ , whereas by hypothesis  $y - x \rightarrow 0$ . Hence  $a = b$ .

**EXAMPLE.** The never ending sequence  $x_1, x_2, \dots, x_p, \dots, x_{p+n}, \dots$  is supposed to have the property that to any given positive number  $\delta$  corresponds a term  $x_p$  such that  $|x_{p+n} - x_p| < \delta$  for  $n = 1, 2, 3, \dots$ . Let  $M_p$  and  $m_p$  denote the least upper and greatest lower bounds of the numbers  $x_{p+n}$  and show that as  $\delta \rightarrow 0$ ,  $M_p, m_p$ , and therefore  $x_p$ , approach a common limit.

## PROPERTIES OF CONTINUOUS FUNCTIONS

### 329. Oscillation of a function $f(x)$ . Continuous functions.

1. Let the set of the values of the function  $f(x)$  in the interval  $(a, b)$  be a bounded set, § 327, and let  $M$  and  $m$  be the least upper and greatest lower bounds of this set. We call  $M - m$  the *oscillation* of  $f(x)$  in  $(a, b)$ .

Using this notion, we may replace the definition of a continuous function in § 16 by that which follows.

2. The function  $f(x)$  is said to be continuous at the point  $x = c$  when  $f(c)$  has a definite value and when to every given positive number  $\epsilon$  corresponds a second positive number  $\delta$  such that in every interval of length  $\delta$  which contains  $c$  the oscillation of  $f(x)$  is less than  $\epsilon$ .

A point  $c$  where these conditions are not both satisfied is called a point of discontinuity of  $f(x)$ .

We say that  $f(x)$  is continuous in the interval  $(a, b)$  if it is continuous at every point of  $(a, b)$ , it being sufficient at  $a$  and  $b$  that there be intervals  $(a, x')$  and  $(x'', b)$  in which the oscillation of  $f(x)$  is  $< \epsilon$ .

3. Since the sets of numbers  $x$  such that

$$a < x \leq b, \quad \text{or} \quad a \leq x < b, \quad \text{or} \quad a < x < b$$

are sometimes called *open intervals*  $(a, b)$ , we shall now call the interval  $(a, b)$  defined in § 2 a *closed interval*.

**330. Theorem 1.** Let  $f(x)$  be continuous in the closed interval  $(a, b)$ , and let  $\epsilon$  denote any given positive number,

*however small. It is always possible to divide  $(a, b)$  into parts in each of which the oscillation of  $f(x)$  is less than  $\epsilon$ .*

For divide  $(a, b)$  into two equal parts, then each of these parts into two equal parts, and so on. The process must ultimately yield parts in each of which the oscillation of  $f(x)$  is less than  $\epsilon$ .

For if not, it must be the case that the oscillation of  $f(x)$  is not less than  $\epsilon$  in at least one of the halves of  $(a, b)$ , call it  $(a_1, b_1)$ , and again in at least one of the halves of  $(a_1, b_1)$ , call it  $(a_2, b_2)$ , and so on without end. Let  $(a_n, b_n)$  be the  $n$ th subinterval thus obtained. As  $n$  increases,  $a_n$  increases (or is stationary),  $b_n$  decreases (or is stationary), and  $b_n - a_n = (b - a)/2^n \rightarrow 0$ ; hence, § 328, 3.,  $a_n$  and  $b_n$  approach a common limit  $c$ . And since  $c$  is in  $(a_n, b_n)$ , and by the supposition made the oscillation of  $f(x)$  in  $(a_n, b_n)$  never becomes less than  $\epsilon$  however great  $n$  is taken,  $f(x)$  is not continuous at  $c$ , § 329, 2.

But this conclusion contradicts the hypothesis that  $f(x)$  is continuous in  $(a, b)$ , and therefore at  $c$ . Hence the process of dividing  $(a, b)$  into halves, and so on, must ultimately yield parts in each of which the oscillation of  $f(x)$  is less than  $\epsilon$ .

**331. Theorem 2.** *If  $f(x)$  is continuous in the closed interval  $(a, b)$ , it is "uniformly continuous" in  $(a, b)$ : that is, to any given positive number  $\epsilon$  corresponds another positive number  $\delta$  such that in every part of  $(a, b)$  of length  $\delta$  the oscillation of  $f(x)$  is less than  $\epsilon$ .*

For, by Theorem 1., we can divide  $(a, b)$  into equal parts in each of which the oscillation of  $f(x)$  is less than  $\epsilon/2$ . Let  $\delta$  denote the length of one of these parts, and let  $(x', x'')$  denote any part of  $(a, b)$  of length  $\delta$ . Since  $(x', x'')$  can overlap at most two of the consecutive equal parts, the oscillation of  $f(x)$  in  $(x', x'')$  is less than  $2(\epsilon/2)$  or  $\epsilon$ .

**EXAMPLE.** Prove that in the open interval  $0 < x \leq 1$ , the function  $f(x) = 1/x$  is continuous but not uniformly continuous.

**332. Theorem 3.** *If  $f(x)$  is continuous in the closed interval  $(a, b)$ , its values in  $(a, b)$  have a least upper bound  $M$  and a greatest lower bound  $m$ . We call  $M$  and  $m$  the least upper and greatest lower bounds of  $f(x)$  in  $(a, b)$ .*

For since we can divide  $(a, b)$  into  $2^n$  equal parts in each of which the oscillation of  $f(x)$  is less than  $\epsilon$ , the oscillation of  $f(x)$  in  $(a, b)$  itself is less than  $2^n \epsilon$ . Hence all the values of  $f(x)$  in  $(a, b)$  are between the two finite members  $f(a) - 2^n \epsilon$  and  $f(a) + 2^n \epsilon$ ; they therefore have least upper and greatest lower bounds  $M$  and  $m$ , § 327.

**333. Theorem 4.** *If  $f(x)$  is continuous in the closed interval  $(a, b)$ , it has a greatest and a least value in  $(a, b)$ : that is, there exist values,  $x_1$  and  $x_2$ , of  $x$  in  $(a, b)$  such that  $f(x_1) = M$  and  $f(x_2) = m$ .*

For divide  $(a, b)$  into parts as in the proof of Theorem 1. Then  $M$  is the least upper bound of  $f(x)$  in at least one of the halves of  $(a, b)$ , call it  $(a_1, b_1)$ , and again in at least one of the halves of  $(a_1, b_1)$ , call it  $(a_2, b_2)$ , and so on without end. Let  $(a_n, b_n)$  denote the  $n$ th subinterval thus obtained, and let  $x_1$  denote the limit which both  $a_n$  and  $b_n$  approach, as  $n \rightarrow \infty$ . Then  $f(x_1) = M$ .

For if this be not true, then since  $M$  and  $f(x_1)$  have definite values,  $M - f(x_1)$  must be some definite positive number  $d$ . But this is impossible since,  $x_1$  being a number in  $(a_n, b_n)$ ,  $M - f(x_1)$  cannot exceed the oscillation of  $f(x)$  in  $(a_n, b_n)$ ; and  $f(x)$  being continuous at  $x = x_1$ , we can take  $n$  great enough to make the oscillation of  $f(x)$  in  $(a_n, b_n)$  less than  $d$ , § 329, 2. Hence  $d = 0$ , that is,  $f(x_1) = M$ .

Similarly there is a number  $x_2$  in  $(a, b)$  such that  $f(x_2) = m$ .

**334. Theorem 5.** *If  $f(x)$  is continuous in the closed interval  $(a, b)$ , and  $f(a)$  and  $f(b)$  have opposite signs, there is a value  $x_0$  of  $x$  in  $(a, b)$  such that  $f(x_0) = 0$ .*

Suppose that  $f(a)$  is  $+$  and  $f(b)$  is  $-$ . Divide  $(a, b)$  into parts as in the proof of Theorem 1. Then, unless  $f(x)$  is 0 at one of the points of division, — in which case the theorem is true, — one of the halves of  $(a, b)$ , call it  $(a_1, b_1)$ , is such that  $f(a_1)$  is  $+$  and  $f(b_1)$  is  $-$ ; again one of the halves of  $(a_1, b_1)$ , call it  $(a_2, b_2)$ , is such that  $f(a_2)$  is  $+$  and  $f(b_2)$  is  $-$ ; and so on without end. Let  $(a_n, b_n)$  denote the  $n$ th sub-interval obtained by this process, and let  $x_0$  denote the limit which both  $a_n$  and  $b_n$  approach, as  $n \rightarrow \infty$ . Then  $f(x_0) = 0$ .

For since  $f(x)$  is continuous at  $x = x_0$ , we have

$$f(x_0) = \lim_{n \rightarrow \infty} f(a_n) = \lim_{n \rightarrow \infty} f(b_n)$$

But since  $f(a_n)$  is always  $+$ ,  $\lim_{n \rightarrow \infty} f(a_n)$  cannot be  $-$ ; and since  $f(b_n)$  is always  $-$ ,  $\lim_{n \rightarrow \infty} f(b_n)$  cannot be  $+$ . Hence both of these limits, and therefore  $f(x_0)$ , must be 0.

**335. Theorem 6.** *If  $f(x)$  is continuous in the closed interval  $(a, b)$ , then between any two values,  $x_1$  and  $x_2$ , of  $x$  in  $(a, b)$ ,  $f(x)$  takes every value between  $f(x_1)$  and  $f(x_2)$ .*

For let  $C$  denote any given number between  $f(x_1)$  and  $f(x_2)$ . The function  $f(x) - C$  is continuous in the interval  $(x_1, x_2)$  and has opposite signs at the ends  $x_1$  and  $x_2$  of the interval. Hence, by Theorem 5, there is a value  $x_0$  of  $x$  in  $(x_1, x_2)$  such that

$$f(x_0) - C = 0 \quad \text{or} \quad f(x_0) = C$$

**EXAMPLE 1.** Show that if  $f(x)$  is continuous in  $(a, b)$  it takes every value between its least and greatest values in  $(a, b)$ .

**EXAMPLE 2.** By applying Theorem 6 to  $f(x) = x^n - a$ , prove that the set of real numbers contains the  $n$ th root of any positive number  $a$ .

## FUNCTIONS OF TWO OR MORE VARIABLES

**336. Oscillation of  $f(x, y)$ .** Let  $S$  denote a given enclosed region in the  $xy$ -plane (Fig. 147); let the values of  $f(x, y)$  at the points of  $S$ , the contour points included, be a bounded

set; and let  $M$  and  $m$  denote the least upper and greatest lower bounds of this set: we call  $M - m$  the *oscillation* of  $f(x, y)$  in  $S$ .

**337. Continuous functions.** Using the notion of oscillation, we may replace the definition of a continuous function  $f(x, y)$  given in § 148 by that which follows.

1. The function  $f(x, y)$  is said to be continuous at the point  $P(a, b)$  of the region  $S$ , when  $f(x, y)$  has a definite value at  $P$  and when to every given positive number  $\epsilon$  corresponds a second positive number  $\rho$  such that in the part of  $S$  within every circle of radius  $\rho$  that encloses  $P$  the oscillation of  $f(x, y)$  is less than  $\epsilon$ .

A point  $P$  where these conditions are not both satisfied is called a point of discontinuity of  $f(x, y)$ .

2. If  $f(x, y)$  is continuous at all points of  $S$ , contour points included, then  $f(x, y)$  is said to be *continuous* in  $S$ .

**338. Theorem 7.** *Let  $f(x, y)$  be continuous within  $S$  and on its contour, and let  $\epsilon$  denote any given positive number. It is possible to divide  $S$  into parts in each of which the oscillation  $f(x, y)$  is less than  $\epsilon$ .*

For, as in Fig. 147, enclose  $S$  in a rectangle  $R$  whose sides are parallel to  $Ox$  and  $Oy$ . Then by parallels to  $Ox, Oy$  divide  $R$  into four equal rectangles. Deal similarly with such of these quarter rectangles as contain points of  $S$ , and so on. This process must ultimately yield parts of  $S$  in each of which the oscillation of  $f(x, y)$  is less than  $\epsilon$ .

For, if not, there is a quarter of  $R$ , call it  $R_1$ , which contains a part of  $S$  in which the oscillation of  $f(x, y)$  is not less than  $\epsilon$ ; the like is true of one of the quarters of  $R_1$ , call it  $R_2$ , and so on without end. But this is impossible. For if  $R_n$  denote the  $n$ th rectangle thus obtained, it is readily shown, by § 328, 3., that as  $n \rightarrow \infty$ , all the points of  $R_n$  approach a definite point  $(a, b)$  in  $R_n$  as limit. And were it true that, as  $n \rightarrow \infty$ , the oscillation of  $f(x, y)$  in the part of  $S$  in  $R_n$  re-

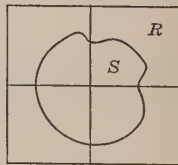


FIG. 147.



mained  $\geq \epsilon$ , then, by § 337, 1.,  $f(x, y)$  would not be continuous at  $(a, b)$ ; whereas, by hypothesis,  $f(x, y)$  is continuous at all points of  $S$  and therefore at  $(a, b)$  which is in  $S$ .

Hence this process must ultimately yield parts of  $S$  in each of which the oscillation of  $f(x, y)$  is less than  $\epsilon$ .

From this theorem, by the reasoning in §§ 331–333, we deduce the following:

**339. Theorem 8.** *If  $f(x, y)$  is continuous within  $S$  and on its contour, it is “uniformly continuous” in  $S$ : that is, to every given positive number  $\epsilon$  corresponds another positive number  $\rho$  such that in every part of  $S$  which can be enclosed in a circle of radius  $\rho$  the oscillation of  $f(x, y)$  is less than  $\epsilon$ .*

**340. Theorem 9.** *If  $f(x, y)$  is continuous within  $S$  and on its contour, it has a greatest and a least value in  $S$ : that is, the set of values of  $f(x, y)$  in  $S$  is a bounded set, and if  $M$  and  $m$  denote the least upper and greatest lower bounds of the set, there are points  $(x_1, y_1)$  and  $(x_2, y_2)$  within  $S$  or on its contour such that  $f(x_1, y_1) = M$  and  $f(x_2, y_2) = m$ .*

This discussion may be extended to functions of more than two variables, as  $f(x, y, z)$ ,  $f(x, y, z, w)$ , and yields theorems for such functions which are the counterparts of those just proved.



### XXX. FUNCTIONS OF A COMPLEX VARIABLE

**341. Complex numbers.** There being no real number whose square is  $-1$ , we invent a new number which we call the *unit of imaginaries* and represent by  $i$ , subsequently formulating a definition of multiplication according to which  $i^2 = -1$  and therefore  $i = \sqrt{-1}$ .

Then, taking any pair of real numbers,  $a$  and  $b$ , we form symbols of the type  $a + bi$ , which we call *complex numbers*, with the understanding that by definition

$$1. a + 0i = a \qquad 2. 0 + bi = bi \qquad 3. 1i = i$$

so that the set of complex numbers contains the set of real numbers  $a$  and the set of *pure imaginaries*  $bi$ , including  $i$ ; and that

$$4. a + bi = c + di \text{ is to mean } a = c \text{ and } b = d$$

**342. The fundamental operations.** If we assume that the elementary rules for reckoning with complex numbers are the same as for real numbers, except that  $i^2 = -1$ , and then seek to express the sum, difference, product, and quotient of two complex numbers, each in the form of a single complex number, we obtain :

$$5. (a + bi) + (c + di) = (a + c) + (b + d)i$$

$$6. (a + bi) - (c + di) = (a - c) + (b - d)i$$

$$7. (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

$$8. \frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$$

the last being got by multiplying numerator and denominator of  $(a + bi)/(c + di)$  by  $c - di$  and simplifying the result.

We make the formulas 5.-8. our *definitions* of the sum, difference, product, and quotient of  $a + bi$  and  $c + di$ . To justify them, it is only necessary to show that they are consistent with 1., 2., 3. and the corresponding definitions for real numbers — which is easily done. That 7. gives  $i^2 = -1$ , we find by setting  $a = c = 0$ ,  $b = d = 1$ , and applying 1.-3.

**343. Graphical representation.** 1. Any complex number  $\alpha = a + bi$  being given, take rectangular axes  $Ox$ ,  $Oy$ , and plot the point  $A$  whose coordinates are  $(a, b)$ . We may represent  $\alpha$  by this point  $A$ .

2. We call  $r = (a^2 + b^2)^{1/2}$  the *absolute value* of  $\alpha$  and write

$$|\alpha| = |a + bi| = (a^2 + b^2)^{1/2}.$$

We call  $\theta = xOA$  the *angle* of  $\alpha$ . Since  $a = r \cos \theta$ ,  $b = r \sin \theta$ ,

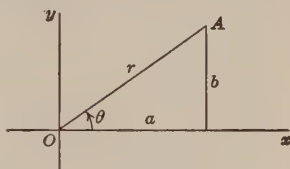


FIG. 148.

$$a + bi = r(\cos \theta + i \sin \theta) \quad (1)$$

3. We may also represent  $a + bi$  by the vector  $OA$ . The factor  $r$  of (1) is the length of  $OA$ . We may call  $\cos \theta + i \sin \theta$  the *direction factor* of  $OA$ , or of  $a + bi$ .

**344. Sums and differences.** Let  $A$  and  $B$  represent  $a + bi$  and  $c + di$ , and construct the parallelogram  $OACB$ . The point  $C$  represents

$a + bi + (c + di)$ . For  
 $OF = OE + EF = OE + OD = a + c$   
 $FC = FG + GC = EA + DB = b + d$   
 Or, representing  $\alpha = a + bi$  and  $\beta = c + di$  by the vectors  $OA$ ,  $OB$ , then, § 83,

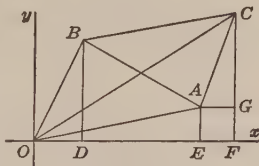


FIG. 149.

$$\alpha + \beta = OA + OB = OA + AC = OC \quad (2)$$

We also have by subtraction of vectors, § 83 (3),

$$\alpha - \beta = OA - OB = BA \quad (2)$$

We may get the point  $\alpha - \beta$  by shifting the initial point of  $B$  of  $BA$  to  $O$ .

If  $\overline{OC}$  denote the length of the vector  $OC$ , and so on, we have  $\overline{OC} < \overline{OA} + \overline{AC} = \overline{OA} + \overline{OB}$ . Hence

$$|\alpha + \beta| \leq |\alpha| + |\beta| \quad (3)$$

the sign = being taken when  $OA$  and  $OB$  have the same direction.

**345. Products, powers, quotients.** 1. By aid of the formulas for  $\cos(\theta + \theta')$  and  $\sin(\theta + \theta')$  in terms of  $\sin \theta$ ,  $\cos \theta$ ,  $\sin \theta'$ ,  $\cos \theta'$ , we obtain

$$r(\cos \theta + i \sin \theta) \cdot r'(\cos \theta' + i \sin \theta') = rr'[\cos(\theta + \theta') + i \sin(\theta + \theta')] \quad (4)$$

*The absolute value of the product of two complex numbers is the product of their absolute values, and its angle is the sum of their angles.*

The same is true of a product of  $n$  factors, as may be shown by repeated applications of (4). When these  $n$  factors are equal, we have

$$[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta) \quad (5)$$

which (when  $r = 1$ ) is called *De Moivre's theorem*.

2. The quotient of  $r(\cos \theta + i \sin \theta)$  by  $r'(\cos \theta' + i \sin \theta')$  is

$$(r/r')[\cos(\theta - \theta') + i \sin(\theta - \theta')] \quad (6)$$

For (6), when multiplied by  $r'(\cos \theta' + i \sin \theta')$ , gives  $r(\cos \theta + i \sin \theta)$ .

3. The  $n$   $n$ th roots of  $r(\cos \theta + i \sin \theta)$  are the  $n$  numbers  $r^{1/n} \left( \cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right)$ ,  $k = 0, 1, 2, \dots, n-1$  (7)

For, by (5), the  $n$ th power of each of these  $n$  numbers is  $r(\cos \theta + i \sin \theta)$

EXERCISE LXIII

1. Construct the points which represent the square roots of  $4 + 3i$ .
2. Find the points which represent the roots of  $x^3 - i = 0$ ; of  $x^5 + 32 = 0$ .
3. Show that  $(a + ib)(c + id)$  is 0 when, and only when,  $a + ib$  or  $c + id$  is 0.
4. By De Moivre's theorem show that  $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$ .
5. Show, by (7), that the three cube roots of 1 are  $1, (-1 \pm i\sqrt{3})/2$ .
6. Since  $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$ , and  $\sin 3\theta$  vanishes when  $\theta = 0$  or  $\pm \pi/3$ , show that  $\sin 3\theta = -4 \sin \theta [\sin^2 \theta - \sin^2(\pi/3)]$ .
7. Show that when  $n$  is odd,  $\sin n\theta$  equals a polynomial of the  $n$ th degree in  $\sin \theta$  which vanishes when  $\theta = 0, \pm \frac{\pi}{n}, \dots, \pm \frac{n-1}{2} \frac{\pi}{n}$ .

**346. The complex variable.** When  $x$  and  $y$  are real variables, we call  $z = x + iy$  a *complex variable*.

Let  $\alpha = a + ib$  denote a given constant, and suppose that  $z$  varies in such a manner that  $|z - \alpha|$ , the distance of the point  $z$  from the point  $\alpha$ , approaches the limit 0; we then say that  $z$  *approaches*  $\alpha$  as *limit*: in other words,

$$z \rightarrow \alpha \quad \text{means} \quad |z - \alpha| \rightarrow 0 \quad (1)$$

The statement  $z \rightarrow \alpha$  implies  $x \rightarrow a, y \rightarrow b$ ; and conversely.

*The theorems respecting limits of sums, products, and quotients of real numbers, § 7, hold good for complex numbers.*

For they depend only on the definition (1) and the relations  $|\alpha + \beta| \leq |\alpha| + |\beta|$ ,  $|\alpha\beta| = |\alpha| \cdot |\beta|$ , which hold good for complex as for real numbers.

**347. Rational functions.** By aid of the fundamental operations, § 342, we may form polynomials in  $z$  with complex coefficients, also fractions whose numerators and denominators are two such polynomials without common factors. Such polynomials and fractions are functions of  $z$  in the sense of § 14. They are called *rational functions, integral or fractional*.

An integral function,  $f(z) = a_0 z^n + \cdots + a_n$ , is *continuous* at every finite point  $z = \alpha$ ; that is, § 16,  $f(\alpha)$  is finite and  $\lim_{z \rightarrow \alpha} f(z) = f(\alpha)$ .

A fractional function  $f(z)$  is continuous except at the points where its denominator vanishes; but if  $\beta$  is such a point, then  $\lim_{z \rightarrow \beta} |f(z)| = \infty$ .

The definition of *derivative*, § 23, applies without modification to rational functions  $f(z)$ ; and we may prove as in § 29, for positive integral values of  $n$ , that  $D_z z^n = n z^{n-1}$ .

**348. Infinite series with complex terms.** Consider the series

$$\begin{aligned} \Sigma u_n = \Sigma(a_n + ib_n) &= (a_1 + ib_1) + (a_2 + ib_2) + \cdots \\ &\quad + (a_n + ib_n) + \cdots \end{aligned}$$

If the real series  $\Sigma a_n$  and  $\Sigma b_n$  converge to the sums  $A$  and  $B$ , then  $\Sigma u_n$  converges to the sum  $A + iB$ . But if either  $\Sigma a_n$  or  $\Sigma b_n$  is divergent, so also is  $\Sigma u_n$ .

*If  $\Sigma |u_n|$  is convergent, then  $\Sigma u_n$  is convergent, and it is said to be absolutely convergent.*

For  $|a_n| \leq (a_n^2 + b_n^2)^{1/2}$  and  $|b_n| \leq (a_n^2 + b_n^2)^{1/2}$ . Therefore since  $\Sigma (a_n^2 + b_n^2)^{1/2}$  converges, so also do  $\Sigma a_n$  and  $\Sigma b_n$ , §§ 168, 176, hence also  $\Sigma u_n$ .

**349. Power series in  $z$ .** This name is given to series of the form  $\Sigma a_n(z - c)^n$ , where  $z$  is a complex variable, and  $c$ ,  $a_0$ ,  $a_1$ ,  $\cdots$  are complex constants.

**350. Theorem 1.** *If the power series  $\Sigma a_n z^n$  converges when  $z = b$ , it converges absolutely when  $|z| < |b|$ .*

For since  $\Sigma a_n b^n$  converges, all its terms are finite; that is, we can find a finite positive number,  $M$ , such that  $|a_n b^n| < M$  for every  $n$ . We then have

$$|a_n z^n| = |a_n b^n| \cdot \left| \frac{z}{b} \right|^n < M \left| \frac{z}{b} \right|^n, \text{ for every } n.$$

Hence each term of  $\Sigma |a_n z^n|$  is less than the corresponding term of the geometric series  $\Sigma M \left| \frac{z}{b} \right|^n$  and therefore converges when  $\Sigma M \cdot \left| \frac{z}{b} \right|^n$  converges, that is, when  $|z/b| < 1$ , or  $|z| < |b|$ .

**351. Corollary.** *If  $\Sigma a_n z^n$  diverges when  $z = b$ , it diverges when  $|z| > |b|$ .*

For were  $\Sigma a_n z^n$  to converge for a value  $z = c$  such that  $|c| > |b|$ , it would also converge for  $z = b$ , § 350.

**352. Circle of convergence.** It follows from §§ 350, 351 that every  $|z|$  for which a given series  $\Sigma a_n z^n$  converges is less than every  $|z|$  for which it diverges. Therefore, unless it never converges or always converges, there is a definite number  $R$  which is either the greatest  $|z|$  for which  $\Sigma a_n z^n$  converges or the least  $|z|$  for which it diverges, § 320. In the  $z$ -plane, draw about  $O$  as center the circle of radius  $R$ , and call it  $O_R$ . At every point inside of  $O_R$ ,  $\Sigma a_n z^n$  converges absolutely, at every point outside it diverges, at a point on the circumference it may converge or diverge. We therefore call  $R$  the *radius* and  $O_R$  the *circle of convergence* of  $\Sigma a_n z^n$ . In case  $\Sigma a_n z^n$  always converges,  $R = \infty$ ; thus,  $\Sigma z^n/n!$ . In case  $\Sigma a_n z^n$  converges only for  $z = 0$ , then  $R = 0$ ; thus,  $\Sigma z^n n!$ .

The series  $\Sigma a_n(z - c)^n$  converges when  $|z - c| < R$ , diverges when  $|z - c| > R$ . Hence its region of convergence is the circle  $C_R$  of radius  $R$  about the point  $z = c$  as center.

**EXAMPLE.** Draw the circles of convergence of  $\Sigma (2z)^n$ ,  $\Sigma z^n/n$ ,  $\Sigma [z - (1 + i)]^n$ .

**353. Functions defined by power series in  $z$ .** At each point  $z$  within its circle of convergence  $C_R$ , a given power series  $\Sigma a_n(z - c)^n$  has a definite sum. Hence  $\Sigma a_n(z - c)^n$  defines a function of  $z$  in  $C_R$ , § 14. This function  $f(z)$  we call *the sum of  $\Sigma a_n(z - c)^n$  in  $C_R$*  and write

$$f(z) = a_0 + a_1(z - c) + \cdots + a_n(z - c)^n + \cdots, \quad |z - c| < R$$

**354. Theorem 2.** Let  $R$  be the radius of convergence of  $\Sigma a_n z^n$ , and let  $\rho$  be any positive number  $< R$ . The function  $f(z) = \Sigma a_n z^n$  is continuous<sup>1</sup> in the circle  $O_\rho$  of radius  $\rho$  about  $O$ .

The proof of this theorem is the same as that for real series, § 181. And by the same reasoning as for real series, § 309, we may prove the following :

**355. Theorem 3.** Let the terms of the series  $\Sigma u_n$  be functions of  $z$  which within the circle  $O_R$  are continuous and numerically less than the corresponding terms of a given positive series  $\Sigma M_n$ ; in any circle  $O_\rho$ ,  $\rho < R$ , the sum  $f(z)$  of  $\Sigma u_n$  is a continuous function of  $z$ .

**356. Theorem 4.** Let  $f(z) = \Sigma a_n z^n$  converge within the circle  $O_R$ . The series  $\Sigma n a_n z^{n-1}$ , got by differentiating  $\Sigma a_n z^n$  term by term, will also converge within  $O_R$ , and its sum will be  $f'(z)$ , the derivative of  $f(z)$ .

For let  $z$  denote any fixed point within  $O_R$ , and  $h = \Delta z$  any variable increment such that  $|z| + |h| < R$ .

Since  $f(z) = \Sigma a_n z^n$  and  $f(z+h) = \Sigma a_n (z+h)^n$  both converge, so also will the series got by subtracting the first from the second and dividing the result by  $h$ , namely

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \Sigma a_n \frac{(z+h)^n - z^n}{h} \\ &= \Sigma a_n \left[ n z^{n-1} + \frac{n(n-1)}{1 \cdot 2} z^{n-2} h + \dots \right] \end{aligned}$$

The terms of this series are continuous functions of  $h$  and they are numerically  $\leq$  the corresponding terms of the convergent positive series

$$\Sigma |a_n| \left[ n |z|^{n-1} + \frac{n(n-1)}{1 \cdot 2} |z|^{n-2} |h| + \dots \right]$$

Hence, by Theorem 3., their sum is a continuous function of  $h$ , and therefore, when  $h \rightarrow 0$ , it approaches as limit the value it has when  $h = 0$ . Hence

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \Sigma n a_n z^{n-1}$$

<sup>1</sup> As in § 16, we say that  $f(z)$  is continuous at  $x = c$  if  $f(c)$  has a definite finite value and if  $f(z) \rightarrow f(c)$  when  $z \rightarrow c$ ; also that  $f(z)$  is continuous in a region  $S$ , if it is continuous at all points of  $S$ . See also § 337.



Hence a function  $f(z)$  defined by a power series has continuous derivatives of every order.

**357. The function  $e^z$ .** 1. For all finite real values of  $z$ , by § 189,

$$e^z = 1 + \frac{z}{1} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots \quad (1)$$

We make (1) our *definition* of  $e^z$  when  $z$  is a complex number. To justify so doing, it is only necessary to show that  $e^z$  as thus defined has the functional properties which characterize  $e^z$  when  $z$  is real, namely :

$$e^z \cdot e^{z_1} = e^{z+z_1} \quad De^z = e^z \quad (2)$$

But by the rule for multiplication of series, § 300, 1., extended to two variables, it is readily shown that  $[\Sigma z^n/n!][\Sigma z_1^n/n!] = \Sigma(z+z_1)^n/n!$ ; which gives, by (1),  $e^z \cdot e^{z_1} = e^{z+z_1}$ . And since  $\Sigma z^n/n!$  reproduces itself when differentiated term by term, we have, by (1) and § 356,  $De^z = e^z$ .

2. When  $z = i\theta$ ,  $\theta$  being real, (1) becomes

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$\text{Hence, § 190,} \quad e^{i\theta} = \cos \theta + i \sin \theta \quad (3)$$

$$\text{when } \theta = 2\pi, (3) \text{ gives } e^{2\pi i} = 1 \quad (4)$$

$$\text{Hence, by (2),} \quad e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z \quad (5)$$

which shows that  $e^z$  is a *periodic function* with the period  $2\pi i$ .

3. It also follows from (3) and § 343 (1), that any complex number  $a + bi$  may be expressed in terms of its absolute value  $r = (a^2 + b^2)^{1/2}$  and its angle  $\theta$  by the formula

$$a + bi = re^{i\theta} \quad (6)$$

4. Hence also, § 343, 3., any *vector* in the  $z$ -plane whose length is  $r$  and whose direction makes the angle  $\theta$  with  $Ox$  may be written  $re^{i\theta}$ .

**358. Logarithms.** 1. Given a complex number  $\alpha = r(\cos \phi + i \sin \phi)$ ,  $r \neq 0$ , it is required to find  $z = x + iy$  such that  $e^z = \alpha$ .

The equation to be solved is

$$e^x(\cos y + i \sin y) = r(\cos \phi + i \sin \phi)$$

which means, § 341, 4.,

$$e^x \cos y = r \cos \phi, \quad e^x \sin y = r \sin \phi.$$

Hence  $e^x = r$ , or  $x = \log r$ ; and  $y = \phi + 2k\pi$ ,  $k$  being any integer.

We call  $z$  the *logarithm* of  $\alpha$  and write it  $\text{Log } \alpha$ . Hence

$$\text{Log } re^{i\phi} = \log r + i(\phi + 2k\pi) \quad (1)$$

Observe that  $\text{Log } re^{i\phi}$  is infinitely many valued. As its *principal value*, we take that for which  $-\pi < \phi + 2k\pi \leq \pi$ .

2. We define  $z^m$ , when  $z$  and  $m$  are complex, by the formula

$$z^m = e^{m \text{Log } z} \quad (2)$$

Hence, except when  $m$  is rational,  $z^m$  has infinitely many values.

**EXAMPLE.** Since  $i = e^{\pi i/2}$ , we have

$$i^i = e^{i \text{Log } i} = e^{i \cdot i(\pi/2 + 2k\pi)} = e^{-(\pi/2 + 2k\pi)}.$$

**359. The functions  $\sin z$ ,  $\cos z$ .** 1. By § 190, when  $z$  is real,

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \quad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \quad (1)$$

We make the equations (1) our definitions of  $\sin z$ ,  $\cos z$  when  $z$  is complex. For it follows from § 357 that  $\sin z$ ,  $\cos z$  as thus defined have all the functional properties which characterize them when  $z$  is real.

Thus, § 357 (1),  $e^{iz} = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)$

$$\text{Hence, by (1),} \quad e^{iz} = \cos z + i \sin z \quad (2)$$

$$\text{Similarly,} \quad e^{-iz} = \cos z - i \sin z \quad (3)$$

$$\text{Solving (2), (3),} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad (4)$$

By (4), we obtain

$$\sin^2 z + \cos^2 z = 1 \quad \sin(z + z_1) = \sin z \cos z_1 + \cos z \sin z_1$$

and so on; also, § 356,

$$D \sin z = \cos z, \quad D \cos z = -\sin z.$$

2. We define  $\tan z$  by the formula  $\tan z = \sin z / \cos z$ ; and  $\cot z$ ,  $\sec z$ ,  $\operatorname{cosec} z$  similarly.

3. It follows from the second of the formulas (4) that

$$\operatorname{arc} \sin z = \frac{1}{i} \operatorname{Log} (iz \pm \sqrt{1 - z^2}) \quad (5)$$

For, if  $w$  satisfies the equation  $\sin w = z$ , we have by (4)

$$z = \frac{e^{iw} - e^{-iw}}{2i} = \frac{e^{2iw} - 1}{2ie^{iw}} \quad \therefore e^{2iw} - 2ize^{iw} - 1 = 0$$

$$\therefore e^{iw} = iz \pm \sqrt{1 - z^2} \quad \therefore w = \frac{1}{i} \operatorname{Log} (iz \pm \sqrt{1 - z^2})$$

#### EXERCISE LXIV

1. Prove that  $\cos 2z = \cos^2 z - \sin^2 z$  when  $z$  is complex.
2. Prove that  $\arctan z = \frac{1}{2i} \operatorname{Log} [(1 + iz)/(1 - iz)]$ . Verify geometrically for the case in which  $z$  is real.
3. If  $z = Re^{i\theta}$ , show that  $e^{iz} = e^{-R \sin \theta} \cdot e^{iR \cos \theta}$ .
4. Reduce each of the following to the form  $re^{i\theta}$ :  
(1)  $e^{2+3i}e^{-1+2i}$  (2)  $2^i$  (3)  $(1 + i)^i$  (4)  $(1 + \sqrt{3}i)^{1+i}$
5. Show that  $z^{1/2}$  changes sign when  $z$  traces a closed curve which includes the origin. What happens to  $\operatorname{Log} z$  when  $z$  traces this curve?
6. A point  $P$ ,  $z = re^{i\theta}$ , is moving in the  $z$ -plane. Find the components of its velocity and acceleration, along and perpendicular to  $OP$ , from  $dz/dt$  and  $d^2z/dt^2$ . Compare § 272.

**360. Analytic functions.** 1. By substituting  $x + iy$  for  $z$ , any of the functions of  $z$  thus far considered may be reduced to the form  $u(x, y) + iv(x, y)$ , where  $u$  and  $v$  denote real functions of  $x, y$ . Thus

$$z^2 = (x + iy)^2 = (x^2 - y^2) + 2xyi.$$

If, as in § 14, we say that  $w$  is a function of  $z$  when to each value of  $z$  there corresponds a definite value of  $w$ , then *every*

one-valued function of  $x, y$  of the form  $w = u + iv$  is a function of  $z$ ; for when a value of  $z$  is given, the values of  $x$  and  $y$  are given, and therefore the value of  $u + iv$  is determined. Thus, if  $w = x + 2yi$ , then when  $z = 3 + 5i$ , we have  $w = 3 + 10i$ .

But nothing is gained by regarding a function  $u + iv$  as a function of  $z$  unless it has a  $z$ -derivative. The functions of  $z$  considered in the preceding sections possess this property, but the function  $w = x + 2yi$  does not. We proceed to show that it is characteristic of the functions  $u + iv$  which have  $z$ -derivatives, that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x} \qquad (1)$$

For, using  $z$  and  $w$  to denote corresponding values of  $z = x + iy$  and  $w = u + iv$ , give  $z$  any increment  $\Delta z = \Delta x + i \Delta y$ , and let  $\Delta w = \Delta u + i \Delta v$  denote the resulting increment of  $w$ . Suppose that for all manners of approach of  $\Delta z$  to 0 as limit, the ratio  $\Delta w / \Delta z$  approaches one and the same limit. We then call that limit the  $z$ -derivative of  $w$  at the point  $z$  and represent it by  $dw/dz$ . If  $\Delta w / \Delta z$  does not approach such a limit, we say that  $w$  has no  $z$ -derivative at the point  $z$ .

*If  $w$  has a  $z$ -derivative at the point  $z$ , then the partial derivatives  $u_x, u_y, v_x, v_y$  exist at the point  $z$  and satisfy the equations (1).*

For among the possible modes of approach of  $\Delta z$  to 0 are those for which  $\Delta z$  takes real values only ( $\Delta z = \Delta x$ ) and those for which it takes pure imaginary values only ( $\Delta z = i \Delta y$ ). Hence the statement that  $dw/dz$  exists at the point  $z$  implies in particular that at  $z$  the limits

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta w}{\Delta x} \quad \text{and} \quad \lim_{\Delta y \rightarrow 0} \frac{\Delta w}{i \Delta y} \quad \text{exist and are equal.}$$

By § 215, these limits exist when, and only when,  $u_x, v_x, u_y, v_y$  exist, and we then have

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{\Delta w}{\Delta x} &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \\ \lim_{\Delta y \rightarrow 0} \frac{\Delta w}{i \Delta y} &= \frac{1}{i} \left[ \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right] = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \end{aligned}$$

Therefore, since these limits must be equal, we have

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \quad \therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

*Conversely, if at the point  $z$  the partial derivatives  $u_x, u_y, v_x, v_y$  exist, are continuous, and satisfy the equations (1), then  $dw/dz$  exists and is continuous at  $z$ . Moreover*

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad (2)$$

For since  $u_x, u_y, v_x, v_y$  exist and are continuous, we have, § 216 (6),

$$\Delta u = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$\Delta v = \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y$$

where  $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \rightarrow 0$  when  $\Delta x, \Delta y \rightarrow 0$ .

But since  $u_x, u_y, v_x, v_y$  satisfy the equations (1), we may replace  $\partial u/\partial y$  by  $-\partial v/\partial x$  and  $\partial v/\partial y$  by  $\partial u/\partial x$ . This having been done, add  $i$  times the second equation to the first, and in the result represent  $\epsilon_1 + i\epsilon_3$  by  $\zeta_1$  and  $\epsilon_2 + i\epsilon_4$  by  $\zeta_2$ . We thus obtain

$$\begin{aligned} \Delta u + i \Delta v &= \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \Delta x + \left( -\frac{\partial v}{\partial x} + i \frac{\partial u}{\partial x} \right) \Delta y + \zeta_1 \Delta x + \zeta_2 \Delta y \\ &= \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) (\Delta x + i \Delta y) + \zeta_1 \Delta x + \zeta_2 \Delta y \end{aligned}$$

$$\text{Hence} \quad \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + \zeta_1 \frac{\Delta x}{\Delta z} + \zeta_2 \frac{\Delta y}{\Delta z}$$

When  $\Delta z \rightarrow 0$ , the last two terms  $\rightarrow 0$ ; for  $\left| \frac{\Delta x}{\Delta z} \right|, \left| \frac{\Delta y}{\Delta z} \right| \leq 1$ , and  $\zeta_1, \zeta_2 \rightarrow 0$ .

$$\text{Therefore} \quad \lim_{z \rightarrow 0} \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

2. A function  $f(z)$  is said to be *analytic* at the point  $z = c$  if it has a continuous derivative  $f'(z)$  at  $z = c$ . If  $f(z)$  is analytic at all points of a region  $S$ , it is said to be analytic in  $S$ . If it is analytic except at certain points of  $S$ , these points are called its *singular points* in  $S$ .<sup>1</sup>

<sup>1</sup> Goursat has shown that if  $f'(z)$  exists at all points of a region  $S$ , it is necessarily continuous, and therefore analytic, in  $S$ . See Goursat-Hedrick, *Mathematical Analysis*, Vol. II, §§ 28, 33.

Thus  $1/(z^2 + 1)$  is analytic except at the singular points  $z = i$  and  $z = -i$ . And, by §§ 354, 356, a function defined by a power series is analytic within the circle of convergence of the series.

By what has just been proved, a function  $u(x, y) + iv(x, y)$  is an analytic function of  $z$  in a region  $S$ , when and only when, throughout  $S$ , the partial derivatives  $u_x, u_y, v_x, v_y$  exist, are continuous, and satisfy the equations  $u_x = v_y, u_y = -v_x$ .

In what follows,  $f(z)$  will denote an analytic function, whether given in the form  $f(z)$  or the form  $u + iv$ . In § 369 it is proved that the class of such functions is the same as that of functions  $f(z)$  defined by power series.

**361. The integral  $\int_{AB} f(z) dz$ .** 1. Suppose that  $f(z) = u + iv$  is analytic in a simply connected region  $R$  of the  $z$ -plane, § 287, 4. Let  $AB$  be any curve arc in  $R$  of the kind described in § 286. Divide  $AB$  into  $n$  parts all of which  $\rightarrow 0$  when  $n \rightarrow \infty$ . Let  $\delta s$  denote any one of these parts, and  $\delta x$  and  $\delta y$  the projections of  $\delta s$  on  $Ox$  and  $Oy$ , and let  $\delta z = \delta x + i \delta y$ ; so that  $|\delta z|$  is the length of the chord of  $\delta s$ . Then, if  $z$  be any point of  $\delta s$ , we have

$$\begin{aligned}\Sigma f(z) \delta z &= \Sigma [u(x, y) + iv(x, y)](\delta x + i \delta y) \\ &= \Sigma (u \delta x - v \delta y) + i \Sigma (v \delta x + u \delta y)\end{aligned}$$

and, by § 286, when  $n \rightarrow \infty$  this sum approaches the limit

$$\int_{AB} (u dx - v dy) + i \int_{AB} (v dx + u dy)$$

Hence, if we represent  $\lim \Sigma f(z) \delta z$  by  $\int_{AB} f(z) dz$ , we have

$$\int_{AB} f(z) dz = \int_{AB} (u dx - v dy) + i \int_{AB} (v dx + u dy) \quad (1)$$

2. Each of the integrals in the second member of (1) is of the type  $\int P dx + Q dy$  where, as follows from § 360 (1),  $\partial Q / \partial x = \partial P / \partial y$ . Hence, § 288, 1., the value of  $\int_{AB} f(z) dz$  is the same for all paths in  $R$  from  $A$  to  $B$ , and it may be represented by  $\int_A^B f(z) dz$ .

**362. The integral  $\int_{z_0}^z f(z) dz$ .** 1. It follows from § 361, 2., that the integral  $\int_{z_0}^z f(z) dz$  taken on any path in  $R$  from the fixed point  $z_0 = a + ib$  to the variable point  $z = x + iy$  is a one valued function of this variable  $z$ . We are to prove that it is an analytic function, and that its derivative is  $f(z)$ .

For set 
$$\int_{z_0}^z f(z) dz = P + iQ$$

where 
$$P = \int_{a,b}^{x,y} u dx - v dy \quad Q = \int_{a,b}^{x,y} v dx + u dy \quad (1)$$

Since  $\partial P/\partial x = u$ , and  $\partial Q/\partial y = u$ , § 288, we have  $\partial P/\partial x = \partial Q/\partial y$ ; and since  $\partial P/\partial y = -v$ , and  $\partial Q/\partial x = v$ , we have  $\partial P/\partial y = -\partial Q/\partial x$ .

Hence  $P + iQ$  is an analytic function, § 360. Call it  $F(z)$ . Then, § 360 (2),

$$F'(z) = \partial P/\partial x + i \partial Q/\partial x = u + iv = f(z)$$

2. If  $f(z) = \Sigma a_n z^n$ , and  $\phi(z) = \Sigma a_n z^{n+1}/(n+1)$ , then

$$\int_{z_0}^z f(z) dz = \phi(z) - \phi(z_0) \quad (2)$$

For set  $F(z) - \phi(z) = W = U + iV$ . Since  $D_z[F(z) - \phi(z)] = f(z) - f(z) \equiv 0$ , we have  $U_z, V_z, U_y, V_y$ , all  $\equiv 0 \quad \therefore \Delta W = 0$ , § 223 (1),  $\therefore W$  is a constant, and therefore  $F(z) = \phi(z) + \text{const.}$  But from this (2) follows as in § 126, 2.

**363. Cauchy's theorem.** Let  $C$  denote any closed curve in the  $z$ -plane, of the kind described in § 287, and  $S$  the region within  $C$ . If the function  $f(z)$  is analytic on  $C$  and in  $S$ , then

$$\int_C f(z) dz = 0 \quad (1)$$

For replacing  $AB$  by  $C$  in § 361 (1), we have

$$\int_C f(z) = \int_C (u dx - v dy) + i \int_C (v dx + u dy)$$

And since each integral in the second member is of the type  $\int P dx + Q dy$  where  $\partial Q/\partial x = \partial P/\partial y$ , each is 0 by Green's formula, § 287, 2.

**364. Corollary.** Let  $S$  be a region bounded by an outer contour  $C$  and an inner contour  $C_1$ . If  $f(z)$  be analytic in  $S$  and on  $C, C_1$ , then

$$\int_C f(z) dz = \int_{C_1} f(z) dz \quad (1)$$

where  $C$  and  $C_1$  are supposed traced in the same sense.



For join  $C$  and  $C_1$  by the line  $ad$ . The integral  $\int f(z) dz$  taken along the single contour  $adefadabca$  is 0 by § 363 (1). The parts along  $ad$  and  $da$  cancel each other; the part along  $abca$  is  $\int_C f(z) dz$ ; and that along  $defd$  is  $-\int_{C_1} f(z) dz$ .

Similarly if there be two interior contours  $C_1$  and  $C_2$ , we have

$$\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz,$$

all three contours being traced in the same sense.

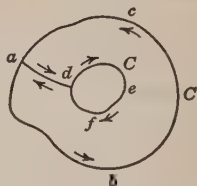


FIG. 150.

**365. The integral  $\int_{\gamma} dz/(z - c)$ .** 1. Let  $c$  be any point in the  $z$ -plane, and  $\gamma$  a circle of radius  $r$  about  $c$  as center. Then

$$\int_{\gamma} \frac{dz}{z - c} = 2\pi i \quad (1)$$

For set  $z - c = re^{i\theta}$ , § 357 (6). Then since  $r$  is constant,

$$dz = re^{i\theta} i d\theta.$$

Hence  $\int_{\gamma} \frac{dz}{z - c} = \int_0^{2\pi} i d\theta = 2\pi i$ , which gives (1).

On the other hand, if  $n$  be any integer  $\neq 1$ , then, § 357 (3),

$$\int_{\gamma} \frac{dz}{(z - c)^n} = \frac{i}{r^{n-1}} \int_0^{2\pi} e^{(-n+1)i\theta} d\theta = 0 \quad (2)$$

It follows from § 364 (1) that in the integrals (1) and (2) we may replace the circle  $\gamma$  by any closed curve  $C$  which contains the point  $c$ .

**366. Residues.** It follows from § 365 (1), (2) and § 362, 2., that if  $c$  be a singular point of  $f(z)$  such that, near  $c$ ,

$$f(z) = \frac{a_{-m}}{(z - c)^m} + \cdots + \frac{a_{-1}}{z - c} + a_0 + a_1(z - c) + \cdots \quad (1)$$

and the contour  $C$  encloses  $c$  and no other singular point of  $f(z)$ , then

$$\int_C f(z) dz = 2\pi i a_{-1} \quad (2)$$

The coefficient  $a_{-1}$  is called the *residue* of  $f(z)$  at  $z = c$ .  
By § 364,

If the contour  $C$  encloses one or more singular points of  $f(z)$ , and if  $\Sigma R_k$  denote the sum of the residues of  $f(z)$  at these points, then

$$\int_C f(z) dz = 2\pi i \Sigma R_k \quad (3)$$

EXAMPLE 1. The function  $f(z) = 3/z + i/z^2 + (2 + 3i)/(z - 2i)$  has the singular points 0 and  $2i$ , with the residues 3 and  $2 + 3i$ . Hence if  $C$  encloses 0 and  $2i$ , then

$$\int_C f(z) dz = 2\pi i [3 + (2 + 3i)] = 2\pi i (5 + 3i)$$

**367. Applications to definite integrals.** The following examples illustrate the use that may be made of the theorems of §§ 363–366 in evaluating certain real definite integrals. But before turning to them, observe that

1. If  $C$  denote a given curve arc or contour,  $L$  the length of  $C$ , and  $M$  the greatest value of  $|f(z)|$  on  $C$ , then  $|\int_C f(z) dz| \leq LM$ .

For  $\int_C f(z) dz = \lim \sum_C f(z) \delta z$ , and  $|\sum_C f(z) \delta z| \leq M \sum_C |\delta z| \leq ML$ , since  $|\sum_C \delta z| \rightarrow L$ , § 361.

2. If  $C$  be a circle of radius  $R$  about the point  $c$  as center, and  $M$  be the greatest value of  $|f(z)(z - c)|$  on  $C$ , then

$$|\int_C f(z) dz| \leq 2\pi M$$

For setting  $z - c = Re^{i\theta}$ , we have

$$\int_C f(z) dz = \int_C f(z)(z - c) \frac{dz}{z - c} = i \int_0^{2\pi} f(z)(z - c) d\theta$$

Hence 
$$|\int_C f(z) dz| \leq M \int_0^{2\pi} d\theta = 2\pi M$$

Therefore if  $M \rightarrow 0$  when  $R \rightarrow 0$  or  $\infty$ , then  $\int_C f(z) dz \rightarrow 0$  when  $R \rightarrow 0$  or  $\infty$ .

EXAMPLE 1. Let  $F(z) = \phi(z)/f(z)$  be a rational fraction in its lowest terms, with real coefficients, and such that the degree of  $f(z)$  exceeds that of  $\phi(z)$  by at least two units and all the roots of  $f(z) = 0$  are imaginary. Let  $\Sigma R_k$  denote the sum of the residues of  $F(z)$  at the roots of  $f(z) = 0$  in the upper half of the  $z$ -plane. Then

$$\int_{-\infty}^{\infty} F(x) dx = 2\pi i \Sigma R_k \quad (1)$$

For draw a semicircle  $ABA'$  great enough to enclose, with  $A'O A$ , all the roots of  $f(z) = 0$  in the upper half of the  $z$ -plane. By § 366 (3), the integral  $\int F(z) dz$  taken along the contour  $A'OABA'$  equals  $2\pi i \Sigma R_k$ . Hence, if  $OA = R$ ,

$$\int_{-R}^R F(x) dx + i \int_0^\pi F(Re^{i\theta}) Re^{i\theta} d\theta = 2\pi i \Sigma R_k$$

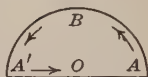


FIG. 151.

Let  $R \rightarrow \infty$ . Then  $\int_{-R}^R F(x) dx \rightarrow \int_{-\infty}^{\infty} F(x) dx$ ; and,  $F(Re^{i\theta})R$  being a proper fraction with respect to  $R$ ,  $\int_0^\pi F(Re^{i\theta}) Re^{i\theta} d\theta \rightarrow 0$ , by § 367, 2. Hence (1).

**EXAMPLE 2.** Prove that  $\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$ .

First consider the integral  $\int \frac{e^{iz}}{z} dz$  taken on the contour  $ABDB'A'EA$ ,

where  $BDB'$  and  $A'EA$  are semicircles with radii  $R$  and  $r$  about  $O$  as center. Since the contour encloses no singular point,

$$\int_r^R \frac{e^{ix}}{x} dx + \int_{BDB'} \frac{e^{iz}}{z} dz + \int_{-R}^{-r} \frac{e^{ix}}{x} dx + \int_{A'EA} \frac{e^{iz}}{z} dz = 0$$



$$\therefore 2i \int_r^R \frac{\sin x}{x} dx = - \int_{A'EA} \frac{e^{iz}}{z} dz - \int_{BDB'} \frac{e^{iz}}{z} dz \quad (1)$$

FIG. 152.

It will be shown that when  $r \rightarrow 0$  and  $R \rightarrow \infty$ , the first term on the right  $\rightarrow \pi i$ , the second  $\rightarrow 0$ .

1. For on  $A'EA$  we have  $z = re^{i\theta}$ , and  $e^{iz} = 1 + (e^{iz} - 1)$ , where  $|e^{iz} - 1| = |iz + \dots| \rightarrow 0$  when  $r \rightarrow 0$ .

$$\text{Hence } \int_{A'EA} \frac{e^{iz}}{z} dz = \int_\pi^0 i d\theta + \int_\pi^0 (e^{iz} - 1) i d\theta \rightarrow -\pi i, \text{ when } r \rightarrow 0.$$

2. On  $BDB'$  we have  $z = Re^{i\theta} = R(\cos \theta + i \sin \theta)$

$$\therefore e^{iz} = e^{-R \sin \theta} \cdot e^{iR \cos \theta}$$

Hence

$$\left| \int_{BDB'} \frac{e^{iz}}{z} dz \right| = \left| \int_0^\pi e^{-R \sin \theta} \cdot e^{iR \cos \theta} i d\theta \right| < \int_0^\pi e^{-R \sin \theta} d\theta = 2 \int_0^{\frac{\pi}{2}} e^{-R \sin \theta} d\theta.$$

$$\text{But when } \theta < \frac{\pi}{2} \text{ we have } \frac{\sin \theta}{\theta} \geq \frac{2}{\pi} \therefore R \sin \theta \geq \frac{2R\theta}{\pi}$$

$$\text{Hence } \int_0^{\pi/2} e^{-R \sin \theta} d\theta < \int_0^{\pi/2} e^{-\frac{2R\theta}{\pi}} d\theta = \frac{\pi}{2R} (1 - e^{-R}) \text{ which } \rightarrow 0 \text{ when } R \rightarrow \infty.$$

Therefore  $\int_{BDB'} \frac{e^{iz}}{z} dz \rightarrow 0$  when  $R \rightarrow \infty$ . Therefore, by (1),

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

**368. Cauchy's formula.** *Let  $f(z)$  denote a function which is analytic at all points on and within a given closed curve  $C$ . The value of  $f(z)$  at any point  $z = c$  within  $C$  may be expressed in terms of its values on  $C$  by the formula*

$$f(c) = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{z - c} \quad (1)$$

For let  $\gamma$  denote a circle of radius  $r$  about  $c$  and within  $C$ . Then, since  $f(z) = f(c) + [f(z) - f(c)]$ , we have, by § 365, (1)

$$\int_C \frac{f(z) dz}{z - c} = \int_\gamma \frac{f(z) dz}{z - c} = 2\pi i f(c) + \int_\gamma \frac{f(z) - f(c)}{z - c} dz$$

But  $f(z)$  being continuous, if any  $\epsilon$  be assigned we can take  $r$  small enough to make  $|f(z) - f(c)| < \epsilon$  at all points on  $\gamma$  and therefore, § 367, 2.,  $\left| \int_\gamma \frac{f(z) - f(c)}{z - c} dz \right| < 2\pi\epsilon$ .

Therefore, since  $\int_C \frac{f(z) dz}{z - c}$  and  $2\pi i f(c)$  are constants, and it has been proved that they differ by less than any number  $2\pi\epsilon$  that may be assigned, they are equal.

**369. Taylor's series.** *Let  $f(z)$  be analytic at the point  $z = c$  and in its neighborhood; let  $R$  be the distance from  $c$  to the nearest singular point or points of  $f(z)$ ; and let  $C_R$  be the circle of radius  $R$  about  $c$  as center. There exists a power series  $\Sigma a_n(z - c)^n$ , and but one, such that  $f(z) = \Sigma a_n(z - c)^n$  at all points  $z$  within  $C_R$ .*

For let  $C$  be the circle with center  $c$  and any radius  $AB < R$ . If  $t$  be any point within  $C$ , then, by § 368, (1),

$$f(t) = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{z - t} \quad (1)$$

Since  $z - t = (z - c) \left[ 1 - \frac{t - c}{z - c} \right]$  we have

$$\frac{1}{z - t} = \frac{1}{z - c} + \frac{t - c}{(z - c)^2} + \cdots + \frac{(t - c)^{n-1}}{(z - c)^n} + \frac{(t - c)^n}{(z - t)(z - c)^n} \quad (2)$$

Substituting this expression for  $1/(z - t)$  in (1), we obtain

$$f(t) = a_0 + a_1(t - c) + \cdots + a_{n-1}(t - c)^{n-1} + R_n \quad (3)$$

where

$$a_0 = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{z - c}, \quad a_1 = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{(z - c)^2}, \quad \dots,$$

$$R_n = \frac{1}{2\pi i} \int_C \frac{f(z)(t - c)^n dz}{(z - t)(z - c)^n}$$

Here  $a_0 = f(c)$ ;  $a_1, a_2, \dots$  have definite finite values; and if  $M$  denote the greatest value of  $|f(z)|$  on  $C$ , then, § 367, 2.,

$$|R_n| < \frac{M \cdot AB}{TB} \left( \frac{AT}{AB} \right)^n, \text{ and therefore } \rightarrow 0 \text{ when } n \rightarrow \infty$$

We therefore have, since  $t$  is any value of  $z$  within  $C_R$ ,

$$f(z) = a_0 + a_1(z - c) + \cdots + a_n(z - c)^n + \cdots, \quad |z - c| < R \quad (4)$$

In § 186 it was proved, for functions of real variables, that if a power series exists of which such a function is the sum, the series is a Taylor series. But the proof depended solely on the theorem on the differentiation of a power series term by term, — which, by § 356, holds good for series in the complex variable also. Hence (4) is a Taylor series: that is, for  $|z - c| < R$ ,

$$f(z) = f(c) + f'(c) \frac{(z - c)}{1} + \cdots + f^{(n)}(c) \frac{(z - c)^n}{n!} + \cdots \quad (5)$$

**370. General conclusions.** 1. It follows from the theorems of § 369 and § 356 that if  $f(z) = u + iv$  is analytic in a region  $S$ , then  $u$  and  $v$  have continuous partial derivatives of every order within  $S$ .

2. The §§ 349–356 also show that, apart from its geometrical interpretations, the entire theory of operations with functions defined by power series which was developed for

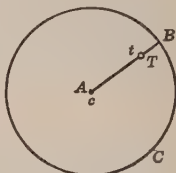


FIG. 153.

real series in §§ 191, 299–310, holds good for complex series also. In particular it follows from § 301 that

(1) *If  $w$  is an analytic function of  $z$  at the “point”  $(z_0, w_0)$  and in its neighborhood, and if  $dw/dz \neq 0$  at  $(z_0, w_0)$ , then  $z$  is an analytic function of  $w$  at  $(z_0, w_0)$  and in its neighborhood.*

(2) *An algebraic equation  $F(z, w) = 0$  defines  $w$  as an analytic function of  $z$  at and in the neighborhood of any point  $(z_0, w_0)$  where  $F$  is 0 and  $\partial F/\partial w \neq 0$ .*

### EXERCISE LXV

1. Show that the radii of the circles of convergence of the series for  $1/(z^2 + 1)$  and  $1/(z^4 + 1)$  in powers of  $z + 1$  are  $\sqrt{2}$  and  $(2 - \sqrt{2})^{1/2}$ .

2. Prove that the Maclaurin series for  $\sec x$  converges when  $|x| < \pi/2$ .

3. Show that the series for  $(x + 2)/(x^2 - 2x + 2)$  in powers of  $x + 1$  converges when  $x$  is between  $-\sqrt{5} - 1$  and  $\sqrt{5} - 1$ .

4. Show that if  $u + iv$  is analytic, then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

5. When  $u + iv$  is analytic,  $u$  and  $v$  are called *conjugate functions*.

Show that if  $u$  is a given function such that  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ , then the

integral  $v = \int_{a,b}^{x,y} -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$  is a function conjugate to  $u$ . Find

a function conjugate to  $u = 2xy + y$ .

6. Let  $z = re^{i\phi}$ . Show that  $\int_1^z (1/z) dz = \log r + i\phi = \log z$ , if the path of integration be from  $A(1, 0)$  to  $B(r, 0)$  on  $Ox$ , and then from  $B$  to  $z$  on the circle of radius  $r$  about  $O$ . Hence show that  $\frac{d}{dz} \log z = \frac{1}{z}$ .

7. Prove that  $\log(1 + z) = z - z^2/2 + z^3/3 - \dots$ ,  $|z| < 1$ .

8. Let  $F(z) = \phi(z)/f(z)$  be a proper fraction in its lowest terms, and let  $c$  be a simple root of  $f(z) = 0$ . Show that the residue of  $F(z)$  at  $z = c$  is  $\phi(c)/f'(c)$ .

9. Find  $\int_C [z^2/(z^3 + 1)] dz$ , when  $C$  is the circle of radius 2 about  $O$ .

10. By the method of § 367, Ex. 1, show that  $\int_{-\infty}^{\infty} 1/(x^4 + 1) dx = \pi/\sqrt{2}$ .

11. By the same method find

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 4}, \quad \int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)}, \quad \int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^2}$$

12. Show that if  $f(z) = a_0(z - z_1)(z - z_2) \cdots (z - z_n)$ , then

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \cdots + \frac{1}{z - z_n}.$$

Hence show that the number  $N$  of the roots  $z_1, z_2, \dots$  of  $f(z) = 0$  within a contour  $C$  (a multiple root of order  $r$  being counted  $r$  times) is given by

$$N = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz.$$

13. Supply the details of the following proof that every algebraic equation  $f(z) = 0$  has a root.

(1) Let  $C$  be a circle of radius  $R$  about  $O$ . By § 367, 2., if  $P$  and  $Q$  be polynomials in  $z$  of degrees  $n - 2$  and  $n$ , then  $\lim_{R \rightarrow \infty} \int_C (P/Q) dz = 0$ .

(2) Hence, if  $f(z)$  be of degree  $n$ , then  $\lim_{R \rightarrow \infty} \int_C \left[ \frac{f'(z)}{f(z)} - \frac{n}{z} \right] dz = 0$ .

(3) But  $\int_C (n/z) dz = 2\pi n i$ , § 365. Hence, § 363,  $f'(z)/f(z)$  has at least one singular point in the  $z$ -plane, and therefore  $f(z)$  at least one zero point.

14. Using the contour in Fig. 151, show that if  $b > 0$ ,

$$\int_{-\infty}^{\infty} \frac{e^{ix} dx}{x - ib} = 2\pi i e^{-b}, \quad \int_{-\infty}^{\infty} \frac{e^{ix} dx}{x + ib} = 0, \quad \therefore \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + b^2} dx = \frac{\pi e^{-b}}{b}.$$



# INDEX

References are to pages

- Acceleration, 27, 90, 92, 266
- Algebra, fundamental theorem of, 13, 418
- Algebraic functions, 14, 35, 360
  - continuity of, 369
  - infinite values of, 370
- Approximate values, 24, 214, 218
- Approximations, successive, 112, 363, 367, 370
- Area, definitions of, 136, 153, 341
- Areas of plane figures, 142, 154
  - of surfaces, 162, 341, 354
- Asymptotes, 52, 80, 97, 372
- Atmospheric pressure, 121
- Attraction of gravitation, 189
- Bounds, least upper and greatest lower, 391, 394
- Cardioid, 78, 102, 161
- Cauchy's formula, 415
- Cauchy's theorem, 411
- Centroids, 178
- Concavity of a curve, 43
- Conics, equations of, 80, 99
- Conjugate point, 96
- Continuity of a function, 9, 168, 238, 392, 396, 404
  - in neighborhood of a point, 238
  - uniform, 137, 168, 393, 397
- Continuity of the set of real numbers, 388
- Continuous functions, properties of, 11, 392
- Convergence of infinite series, 192, 402
  - absolute, 198, 402
  - limit of, 199
  - radius of, 403
  - tests of, 193-198
  - uniform, 375
- Coordinates, transformation of, 237
- Curvature of a curve, 81, 268
  - center of, 83
  - circle of, 84
  - radius of, 82
- Curvature of a surface, 269
- Curves in space, 231, 261, 263, 265, 267, 268
- Curve tracing, 51, 79, 95, 368, 372
- Cusp, 96
- Cycloid, 101, 103, 149
- Cylindrical surfaces, 231
- De Moivre's theorem, 400
- Derivatives, 16, 34, 402, 408
  - directional, 265
  - normal, 266
  - partial, 37, 239, 244
- Determinants, 221-229
  - elimination by, 226
- Differentials, 24, 239
  - exact or total, 239, 292, 317, 339, 349
  - of arc, 81, 264, 353
  - of composite functions, 242
- Differential equations, 120, 283
  - exact, 291, 317
  - geometrical interpretations of, 287, 296, 320, 324
  - homogeneous, 288
  - integrable, 318
  - linear, 290, 301
  - linear with constant coefficients, 303
  - of first order and higher degree, 296
  - of second order, 310
  - partial, 324
  - simultaneous, 322
  - solutions of, 283; their existence, 376
  - singular solutions of, 299

- Differentiation, 18  
   of algebraic functions, 20, 30  
   of composite functions, 32, 241  
   of exponential functions, 72  
   of functions defined by parametric equations, 57  
   of hyperbolic functions, 75  
   of implicit functions, 35, 250, 256  
   of inverse functions, 56  
   of logarithmic functions, 72  
   of sums, products, quotients, 20, 30  
   of trigonometric functions, 62, 67  
 Differentiation, logarithmic, 73  
 Differentiation, partial, 241, 250, 256  
 Differentiation under the integral sign, 331  
 Direction cosines, 232  
 Direction ratios, 233  
 Discontinuity, points of, 10  
  
 Ellipse, 63, 86, 99, 145, 220  
 Ellipsoid, 158  
 Envelopes, 272  
   of surfaces, 274  
 Epicycloid, 101  
 Euler's theorem, 242  
 Evolute, 84, 276  
  
 Fluid pressure, 165  
 Folium of Descartes, 100  
 Fourier series, 378  
 Functional determinant, 252  
 Functions, 8, 168, 238, 401, 403  
   algebraic, 14, 35, 361  
   analytic, 407  
   conjugate, 417  
   continuous, 9, 168, 238, 392, 396  
   exponential, 68, 405  
   gamma, 332  
   hyperbolic, 75  
   homogeneous, 242  
   implicit, 35, 247, 361  
   increasing or decreasing, 23  
   inverse, 56  
   logarithmic, 68, 406  
   rational, 14, 401  
   trigonometric, 61, 65, 406  
 Functions, interdependence of, 257  
  
 Gamma function, 332  
 Gradient of a function, 266  
 Greatest and least values, 45, 279  
  
 Green's formula, 337  
   for space, 346  
  
 Heat, flow of, 328  
 Huygen's formula, 220  
 Hyperbolic paraboloid, 231  
 Hypocycloid, 101, 161  
  
 Implicit functions, 35, 247  
   differentiation of, 36, 250, 256  
   existence theorems for, 248, 250, 253  
   expression by Taylor's series, 252  
 Indeterminate forms, 7, 106, 213  
 Indicatrix, 269  
 Infinitesimals, 3  
 Infinity as limit, 6  
 Inflection, point of, 43  
 Interval, 1, 392  
 Integrals, definite, 138–141  
   change of variable in, 148  
   evaluation of, 141  
   properties of, 145–147  
   with infinite limits or integrand, 149  
    $\int_0^\infty e^{-x^2} dx$ , 344  
    $\int_0^\infty [\sin x/x] dx$ , 414  
 Integrals, double and triple, 168, 174  
   evaluation of, 170, 174  
   properties of, 331  
   transformation of, 340, 357  
 Integrals, indefinite, 114  
   theorems respecting, 115, 116, 135  
 Integrals, line, 335, 410, 411  
 Integrals, surface, 344  
   transformation of, 355  
 Integrating factors, 292, 295, 318, 326  
 Integration, 114  
   by parts, 127  
   by substitution, 117, 129, 130  
   formulas of, 122  
   of irrational functions, 125  
   of rational fractions, 131  
   of trigonometric functions, 124, 128  
  
 Jacobians, 333  
  
 Lagrange's method of multipliers, 282  
 Lemniscate, 79  
 Length of arc, 59, 159, 263  
 Limaçon, 80

- Limits of variables*, 2, 401  
    existence of, 3, 391  
    theorems respecting, 4, 5, 401  
*Line*, equations of, 234  
*Logarithms*, 70, 72, 406  
*Maclaurin's series*, 205, 208, 214  
*Mass center*, 180  
*Matrix*, 227  
*Maxima and minima*, 42, 277  
    of implicit functions, 49, 280  
*Mean values*, 176  
*Mean value theorems*, 104, 107, 207, 247  
    for definite integrals, 146, 331  
*Measurement*, 390  
*Moment of inertia*, 183  
*Motion*, equations of, 311  
    central, 314  
    constrained, 317  
    curvilinear, 87, 266  
    rectilinear, 27  
    simple harmonic, 313; with  
        damping, 314; with forced  
        vibrations, 314  
*Newton's diagram*, 364  
*Newton's method of approximation*,  
    112  
*Node*, 96  
*Numbers*, real, 386  
    complex, 398  
*Orthogonal trajectories*, 120, 293  
*Oscillation of a function*, 137, 168,  
    392, 395  
*Osculating circle*, 106  
*Osculating plane*, 267  
*Pappus*, theorems of, 180  
*Parametric equations of curves*, 57,  
    261  
    of surfaces, 352  
*Pendulum*, simple, 317  
*Plane*, equation of, 235  
*Polar coordinates*, 77  
*Prismoid formula*, 218  
*Projectiles*, motion of, 94, 119  
*Proportional parts*, 215  
*Radius of gyration*, 184  
*Rate of change*, 25, 39  
*Region*, 238  
    simply connected, 338  
*Residues*, 412  
    applications to definite integrals,  
        413  
*Rolle's theorem*, 104  
*Rotation*, 92, 186  
*Series*, infinite, 192  
    alternating, 198  
    convergent, *see* Convergence  
    double, 359  
    with complex terms, 402  
*Series*, power, 199, 402  
    binomial, 203  
    convergence of, 199, 375, 402  
    cosine, 209  
    double, 375  
    exponential, 208  
    logarithmic, 203  
    Maclaurin's, 205  
    sine, 209  
    Taylor's, 205  
*Series*, power, functions defined by,  
    201, 403  
    continuity of, 201, 404  
    differentiation of, 202, 404  
    integration of, 202, 411  
    operations with, 210, 360, 416  
*Simpson's rule*, 218  
*Singular points of curves*, 37, 96, 368  
    of functions, 409  
*Spiral*, logarithmic, 79  
    of Archimedes, 80  
*Stokes's theorem*, 347  
*Subnormal*, 120  
*Subtangent*, 120  
*Surfaces*, equations of, 231, 352  
*Tangents*, 22, 38, 261  
*Tangent planes*, 260  
*Taylor's series*, 205, 208, 415  
    with remainder term, 208, 214  
    for functions of two variables, 246  
*Torsion of space curve*, 268  
*Vallée Poussin*, de la, 217, 263  
*Variable*, 1, 401  
*Vectors*, 87, 89  
*Velocity*, 27, 88, 92, 266  
*Volumes of solids*, 155, 158, 171  
    of revolution, 155  
*Wilson*, E. B., 342  
*Work*, 166, 350



**ANSWERS**  
**TO THE EXAMPLES IN**  
**FINE'S CALCULUS**

## ERRATA

Page 6, last line: for "exceed" read "continually exceed."

Page 49, last line: for "minimum" read "maximum."

Page 161, XXVIII, 2.: replace  $1/2$  by  $0$  and  $3/2$  by  $\pi/4$ .

Page 164, XXIX, 10.: for "arc" read "arch."

Page 173, XXXII, 14: replace  $2 \int_0^{\pi/2} \int_0^{1-\cos \theta}$  by  $\int_0^{2\pi} \int_0^{1-\cos \theta}$

Page 221, 6th line: for "follows it" read "it follows."

Page 236, XLI, 3: replace  $zy -$  by  $xy -$ .

Page 243, (1): replace  $\frac{\partial u}{\partial s}$  by  $\frac{\partial u}{\partial s}$ .

Page 247, lines 21 and 23: replace  $\phi(x), \psi(x), f(x)$  by  $\phi(x, y), \psi(x, y), f(x, y)$ .

Page 254, 3d line from bottom: replace  $\frac{\partial F_2}{\partial u}$  by  $\frac{\partial F_2}{\partial v}$ .

Page 262, line 13: replace § 220 by § 230.

Page 270, XLVI, 2: replace  $x^2 + y^2$  by  $x^2 - y^2$ .

Page 281, XLVIII, 6: for "greatest" read "minimum."

Page 290, 3d line from bottom: replace  $e^4$  by  $e^x$ .

Page 301, LI, 12: replace  $-a^2p^2$  by  $+a^2p^2$ .

Page 317, 1st line: replace  $k^2/r$  by  $k^2r$ .

Page 324, LV, 9: replace  $-3y$  by  $+3y$ .

Page 336, line 18: replace  $\int_C dy$  by  $\int_C f dy$ .

Page 344, 2d line: replace  $y = -x$  by  $y = 0$ .

Page 352, line 17: change the sign of one member of the equation.

Page 394, line 18: replace  $(a, b_2)$  by  $(a_2, b_2)$ .

Page 417, LXV, 10.: replace  $\pi/4$  by  $\pi/\sqrt{2}$ .

## ANSWERS TO EXAMPLES

Page 8

EXAMPLE 2. 1.  $-2$  2.  $15/4$  3.  $\infty$  4.  $1/25$

Page 13

EXAMPLE 3. 2,  $-1$ ,  $-3$ ,  $2/3$

Page 14

### EXERCISE I

10. 1. At  $x = 0, \pm 1$  2. At  $x = 1$  3. At no real point  
 4. At  $x = 6$  5. At  $x = -2$  6. At  $x = 0, -1$ .  
 11. No. 12. At  $x = \pm 1$ .  
 13. At  $x = 0$ , we have  $y = 0$ ,  $\lim_{x \rightarrow 0} y = -1$ ,  $\lim_{x \rightarrow 0} y = 1$   
 15. No. 18.  $1/2, -1$ . 19.  $y = \sqrt{x}$ ,  $y = -\sqrt{x}$ .

Page 18

§ 23. EXAMPLE 2.  $4x^3, -4$

§ 24. EXAMPLE 3. 1.  $-\frac{1}{2\sqrt{1-x}}$  2.  $-\frac{1}{2x\sqrt{x}}$  3.  $\frac{5}{(3x+1)^2}$   
 4.  $-\frac{2}{x^3}$  5.  $\frac{x^2+2x-1}{(x+1)^2}$

Page 21

EXAMPLE 3. 3.  $3x^2 - 2cx + c^2$  4.  $12x^3 + 18x^2 - 48x$   
 5.  $na_0x^{n-1} + (n-1)a_1x^{n-2} + \dots + 2a_{n-2}x + a_{n-1}$   
 6.  $3x^2 - 12x + 11$  7.  $1 + 2x - 3x^2$

Page 22

EXAMPLE 3.  $(3/2, -9/4)$

EXAMPLE 4. Tangents at points where  $x = -1$  and  $x = 0$  are  
 $y + 9x = 2$ ,  $y + 12x = 0$ . Tangents parallel to  $Ox$  are  $y = 16$ ,  
 $y = -16$ .

EXAMPLE 5.  $f'(x) = 6(x-1)(x-2) \therefore = 0$  when  $x = 1$  or  $2$ ;  $> 0$   
 when  $x < 1$  or  $> 2$ ;  $< 0$  when  $1 < x < 2$ .  $f'(x) = 12$  when  
 $x = 0$  or  $3$ .

Page 24

EXAMPLE 2.  $-10.56$  EXAMPLE 3.  $.003$

Page 26

EXAMPLE 3.  $x = 3$  or  $-1$



EXAMPLE 4.  $c = 240/R \therefore$  when  $R = 12$ , then  $dc/dR = -240/R^2 = -5/3$ .

EXAMPLE 5.  $dA/dr = 2\pi r$ ,  $dr/dA = 1/2\sqrt{A}\pi$

EXAMPLE 6.  $A = h^2/\sqrt{3} \therefore dA/dh = 16/\sqrt{3}$  when  $h = 8$ .

### Page 28

EXAMPLE 3. 1. Forward when  $t < 2$  or  $t > 5$ ; backward when  $2 < t < 5$ . When  $t = 3$ , then  $v = -12$  and  $\alpha = -6$ . 2. Backward when  $t < -1$  and when  $0 < t < 1$ ; forward when  $-1 < t < 0$  and when  $t > 1$ . When  $t = -5$ , then  $v = -480$  and  $\alpha = 296$ .  
EXAMPLE 4. When  $t = 20$  ( $v = 384$ ) and when  $t = 44$  ( $v = -384$ )

### EXERCISE II

- (1)  $dy = (12x^3 - x^2/2 + x - 4)dx$  (2)  $dy = (4x^3 - 1)dx$
- Decreases between  $x = -\infty$  and  $x = 0$ , increases between  $x = 0$  and  $x = \infty$ . Tangent at point where  $x = 1$  is  $y - 2x + 1 = 0$ . The angles whose tangents are 3 and  $3/5$ .
- Increases when  $x < -1$  and when  $x > 3$ , decreases when  $-1 < x < 3$ . Tangent at  $(1, -11)$  is  $y + 12x - 1 = 0$ . Points where slope is  $-9$  are  $(0, 0)$ ,  $(2, -22)$
- 27    5. 6.75    6.  $-10.4$     7. .0002    8. .0002 in.
- Ring,  $d\pi r^2 = 2\pi r dr$ . Shell,  $d(4/3)\pi r^3 = 4\pi r^2 dr$
- $3(x+1)(\Delta x)^2 + (\Delta x)^3$ . Value when  $x = 10$ ,  $\Delta x = .1$ , is .331
- $S = 4\pi r^2$ ,  $V = (4/3)\pi r^3 \therefore$  when  $r = 3$ , then  $dS/dr = 24\pi$ ,  $dV/dr = 36\pi$
- When  $t = 4$ , then  $v = 40$ ; when  $v = 10$ , then  $t = 1$ ,  $s = 5$ ;  $\alpha = 10 \text{ ft/sec}^2$
- Initial velocity 3. Begins rolling downward when  $t = 3/2$
- $s = s_1 - s_2 = t^2 - t + 1 \therefore ds/dt$  changes sign when  $t = 1/2$
- When  $s = 8$ , then  $dv/ds = \sqrt{2}$ ; when  $s = 20$ , then  $dv/ds = 2/\sqrt{5}$

### Page 31

- EXAMPLE 3. 1.  $2acx + (bc + ad)$  2.  $4x^3 + 2(a^2 + b^2)x$  3.  $-2c^2/x^3$   
4.  $\frac{-2x^4 + 16x^3 - 15x^2 - 40x + 80}{(x^3 - 10)^2}$  5.  $\frac{-3x^3 + 10x^2 - 21x + 12}{x^3(x-3)^2}$   
6.  $\frac{ad - bc}{(cx + d)^2}$  7.  $4x^3 - 3x^2 - 6x + 11$  8.  $-12x^2 + 40x - 1$

### Page 33, 34

### EXERCISE III

- $3x^2(5 - 3x)^3(5 - 7x)$
- $-(1 + x)^2(1 - 2x)^3(5 + 14x)$
- $-2x(1 + x^2)^2(1 - 2x^2)^4(7 + 16x^2)$
- $7/(3 - t)^2$
- $6 \left( \frac{1 + 2t}{3 - t} \right)^5 \frac{7}{(3 - t)^2}$
- $\frac{2(2x^2 - x)(x^4 + 8x - 2)}{(x^4 + 2)^2}$
- $(1 - x)(1 + x)^2(1 + x - 6x^2)$
- $\frac{2}{3}(1 + 2x)^{-2/3}$
- $\frac{b}{2}(bx + c)^{-1/2}$

10.  $\frac{1}{4}(x + x^2)^{-3/4}(1 + 2x)$  11.  $-\frac{1}{2}(1 - t - 6t^2)^{-1/2}(1 + 12t)$   
 12.  $\frac{-x}{(a^2 - x^2)^{1/2}}$  13.  $\frac{a^2}{(z^2 + a^2)^{3/2}}$  14.  $\frac{1}{4}\left[\frac{1}{(x+2)^{1/2}} - \frac{1}{x^{1/2}}\right]$   
 15.  $-\left[\frac{1-x^2}{1+x^2}\right]^{-1/2} \frac{2x}{(1+x^2)^2}$  16.  $\left[\frac{cx^2+d}{ax^2+b}\right]^{1/2} \frac{(ad-bc)x}{(cx^2+d)^2}$   
 17.  $\frac{1}{4}(1+y)y^{-1/4}\left(\frac{1}{3} + \frac{y}{5}\right)^{-1/2}$  18.  $-\frac{1}{3}x^{-4/3} - \frac{2}{3}x^{-5/3} + 4x^{-3}$   
 19.  $-\frac{4(1+2x)(2+3x)}{(1+3x)^5}$  20.  $\frac{6x+25}{2(x+5)^{1/2}}$  21.  $\frac{4x^2+13x+10}{(x^2+3x+2)^{1/2}}$   
 24.  $\frac{d}{dt}[x^2 + (x+4)^2]^{1/2} = \frac{d}{dx}[x^2 + (x+4)^2]^{1/2} \frac{dx}{dt} = 14.$

## Page 35

EXAMPLE 3.  $1/(1+x^2)^{3/2}$ EXAMPLE 4.  $\frac{d^n y}{dx^n} = \frac{n!}{(1-x)^{n+1}}$ 

EXAMPLE 5. (a)  $\frac{(-)^n \cdot 2 \cdot n! 3^n}{(3x+4)^{n+1}}$  (b)  $\frac{(-)^n 3 \cdot 4 \cdots (n+2)}{(x+3)^{n+3}}$   
 (c)  $\frac{(-)^{n-1}(1 \cdot 3 \cdot 5 \cdots \text{to } n-1 \text{ factors})}{2^n(1+x)^{n-1/2}}$

EXAMPLE 7.  $-\frac{1}{4}(1-x)^{-3/2}x - (1-x)^{-1/2},$   
 $-\frac{3}{8}(1-x)^{-5/2}x - \frac{3}{4}(1-x)^{-3/2}$

## Page 38

EXAMPLE 5. (§ 42)  $\frac{-4x^2 + 6xy - 2y^2 + 4y}{x^2(2y - 3x - 4)} = \frac{-4y}{x^2(2y - 3x - 4)}$

## EXERCISE IV

1. The equations of the tangents are

1.  $x - 13y + 27 = 0$

2.  $2x - 3y = 0$

3.  $11x - 5y - 7 = 0$

4.  $x + 2y - 2 = 0$

2.  $6(x-2) - (y-3) = 0, \quad 4(x-2) + (y-4) = 0$

3.  $(1 + 1/\sqrt{3}, 2/\sqrt{3}), \quad (1 - 1/\sqrt{3}, -2/\sqrt{3})$

4. Slope 0 at  $(1, 5), (-1, -5)$ ; tangents  $y = \pm 5$

Slope  $\infty$  at  $[\sqrt{10}, \frac{1}{2}\sqrt{10}], [-\sqrt{10}, -\frac{1}{2}\sqrt{10}]$ ; tangents  $x = \pm \sqrt{10}$

7.  $\frac{dy}{dx} = -\frac{1}{3}, \quad \frac{d}{dx}OP = -\frac{\sqrt{5}}{3}, \quad \frac{d^2y}{dx^2} = \frac{20}{27}$

## Page 40

## EXERCISE V

1. Decreasing  $3/\sqrt{5}$  ft./sec.

2.  $OP = \sqrt{10}, OQ = 2\sqrt{10}$

3.  $27\pi/8$  cu. ft./min.

4. Decreasing  $5\sqrt{5}$  ft./sec.

5. Approaching  $44/\sqrt{509}$  m./h.

6.  $192/245$  ft./sec.

7. Approaching foot of pole 1.4 ft./sec.  
 8.  $43/\sqrt{97}$  in./sec. 9.  $11600/\sqrt{11601}$  m./h.  
 10.  $1/400\pi$  ft./min.  $1/5$  sq. ft./min.  
 11.  $816\frac{2}{3}$  ft./min. 12. .64 in./min.  
 13.  $5/8$  ft./sec. 14. 15.5 m./h.  
 15.  $\frac{dy}{dt} = -\frac{5}{\sqrt{3}}, \quad \frac{d}{dt}OP = \frac{5}{\sqrt{3}}.$

## Page 43

## EXAMPLE 3.

1. min. = -1 2. max. = 2 3. No max. or min.  
 4. max. = 3, min. =  $-6\frac{1}{2}$  5. min. =  $1\frac{1}{2}$ , max. = 20 6. max. = 1.

## Page 45

- EXAMPLE 3. 1. Turning points  $(-1, -2), (1, 2)$ ; inflection  $(0, 0)$ .  
 2. Minimum point  $(0, 0)$  3. Inflection  $(0, 0)$   
 4. Turning points  $(-1, 0), (0, 1), (1, 0)$ ; inflections  $(\pm 1/\sqrt{3}, 4/9)$   
 5. Turning points  $(\pm \sqrt{3}, 5), (0, -4)$ ; inflections  $(\pm 1, 1)$   
 6. Inflection  $(0, 0)$ .  
 7. Turning points  $(-1, \frac{1}{2}), (2, -7)$ ; inflection  $(\frac{1}{2}, -\frac{1}{2})$   
 8. Turning points  $(-1, 20/3), (3, -4)$ ; inflection  $(1, 4/3)$ .  
 9. Turning point  $(-1/2, -27/16)$ ; inflections  $(0, -1), (1, 0)$

## Page 47

## EXERCISE VI

1. When the point bisects the hypotenuse.  
 3. Length 12 in., width 8 in. 4. 108 sq. ft. 5. Height 9 ft.  
 6. Area is least when  $PQ$  is bisected at  $(2, 3)$ ; area is then 12.  
 7. When  $DE$  is midway between  $A$  and  $BC$ .  
 8. Greatest area is 4. 9. Greatest area is  $\frac{8}{9}\sqrt{6}$ .  
 10. At about 1.34½ P.M. 11.  $7\frac{1}{2}$  weeks.  
 12. To a point 4 m. from nearest point.  
 13. (1) Altitude =  $\frac{2}{3}\sqrt{3}a$  (2) Altitude =  $2\left[\frac{5 - \sqrt{5}}{10}\right]^{1/2}a$   
 (3) Altitude =  $\frac{1}{3}a$   
 14. Altitude of  $K' = a/3$   
 15. When  $P$  is equidistant from the projections of  $A$  and  $B$  on the line. When  $AP$  and  $BP$  are equally inclined to the line.  
 16. When  $BE = BC/\sqrt{2}$ .  
 17. When base of cone is at distance  $\frac{1}{3}[(h^2 + 3r^2)^{1/2} - h]$  below  $C$ .  
 18. When  $EB = \frac{2}{3}a$ . When  $EB = \frac{1}{3}a$   
 19.  $\left(\frac{4}{3}, \pm \frac{\sqrt{14}}{3}\right)$  20.  $15/\sqrt{2}$  ft. 21.  $4\frac{1}{2}$  m./h.  
 22. If points are  $(a, 0), (b, 0), (c, 0)$ , then  $P$  is  $(\frac{1}{3}(a + b + c), 0)$ .  
 24. When  $OP = a/\sqrt{3}$  26. 18 m./h.

## Page 49

EXAMPLE 2. (1) When altitude equals diameter of base. (2) When altitude equals radius of base.

EXAMPLE 3. When altitude equals  $\sqrt{2}$  times radius of base.

EXAMPLE 4.  $(a^{2/3} + b^{2/3})^{3/2}$ .

EXAMPLE 5. When altitude is  $\frac{2}{3}$  of diameter of sphere.

EXAMPLE 7. By the method of Ex. 6 we find the points  $(\pm 1/2\sqrt{2}, 1/4)$ . The value of  $y$  at these points, namely  $1/4$ , is a maximum value.

## Page 56

EXAMPLE 2. 1.  $x = \frac{1}{3}(y + 2)$  2.  $x = y^{1/2}, x = -y^{1/2}$   
3.  $x = 1 + (y + 1)^{1/2}, x = 1 - (y + 1)^{1/2}$  4.  $x = 2y/(1 - y)$ .

## Page 58

## EXERCISE VIII

1.  $1/10$ . Points where  $\frac{dx}{dy} = 7$  are  $\left[\sqrt{\frac{5}{7}}, -\frac{9}{7}\sqrt{\frac{5}{7}}\right], \left[-\sqrt{\frac{5}{7}}, \frac{9}{7}\sqrt{\frac{5}{7}}\right]$
2.  $x = -1 + (y + 4)^{1/2}, x = -1 - (y + 4)^{1/2}$
3. 1.  $x - 40y - 28 = 0$  2.  $3x + 2y - 155 = 0$
4.  $xy - 2x + 1 = 0$
8.  $(-2, 1)$ . Points where  $t = (3 \pm \sqrt{21})/6$ .
9. Angle whose tangent is  $32/15$ .

## Page 63

## EXERCISE IX

1.  $5 \cos 5x, -\frac{1}{3} \sin \frac{x}{3}, 2 \sec^2 (2x - 4), \frac{2}{3} \sec \frac{2x}{3} \tan \frac{2x}{3},$   
 $-6 \operatorname{cosec}^2 6x, -9 \operatorname{cosec} 9x \cot 9x$
2.  $6x^2 \cos 2x^3, -\frac{1}{2\sqrt{x}} \sin \sqrt{x}, \frac{2}{3} x^{-1/3} \sec^2 x^{2/3}, -5x^4 \operatorname{cosec}^2 x^5,$   
 $-\frac{1}{x^2} \tan \frac{1}{x} \sec \frac{1}{x}, \frac{2}{x^3} \operatorname{cosec} \frac{1}{x^2} \cot \frac{1}{x^2}.$
3.  $3 \sin^2 x \cos x, -\frac{1}{2} \frac{\sin x}{\sqrt{\cos x}}, 12 \tan^3 3x \sec^2 3x,$   
 $3 \cot^2 (1 - x) \operatorname{cosec}^2 (1 - x), 24x^3 \sin^6 x^4 \cos x^4,$   
 $-\frac{x}{\sqrt{\cos x^2}} \sin x^2, -\frac{3}{x^2} \tan^2 \frac{1}{x} \sec^2 \frac{1}{x}.$
4.  $x \cos x$ . 5.  $\tan^2 x$  6.  $4 \tan x \sec^2 x$  7.  $-6 \sin 6x$
8.  $\frac{1}{2}(1 - \cos x)^{-1/2} \sin x$  9.  $\frac{1}{2} \sin 4x$  10.  $\cos \frac{2x}{1 - x^2} \cdot \frac{2(1 + x^2)}{(1 - x^2)^2}$

11.  $\sec^2 \sqrt{\frac{1+x}{1-x}} \cdot \sqrt{\frac{1-x}{1+x}} \cdot \frac{1}{(1-x)^2}$
12.  $\sec \frac{1+x}{1-x} \tan \frac{1+x}{1-x} \cdot \frac{2}{(1-x)^2}$
14. (1) Max.  $\sqrt{2}$  at  $x = \pi/4$ , min.  $-\sqrt{2}$  at  $x = 5\pi/4$   
 (2) Max.  $\sqrt{5}$  at  $x = \tan^{-1} 2$ , min.  $-\sqrt{5}$  at  $x = \tan^{-1} 2 + \pi$   
 (3) Max.  $13/4$  at  $\cos^{-1} (2/3)$  and  $2\pi - \cos^{-1} (2/3)$ , max.  $-3$  at  $x = \pi$ , min.  $3$  at  $x = 0$ .  
 (4) Max.  $9/8$  at  $x = \sin^{-1} (1/4)$  and  $x = \pi - \sin^{-1} (1/4)$ , min.  $0$  at  $x = \pi/2$ , min.  $-2$  at  $x = 3\pi/2$ .  
 (5) Max.  $3/2$  at  $x = \pi/6$  and  $x = 5\pi/6$ , min.  $1$  at  $x = \pi/2$ , min.  $-3$  at  $x = 3\pi/2$ .  
 (6) Max.  $\frac{3}{2}\sqrt{3}$  at  $x = \pi/3$ , min.  $-\frac{3}{2}\sqrt{3}$  at  $x = 5\pi/3$ .
16. Both volume and curved surface are greatest when  $\cos 2\theta = 1/3$ , and therefore when altitude of cone is  $2/3$  diameter of sphere.
17. If the adjacent sides  $a, b$  include the angle  $\theta$ , and the sides  $c, d$  include the angle  $\phi$ , then  $2u = ab \sin \theta + cd \sin \phi$ , where  $a^2 + b^2 + 2ab \cos \theta = c^2 + d^2 - 2cd \cos \phi$ .
18. When the angle is 2 radians.
20.  $OXY$  is least ( $= ab$ ) when  $\phi = \pi/4$ .  $XY$  is least ( $= a + b$ ) when  $\tan \phi = \sqrt{\frac{b}{a}}$ .

Page 67

## EXERCISE X

- |                                                                                            |                                                             |                                |
|--------------------------------------------------------------------------------------------|-------------------------------------------------------------|--------------------------------|
| 1. $\frac{1}{\sqrt{a^2 - x^2}}$                                                            | 2. $\frac{-3}{\sqrt{1 - 9x^2}}$                             | 3. $\frac{a}{x^2 + a^2}$       |
| 4. $\frac{-3}{\sqrt{1 - (2 - 3x)^2}}$                                                      | 5. $\frac{2x}{\sqrt{1 - x^4}}$                              | 6. $-\frac{1}{x^2 + 1}$        |
| 7. $\frac{1}{x\sqrt{x^2 - 1}}$                                                             | 8. $\frac{-1}{x\sqrt{x^2 - 1}}$                             | 9. $-1$                        |
| 10. 1                                                                                      | 11. 2                                                       | 12. $\frac{1}{\sqrt{1 - x^2}}$ |
| 13. $\frac{-1}{2\sqrt{x - x^2}}$                                                           | 14. $\frac{1}{\sqrt{-2x(1 + 2x)}}$ , $-\frac{1}{2} < x < 0$ |                                |
| 15. $\frac{1 - 2x}{\sqrt{2x - 6x^2 + 8x^3 - 4x^4}}$                                        | 16. $\frac{1}{2} \sqrt{\frac{1 + \sin x}{\sin x}}$          | 17. $\frac{2}{x^2 + 1}$        |
| 18. $\frac{1}{x^2 + 1}$                                                                    | 19. $\frac{1}{x^2 + 1}$                                     | 20. $\frac{1}{2}$              |
| 21. $y = \sin^{-1} x + 2k\pi$ and $y = (2k + 1)\pi - \sin^{-1} x$ , $k$ being any integer. |                                                             |                                |

22. At  $x = \pm 2\sqrt{3}$ .

23.  $\frac{d}{dx} \cot^{-1} x = -\frac{1}{x^2 + 1}, \quad \frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}},$   
 $\frac{d}{dx} \operatorname{cosec}^{-1} x = -\frac{1}{x\sqrt{x^2 - 1}}$

Page 70

EXAMPLE.  $3/2 - (49/6) \log 2$

Page 74

### EXERCISE XI

1.  $4e^{4x}, ke^{kx}, \frac{1}{n}e^{x/n}, -\frac{1}{2e^{x/2}}, 6xe^{3x^2}, \cos xe^{\sin x}, 2 \sec^2 2xe^{\tan 2x}, e^{\sin^{-1} x}(1 - x^2)^{-1/2}.$
2.  $\frac{1}{x}, \frac{b}{bx + c}, 3, \frac{\frac{1}{2}e^{x/2} + 1}{e^{x/2} + x}, \frac{3 \log^2 x}{x}, \frac{1}{x \log x}.$
3.  $\frac{2(12x^3 + 5)}{3x^4 + 5x - 1}, -\tan x, \frac{3 \sec^2 3x}{\tan 3x}, \frac{1}{\tan^{-1} x}, \frac{1}{x^2 + 1}, \frac{3}{2(3x - 1)}, -\frac{1}{x}.$
4.  $\frac{2a}{a^2 - x^2}$     5.  $\frac{a}{a^2 - x^2}$     6.  $\frac{2x}{1 - x^4}$     7.  $-\operatorname{cosec} x$     8.  $\sec x$
9.  $\frac{1}{(x^2 + a^2)^{1/2}}$     10.  $\frac{1}{2} \frac{2e^{2x} + 1}{e^{2x} + x}$     11.  $\frac{1}{2}(e^{x/a} + e^{-x/a})$     12.  $\frac{2}{(x^2 + a^2)^{1/2}}$
13.  $\frac{-2e^x}{(1 + e^x)^2}$     14.  $x^{n+n-1}[n \log x + 1]$
15.  $\sin x^{\tan 2x}[2 \sec^2 2x \log \sin x + \tan 2x \cot x]$
16.  $(1 - x)^x \left[ \log(1 - x) - \frac{x}{1 - x} \right]$
17.  $\frac{1}{3} \left[ \frac{1}{x} - \frac{2}{1 - x} - \frac{24x^2}{1 - 2x^3} \right] x^{1/3}(1 - x)^{2/3}(1 - 2x^3)^{4/3}$
18.  $\frac{1}{3} \left[ \frac{2x}{x^2 + 1} - \frac{5}{x} + \frac{4}{1 - x} \right] \sqrt[3]{\frac{x^2 + 1}{x^5(1 - x)^4}}$
19.  $\left[ \frac{4}{2x - 3} - \frac{20}{1 - 5x} - \frac{3x^2}{x^3 + 1} - \frac{8x^3}{x^4 + 2} \right] \frac{(2x - 3)^2(1 - 5x)^4}{(x^3 + 1)(x^4 + 2)^2}$
20.  $\sqrt{2}x\sqrt[3]{2-1} + \sqrt[3]{3}x\sqrt[4]{3-1}$
22. (1) Minimum point  $[-1, -1/e]$ , point of inflection  $[-2, -2/e^2]$ , asymptotic to  $Ox$  at  $x = -\infty$ . (2) Maximum point  $[-2, 4/e^2]$ , minimum point  $(0, 0)$ , inflections where  $x = -2 \pm \sqrt{2}$ , asymptotic to  $Ox$  at  $x = -\infty$ .
23. Oscillates about  $O$ , the amplitude of the oscillations decreasing from  $\infty$  to 0 as  $t$  increases from  $-\infty$  to  $\infty$ .

Page 78

EXAMPLE 2.  $\psi = \pi/3$ 

Page 80

## EXERCISE XII

4. 0      5.  $n$  or  $2n$  according as  $n$  is odd or even  
 10. At  $\theta = \pi/2, 3\pi/2$  and the four points where  $\tan^2 \theta = 1/5$   
 11. At the points  $(\pm \pi/3, a/2), (\pi, 2a)$       12.  $x = a, x = -a$

Page 83

EXAMPLE 3. 1.  $\frac{5}{8}\sqrt{5}$       2.  $-1$       3.  $\frac{5}{4}\sqrt{5}$       4.  $2\sqrt{2}$       5.  $\sqrt{2}$   
 6.  $-\frac{7}{4}\sqrt{97}$       7.  $-\frac{3}{16}\sqrt{2}$

Page 85

## EXERCISE XIII

4.  $ds = \left[1 + \frac{a}{x}\right]^{1/2} dx = \left[1 + \frac{y^2}{4a^2}\right]^{1/2} dy$   
 6.  $R = -4\sqrt{2}, \alpha = 5, \beta = -2$   
 7. 1.  $R = 1, \alpha = \pi, \beta = 0$       2.  $R = \frac{7}{8}\sqrt{73}, \alpha = 9.3, \beta = 22\frac{7}{8}$   
 3.  $R = -\sqrt{2}/3, \alpha = 2/3, \beta = 2/3$ .  
 10. Where  $x = \pm 1/\sqrt{45}$       12.  $R = \frac{5}{4}\sqrt{5}$   
 16.  $\frac{e^{k\theta}(1+k^2)^{3/2}}{1-k^2}, \frac{4}{3}\sin\frac{\theta}{2}$   
 20.  $\frac{d\phi}{dt} = -\frac{4}{13^2}\sqrt{13}, \frac{dx}{dt} = \frac{4}{13}\sqrt{39}, \frac{dy}{dt} = \frac{2}{13}\sqrt{13}$

Page 93

## EXERCISE XIV

3.  $v = 40\pi$  ft./sec.,  $a = 320\pi^2$  ft./sec.<sup>2</sup>;  $v_x = -20\pi$  ft./sec.,  
 $v_y = 20\sqrt{3}\pi$  ft./sec.,  $a_x = -160\sqrt{3}\pi^2$  ft./sec.<sup>2</sup>,  $a_y = -160\pi^2$  ft./sec.<sup>2</sup>;  
 $s = 7200\pi$  ft./sec.; retardation =  $2\pi/45$  ft./sec.<sup>2</sup>  
 5. When  $t = 0$ , then  $v = 2, a = 3, a_T = 0, a_N = 3$ . When  $t = \frac{\pi}{4}$ ,  
 then  $v = \frac{1}{2}\sqrt{26}, a = \frac{1}{2}\sqrt{26}, a_T = \frac{5}{8}\sqrt{26}, a_N = \frac{6}{8}\sqrt{26}$ .  
 6. 1.  $x + 2y - 4 = 0$       2.  $y = x^2 - 1$       3.  $y = 3 \sin x/2$ .  
 7.  $v_x = 4, v_y = -3, a_x = 8, a_y = -6$ . Hodograph is  $3x + 4y = 0$ .  
 9.  $v_x = 1/10^{1/2}, v_y = 3/10^{1/2}, a_x = -3/400, a_y = 1/400$ .  
 10.  $v_y = -24/x^2, a_y = 288/x^3$ . At  $(2, 2), v_x = 6, v_y = -6, a_x = 0,$   
 $a_y = 36$ .  
 11.  $20\sqrt{5}, 10\sqrt{5}, 15\sqrt{10}$  m./h.      12.  $(10, 3), (-10, -3)$ .  
 13. (3)  $v = [v_0^2 - 2v_0gt \sin \alpha + g^2t^2]^{1/2}$  is least when  $t = v_0 \sin \alpha/g$   
 $= T/2$       (4)  $v_0 = (52800g)^{1/2}$  ft./sec., about 1300 ft./sec.  
 14. About 24 ft./sec.      15.  $\tan \alpha = 3/4$  or  $11/2$ .



## Page 102

## EXERCISE XV

7. When  $b/a = 2/3$ , three cusps; when  $b/a = \sqrt{2}$ , infinitely many cusps.  
 9. The path is the cycloid  $x = 2(\theta - \sin \theta)$ ,  $y = 2(1 - \cos \theta)$ ; and  $d\theta/dt = 30$  rad./sec.  $v_x = 60(1 - \cos \theta)$  ft./sec.,  $v_y = 60 \sin \theta$  ft./sec.,  $a_x = 1800 \sin \theta$  ft./sec.<sup>2</sup>,  $a_y = 1800 \cos \theta$  ft./sec.<sup>2</sup>. Hence  $v = 120 \sin \theta/2$  ft./sec.,  $\tan \phi = \cot \theta/2$ ;  $a = 1800$  ft./sec.<sup>2</sup>

## Page 105

EXAMPLE 1. 1.  $7/2$  2.  $(7/3)^{1/2}$  3.  $9/4$ .

## Page 110

## EXERCISE XVI

- |       |             |           |             |              |            |
|-------|-------------|-----------|-------------|--------------|------------|
| 1. 1  | 2. $\cos a$ | 3. 1      | 4. 0        | 5. 0         | 6. 0       |
| 7. 2  | 8. $\infty$ | 9. 2      | 10. $-1/40$ | 11. $5/3$    | 12. $-\pi$ |
| 13. 1 | 14. $1/4$   | 15. 0     | 16. 0       | 17. $\infty$ | 18. $1/2$  |
| 19. 1 | 20. 1       | 21. $e^3$ |             |              |            |

## Page 113

EXAMPLE 2. (1) 0,  $\pm 1.8955$  (2) .61925, 1.5111

EXAMPLE 4. 4.4934

## Page 116

EXAMPLE 2.  $\log y$ ,  $\frac{2}{3}y^{2/3}$ ,  $t^9/9$ ,  $\frac{8}{3}x^{3/3}$ ,  $-1/3x^3$ ,  $2\sqrt{x}$

## Page 117

EXAMPLE 3.  $\frac{1}{3}(x^3 + 5)^{5/3}$ ,  $-\frac{1}{2} \log(1 - x^2)$ ,  $-(1 - x^2)^{1/2}$ ,  $\frac{1}{3} \log(x^3 + 3x^2 + 1)$

EXAMPLE 4.  $\frac{1}{6}(4x + 3)^{3/2}$ ,  $-\frac{8}{15}(2 - 5x)^{2/3}$ ,  $\frac{1}{2}(2x + 7)^{11}$ ,  $\frac{1}{8} \log^4 2x$ ,  $x^3/3 + x^2/2 + 6x + 6 \log(x - 1)$

## Page 118

EXAMPLE 2.  $2[x^{1/2} + \log(x^{1/2} - 1)]$ ,  $\frac{2}{3}(x + 1)^{1/2}(x - 5)$ ,  $3 \left[ \frac{x^{2/3}}{2} - 2x^{1/3} + 4 \log(x^{1/3} + 2) \right]$

## EXERCISE XVII

- |                                              |                                      |                                                       |
|----------------------------------------------|--------------------------------------|-------------------------------------------------------|
| 1. $\frac{x^3}{3} + a \frac{x^2}{2} + a^2 x$ | 2. $\frac{x^6}{3} + 5 \frac{x^4}{4}$ | 3. $\frac{x^2}{2} + \frac{4}{3} a^{1/2} x^{3/2} + ax$ |
| 4. $\frac{x^3}{3} + 2x - \frac{1}{x}$        | 5. $\frac{1}{2} \log(2x - 1)$        | 6. $\frac{1}{1 - x}$                                  |
| 7. $-\frac{1}{x} + \log x$                   | 8. $-\frac{2}{3}(4 - 5x)^{1/2}$      | 9. $x^2 - 7x + \log x^3$                              |

10.  $-[x + \log(x-1)^2]$  11.  $\frac{1}{6}(x^4 + a^4)^{3/2}$  12.  $\frac{3}{10a}(ax^2 + b)^{5/3}$   
 13.  $2x^{1/2} + \frac{2}{3}x^{3/2}$  14.  $\frac{1}{3}\log(x^3 + 2)$  15.  $\frac{2}{3}(x^3 + 2)^{1/2}$   
 16.  $\frac{2}{3}(1 + \log x)^{3/2}$  17.  $\frac{x^2}{2} - 3\log(x+2)$   
 18.  $\frac{x^3}{3} + \frac{3x^2}{2} + 9x + 27\log(x-3)$  19.  $\frac{3}{4b}(bx + c)^{4/3}$   
 20.  $\frac{\sin^3 x}{3}$  21.  $-\frac{1}{4}(x^2 + 2)^{-2}$   
 22.  $(\frac{2}{3}t^2 - \frac{4}{3}t + 2)t^{1/2}$ ,  $t = x + 1$  23.  $(2x + 5)^{3/2} \frac{3x - 5}{15}$   
 24.  $2\left[\frac{t^9}{9} + \frac{3t^7}{7} + \frac{3t^5}{5} + \frac{t^3}{3}\right]$ ,  $t = (x-1)^{1/2}$ .

Page 121

## EXERCISE XVIII

1.  $y = x^3 - 4x + 4$   
 2. (a)  $3y = x^3 + 1$  (b)  $y + 1 = 4e^{x-x^2/2}$  (c)  $xy = 6$   
 3. (a)  $x^2 + y^2 = C$  (b)  $y^2 + 2x^2 = C$  (c)  $x^2 - y^2 = C$  (d)  $xy = C$   
 4.  $x^{1/2} - y^{1/2} = C$  5.  $\frac{1}{3}y^3 = \frac{1}{2}x^2 + 7$  6. 8,  $\sin^{-1}(1/4)$   
 7. 1825 yrs. 8.  $h = \frac{\log 30 - \log 25}{\log 30 - \log 28.87} 1000 \text{ ft.} = 4750 \text{ ft.}$

Page 123

## EXERCISE XIX

1.  $\cos(2-x)$  2.  $-\frac{1}{2}\cos x^2$  3.  $\frac{1}{4}\tan 4x$   
 4.  $-\frac{1}{3}\operatorname{cosec} 3x$  5.  $\frac{1}{3}\log \sin 3x - \frac{1}{2}\log \cos 2x$   
 6.  $\frac{1}{a}e^{ax+b}$  7.  $-e^{\cos x}$  8.  $2(e^{x/2} - e^{-x/2})$   
 9.  $\frac{2}{3}(1 + e^x)^{3/2}$  10.  $\frac{2^x}{\log 2}$  11.  $\frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2}$   
 12.  $-\frac{1}{3}\cos^3 3x$  13.  $\frac{1}{2}\log(\tan x + 1)^2$  14.  $\tan x - \cot x$   
 15.  $\frac{\sqrt{6}}{12}\log \frac{3x - \sqrt{6}}{3x + \sqrt{6}}$  16.  $\frac{1}{6}\tan^{-1} \frac{2x}{3}$  17.  $\sin^{-1} \frac{x}{2}$   
 18.  $\frac{1}{3}\sin^{-1} \frac{3x}{2}$  20.  $\frac{1}{2}\tan^{-1} x^2$  21.  $\frac{1}{2}\log[x^2 + \sqrt{x^4 + 1}]$   
 23.  $x - \log(e^x + 1)$  24.  $\operatorname{cosec} x - \cot x$

Page 124

- EXAMPLE 2. 1.  $\frac{1}{2}\sin^2 x$ , 2.  $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x$  3.  $3\sin^{1/3} x$   
 4.  $\sec x + \cos x$

$$\text{EXAMPLE 4. } -\frac{1}{2} \left( \frac{\cos 6x}{6} + \frac{\cos 4x}{4} \right), \quad \frac{1}{2} \left( \frac{\sin 5x}{5} + \sin x \right), \\ \frac{1}{2} \left( \sin x - \frac{\sin 5x}{5} \right)$$

## Page 125

$$\text{EXAMPLE 5. } \tan x + \frac{\tan^3 x}{3}, \quad \frac{\tan^6 x}{6} + \frac{\tan^4 x}{4}, \quad -\frac{\operatorname{cosec}^3 x}{3}$$

$$\S 113, \text{ EXAMPLE 3. } 1. \tan^{-1}(x+2)$$

$$2. \log [x+2+(x^2+4x+5)^{1/2}] \quad 3. \sin^{-1} \frac{x-1}{2}$$

## Page 126

$$\text{EXAMPLE 6. } 2 \log (x^2+4x+5) - \tan^{-1}(x+2), \\ 4(x^2+4x+5)^{1/2} - \log [x+2+(x^2+4x+5)^{1/2}]$$

## EXERCISE XX

1.  $\frac{1}{5} \cos (4-5x)$
2.  $\frac{x^{e+1}}{e+1} + e^x$
3.  $\frac{2}{7} \cos^{7/2} x - \frac{2}{3} \cos^{3/2} x$
4.  $\operatorname{cosec} x - \frac{1}{3} \operatorname{cosec}^3 x$
5.  $\frac{1}{5} \sec^5 x$
6.  $\frac{\tan^3 x}{3} - \tan x + x$
7.  $-\frac{1}{2} \left[ \frac{\cos 15x}{15} + \frac{\cos 5x}{5} \right]$
8.  $a \tan x + b \sec x$
9.  $\frac{1}{b} \sin^{-1} \frac{bx}{a}$
10.  $\frac{2}{\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}}$
11.  $\frac{1}{4} \log \frac{x-6}{x-2}$
12.  $5(x^2+4x)^{1/2} - 4 \log [x+2+(x^2+4x)^{1/2}]$
13.  $\frac{p}{ab} \tan^{-1} \frac{bx}{a} + \frac{q}{2b^2} \log (a^2+b^2x^2)$
14.  $\sin^{-1} \frac{2x-3}{3}$
15.  $\sin^{-1} (2x-1)$
16.  $\frac{3}{4} \log (2x^2-3x+1) - \frac{7}{4} \log \frac{x-1}{2x-1}$
17.  $4(1+4x-x^2)^{1/2} - 3 \sin^{-1} \frac{x-2}{\sqrt{5}}$
18.  $\sin^{-1} \frac{x}{2} - 2(4-x^2)^{1/2}$
19.  $\sin^{-1} x + (1-x^2)^{1/2}$
20.  $(x^2+ax)^{1/2} + \frac{a}{2} \log \left[ x + \frac{a}{2} + (x^2+ax)^{1/2} \right]$

## Page 128

$$\text{EXAMPLE 5. } 1. e^x(x-1) \quad 2. e^x(x^2-2x+2)$$

$$3. e^{3x}(9x^3-9x^2+6x+2)/27 \quad 4. \frac{x^4}{4} \log x - \frac{x^4}{16}$$

$$5. x \tan^{-1} x - \log (x^2+1)^{1/2} \quad 6. x \sin^{-1} x + (1-x^2)^{1/2}$$

$$7. -\frac{x \cos 2x}{2} + \frac{\sin 2x}{4} \quad 8. \frac{e^x}{2} (\sin x - \cos x)$$

## Page 129

EXAMPLE 2.

$$\begin{aligned}
 \int \cos^5 x \, dx &= \frac{1}{5} \cos^5 x \sin x + \frac{4}{5} \cos^3 x \sin x + \frac{8}{15} (\cos x \sin x + x) \\
 \int \sin^5 x \, dx &= -\frac{1}{5} \sin^4 x \cos x - \frac{4}{15} \sin^2 x \cos x - \frac{8}{15} \cos x \\
 \int \cos^3 2x \, dx &= \frac{1}{3} \cos^2 2x \sin 2x + \frac{1}{3} \sin 2x \\
 \int \sin^2 x \cos^4 x \, dx &= -\frac{1}{5} \cos^5 x \sin x + \frac{1}{24} \cos^3 x \sin x \\
 &\quad + \frac{1}{16} (\cos x \sin x + x)
 \end{aligned}$$

## Page 130

EXAMPLE 5. 1.  $5 \log \left[ \frac{5}{x} - \frac{\sqrt{25-x^2}}{x} \right] + \sqrt{25-x^2}$

2.  $\frac{1}{\sqrt{3}} \log \frac{\sqrt{3} - \sqrt{3-x^2}}{x}$

3.  $\frac{(x+1)(3-2x-x^2)^{1/2}}{2} + 2 \sin^{-1} \frac{x+1}{2}$  4.  $\frac{1}{24} \frac{x^3+6x}{(x^2+4)^{3/2}}$

5.  $\frac{1}{2} \log \frac{\sqrt{x^2+4}-2}{x}$  6.  $\sqrt{x^2-1} - \sec^{-1} x$

7.  $\frac{1}{2} \left[ \frac{\sqrt{x^2-1}}{x^2} + \sec^{-1} x \right]$

§ 117, EXAMPLE 2.

1.  $\log \frac{(x^2+1)^{1/2}-1}{x}$  2.  $\log \frac{t-1-\sqrt{t^2-2t-1}}{2}, t = \frac{1}{x-1}$

3.  $-\frac{(x^2+x+1)^{1/2}}{x} + \frac{1}{2} \log \left[ \frac{x+2+2\sqrt{x^2+x+1}}{x} \right]$

## Page 131

## EXERCISE XXI

1.  $-\frac{x}{3} \cos 3x + \frac{\sin 3x}{9}$  2.  $\left( \frac{x^2}{2} - \frac{1}{4} \right) \sin 2x + \frac{x}{2} \cos 2x$

3.  $x \tan x + \log \cos x$  4.  $x \tan^{-1} \frac{x}{2} - \log(x^2+4)$

5.  $\frac{x^3}{3} \tan^{-1} x - \frac{1}{6} [x^2 - \log(x^2+1)]$  6.  $(x+1) \tan^{-1} \sqrt{x} - \sqrt{x}$

7.  $\left( \frac{x^2}{2} - \frac{1}{4} \right) \sin^{-1} x + \frac{1}{4} x \sqrt{1-x^2}$

8.  $\frac{x^3}{3} \sin^{-1} x + \frac{1}{9} (x^2+2)(1-x^2)^{1/2}$  9.  $e^{4x} \left( \frac{x^2}{4} - \frac{x}{8} + \frac{1}{32} \right)$

10.  $-\frac{1}{x} (\log x + 1)$  11.  $x \log(x^2-1) - 2x + \log \frac{x+1}{x-1}$

12.  $\frac{e^{3x}}{13} [3 \cos 2x + 2 \sin 2x]$

13.  $\frac{1}{4}(4 - x^2)^{3/2}x + \frac{3}{2}(4 - x^2)^{1/2}x + 6 \sin^{-1} \frac{x}{2}$
14.  $\frac{x}{(x^2 + 1)^{1/2}} - \frac{1}{3} \frac{x^3}{(x^2 + 1)^{3/2}}$       15.  $\frac{x}{(x^2 - 1)^{1/2}} - \frac{1}{3} \frac{x^3}{(x^2 - 1)^{3/2}}$
16.  $\frac{1}{\sqrt{2}} \log \frac{(x-1)}{x+1 + (2x^2+2)^{1/2}}$       17.  $\sin^{-1} \left( \frac{\sin x}{\sqrt{2}} \right)$
18.  $-\frac{x}{(x^2+1)^{1/2}} + \log [(x^2+1)^{1/2} + x]$       19.  $\sin^{-1} \frac{x}{\sqrt{2x^2+2}}$

Page 134

## EXERCISE XXII

1.  $\frac{x^2}{2} - 3x + \log \frac{(x+2)^3}{x+1}$       2.  $\log \frac{(x-1)(x-2)^2}{(x-3)^3}$
3.  $\log \frac{(x-1)^3(x+1)}{x^4}$
4.  $\frac{a^2}{(a-b)(a-c)} \log(x-a) + \frac{b^2}{(b-a)(b-c)} \log(x-b)$   
 $+ \frac{c^2}{(c-a)(c-b)} \log(x-c)$
5.  $-\log x + \frac{1}{4} \log(x+3) + \frac{11}{12} \log(2x-1)$
6.  $x - \log(x+2)^4 - \frac{5}{x+2}$       7.  $\frac{1}{4} \log \frac{x-2}{x} - \frac{3}{2} \frac{1}{x-2}$
8.  $\log x + \log(x^2+x+1) - \frac{4}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}}$
9.  $\frac{1}{32} \log \frac{x-2}{x+2} - \frac{1}{16} \tan^{-1} \frac{x}{2}$
10.  $\frac{1}{3} \log(x+2) + \frac{4}{3} \log(x-2) - \frac{4}{3} \log(x-1) - \frac{1}{3} \log(x+1)$
11.  $\log(x-3) - \frac{1}{16} \log x + \frac{17}{16} \log(x+4) + \frac{3}{4x}$
12.  $\frac{1}{4} \log \frac{x+1}{x-1} - \frac{1}{2} \frac{x}{x^2-1}$
13.  $\frac{1}{2} \log(x^2+2x+2) + \frac{3x+2}{x^2+2x+2}$
14.  $\frac{1}{243} \left\{ \frac{27x}{4(x^2+9)^2} + \frac{9x}{8(x^2+9)} + \frac{3}{8} \tan^{-1} \frac{x}{3} \right\}$
15.  $6 \log t - 3 \log(t^2+t+1) + 2\sqrt{3} \tan^{-1} \frac{2t+1}{\sqrt{3}}, \quad t = x^{1/6}$

Page 135

EXAMPLE.  $\frac{2}{\sqrt{3}} \tan^{-1} \frac{\tan(x/2)}{\sqrt{3}}, \quad \frac{1}{5} \log \frac{2 \tan(x/2) + 1}{2 \tan(x/2) - 4}$

## Page 142

## EXERCISE XXIII

- |                     |                                     |                              |
|---------------------|-------------------------------------|------------------------------|
| 1. $-10\frac{1}{2}$ | 2. 2                                | 3. $\frac{1}{2}(1 - e^{-8})$ |
| 4. $\frac{\pi}{4}$  | 5. $\sqrt{3} + 1 - \frac{7\pi}{12}$ | 6. $\frac{\pi}{2} - 1$       |
| 7. $\frac{\pi}{8}$  | 8. $\frac{5\pi}{12}$                | 9. 0                         |
| 10. $4/15$          | 11. $57\frac{2}{3}$                 | 12. 0                        |

## Page 143

EXAMPLE 2.  $10\frac{2}{3}$ ,  $24\frac{2}{3}$ 

EXAMPLE 3. 72

## Page 144

## EXERCISE XXIV

- |                                 |                                          |                    |                    |
|---------------------------------|------------------------------------------|--------------------|--------------------|
| 1. 36                           | 2. $20\frac{5}{8}$                       | 3. $6\frac{3}{4}$  | 4. $13\frac{1}{4}$ |
| 5. $11\frac{1}{8}$              | 6. $16/15$                               | 7. $1/3$           | 8. $4\frac{1}{2}$  |
| 9. $4 - 3 \log 3$               | 10. $10\frac{2}{3}$                      | 11. $2\frac{2}{3}$ | 12. $2\frac{2}{3}$ |
| 13. 12                          | 14. 8                                    | 15. 12             | 16. $7/6$          |
| 17. $a^2/3$                     | 18. $16a^2/3$                            | 19. $4\frac{1}{2}$ | 20. $4\frac{1}{2}$ |
| 21. 4                           | 22. (1) $(8/15)a^{5/2}$ , (2) $(4/3)a^3$ | 23. 12             |                    |
| 26. $\frac{a^2}{2}(e - e^{-1})$ | 27. $a^2/6$                              |                    |                    |

## Page 146

EXAMPLE 7. We have

$$\int_a^b [M - f(x)]\phi(x)dx > 0 \quad \therefore M \int_a^b \phi(x)dx > \int_a^b f(x)\phi(x)dx$$

Similarly,

$$\int_a^b [f(x) - m]\phi(x)dx > 0 \quad \therefore \int_a^b f(x)\phi(x)dx > m \int_a^b \phi(x)dx$$

## Page 149

EXAMPLE 3.  $7\frac{7}{15}$ EXAMPLE 5.  $ab\pi$ EXAMPLE 6.  $a^4\pi/4$ 

## Page 152

## EXERCISE XXV

- |           |                |               |                         |
|-----------|----------------|---------------|-------------------------|
| 12. 0     | 13. $10/3$     | 14. $\pi/4$   | 15. $\frac{\pi}{2} - 1$ |
| 16. $2/3$ | 19. $4\pi a^2$ | 20. $8/3$     | 21. $2\pi$              |
| 22. 4     | 23. $3\pi$     | 24. $\pi a^2$ | 25. $3a^2\pi$           |

Page 154

## EXERCISE XXVI

1.  $\frac{a^2}{2} \left[ \frac{\pi}{4} + 1 - \sqrt{3} \right]$
2.  $\frac{1}{16} a^2 (\pi + 2)$
3.  $\frac{a^2}{2} \left[ \frac{1}{\sqrt{3}} - \frac{\pi}{6} \right]$
4.  $\frac{4}{3} a^2 \pi^3$
5.  $\frac{1}{8} (e^{2\pi} - 1)$
6.  $a^2$
7.  $\frac{1}{12} a^2 \pi$
8.  $\frac{1}{8} a^2 \pi$
9.  $\frac{4}{3} a^2$
10.  $\frac{9}{2} \pi$
11.  $\frac{1}{8} \pi - \frac{1}{4}$
12.  $\frac{1}{32} [5\pi - 9\sqrt{3}]$

Page 157

## EXERCISE XXVII

1.  $V_x = 6\pi, V_y = 4\pi$
2.  $2\pi [12\sqrt{2} - \frac{32}{3}]$
3.  $\frac{62}{15} \pi$
4.  $64\pi$
5.  $\frac{1}{12} \pi$
6.  $2\pi [12\sqrt{2} - \frac{32}{3}]$
7.  $\frac{512}{3} \pi$
8.  $\frac{1}{6} \pi$
9.  $\frac{62}{15} \pi$
10.  $\frac{1}{2} \pi^2$
11.  $\frac{1}{2} \pi^2$
12.  $\frac{20}{3} \pi$
13.  $4\pi^2 a^3$
14.  $\pi a^3 \left[ \frac{2}{3} - \frac{5\sqrt{2}}{12} \right]$
15.  $\frac{27}{4} \pi$
16.  $5a^3 \pi^2$
17.  $2\pi$
18.  $\frac{64}{5} \pi$
19.  $\frac{5}{2} \pi$
20.  $2\pi$
21.  $\frac{8}{3} a^3 \pi$
22.  $4\pi$
23.  $2\pi [8(\log 2)^2 - 8 \log 2 + 3]$
24.  $2[3\sqrt{3}\pi - \frac{4}{3}\pi^2]$
25.  $24\pi$
26.  $14\pi$
27.  $a^3 \pi^2$
28.  $2\pi^2$
29.  $\frac{128}{15} \sqrt{2} \pi$
30.  $\left[ \frac{5}{3} - \frac{\pi}{2} \right] a^3 \pi$
33.  $\left( a - \frac{h}{3} \right) h^2 \pi$ ,  $a$  = radius of sphere,  $h$  = height of segment.

Page 159

EXAMPLE 3.  $25\sqrt{2}\pi$ EXAMPLE 5.  $\frac{4}{3}\sqrt{6}$ 

Page 161

## EXERCISE XXVIII

1.  $38/3$
2. The arc length between  $x = 0$  and  $x = \pi/4$  is  $\log(\sqrt{2} + 1)$ .
3.  $\frac{2}{3} \pi$
4.  $\frac{3}{2} \pi$
5.  $2 \log 3 - 1$
6.  $2 - \pi/4$
7.  $\frac{1}{2} [1 + 4a^2 x^2]^{1/2} x + \frac{1}{4a} \log [2ax + (1 + 4a^2 x^2)^{1/2}]$
8.  $3\sqrt{13} + 4 \log \frac{3 + \sqrt{13}}{2}$
9.  $\left[ e - \frac{1}{e} \right] \frac{a}{2}$
10.  $6a$
11.  $\sqrt{5} - \sqrt{2} - \log \frac{\sqrt{5} + 1}{2} + \log(\sqrt{2} + 1)$
12.  $15\frac{1}{3}$
13.  $4\sqrt{3}a$



14.  $\frac{4}{3}\sqrt{3}$       15.  $\sqrt{2}(e^{2\pi} - 1)$       16.  $a\pi/4$       17.  $2a$   
 18.  $(a^2 + b^2)^{1/2}\pi$       19.  $a\pi^2/2$       20.  $8a$

Page 164

## EXERCISE XXIX

1.  $(10^{3/2} - 1)\pi/27$       2.  $4\sqrt{2}\pi$       3.  $3\pi$       4.  $\pi/4$   
 5.  $\frac{1}{4}(e^2 + 4 - e^{-2})a^2\pi$       6.  $\frac{1}{6}[5^{3/2} - 1]\pi$       7.  $\pi(\pi + 2)$   
 8.  $(2\sqrt{7} - 1)\pi - \frac{1}{2}\sqrt{2}\pi \log \frac{2\sqrt{2} + \sqrt{7}}{\sqrt{2} + 1}$       9.  $(\sqrt{2} - 1)a^2\pi$   
 10.  $\frac{6}{3}a^2\pi$       11.  $\frac{3}{5}a^2\pi$       12.  $[\log(\sqrt{2} + 1) + \sqrt{2}]\pi$   
 13.  $\frac{\pi\sqrt{2}}{4}$       14.  $\pi \left[ \frac{10}{3} \frac{1}{26^3} + \frac{1}{3} \frac{5}{26^2} + \frac{5}{2} \frac{1}{26} + \frac{1}{2} \tan^{-1} 5 \right]$   
 15.  $\pi \cdot 116\frac{3}{4}$       16.  $\frac{1}{5}a^2\pi$       17.  $\frac{4}{4}\pi$   
 18.  $\pi[3\sqrt{2} - \log(1 + \sqrt{2})]$       19.  $\pi[5 \log(\sqrt{2} + 1) + \sqrt{2}]$

Page 166

## EXERCISE XXX

1. 31000 lbs., 20666 $\frac{2}{3}$  lbs., 10333 $\frac{1}{3}$  lbs.      2. 6200 $\pi$  lbs.  
 5. 23808 $\pi$  lbs.      6. On ends 1984 lbs., on sides 12400 lbs.  
 7. (a) 69000 $\pi$  lbs.      (b) 20700 $\pi$  lbs.

Page 167

## EXERCISE XXXI

1. 63*k*      2.  $m/2a$ ,  $m/a$       3. 645048 $\pi$  ft. lbs.      6. 1080*m*

Page 172

## EXERCISE XXXII

1. 1. 23/6      2. - 4      3. 11/6  
 2. 1. 43 $\frac{1}{2}$       2. 45/8      3. 2a<sup>3</sup>/3  
 4.  $\int_{-a/\sqrt{2}}^{a/\sqrt{2}} \int_{-x}^{\sqrt{a^2-x^2}} (x+y)dydx + \int_{a/\sqrt{2}}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x+y)dydx = \frac{2}{3}\sqrt{2}a^3$   
 5.  $\int_0^1 \int_x^{2x} xydydx = \frac{8}{3}$   
 6. For solid above the *xy*-plane,  $\int_0^2 \int_{-x}^{2x^{1/2}} (x+y)dydx = \frac{1}{5}\sqrt{2} + \frac{1}{3}$   
 7.  $2 \int_0^2 \int_0^{(2x-x^2)^{1/2}} (x^2 + 3y^2)dydx = 2\pi$       8.  $\int_0^1 \int_{x^2}^{x^{1/2}} xdydx = \frac{3}{2}$   
 9.  $\int_0^a \int_0^{\sqrt{a^2-x^2}} (a^2 - x^2)^{1/2}dydx = 2a^3/3$

$$10. \frac{1}{2} \int_0^6 \int_0^{2-x/3} (6-x-3y) dy dx = 6$$

$$11. \int_0^1 \int_0^{(1-x^{1/2})^2} [1 - (x^{1/2} + y^{1/2})]^2 dy dx = \frac{1}{90}$$

$$12. \int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} (2x-x^2-y^2) dy dx$$

Page 175

$$\text{EXAMPLE 2. (1) } \frac{abc}{3} (a^2 + b^2 + c^2) \quad (2) 9\frac{1}{2} \quad (3) \frac{1}{144} a^6$$

$$\text{EXAMPLE 3. } \int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \int_{-\sqrt{2x-x^2-y^2}}^{\sqrt{2x-x^2-y^2}} F(x, y, z) dz dy dx$$

Page 178

## EXERCISE XXXIII

$$8. \frac{8}{9} \left[ 1 - \frac{2}{3\pi} \right] a \quad 10. a^2/6, a = \text{side of square.}$$

11. Representing the mean value by  $M$  and taking  $0 < y < x < c$ , we have

$$M \cdot \int_0^c \int_0^x dy dx = \int_0^c \int_0^x y(x-y)(c-x) dy dx \therefore M = c^3/60$$

Page 179

$$\text{EXAMPLE 4. } \bar{x} = 1, \bar{y} = -1/5$$

$$\text{EXAMPLE 5. } \bar{x} = 5/9, \bar{y} = 4/7, \bar{z} = 2/7$$

Page 180

EXAMPLE 3. The following numbers refer to the examples on p. 157.

$$4. \bar{x} = 16/5 \quad 5. \bar{x} = 3/5$$

$$6. \bar{x} \int_0^4 y(y^{2/3} - y/2) dy = \frac{1}{2} \int_0^4 y(y^{2/3} - y/2)(y^{2/3} + y/2) dy$$

$$7. \bar{x} = 8 \quad 8. \bar{x} = 1/2 \quad 9. \bar{x} = 315/248$$

$$17. \bar{y} = 2/3 \quad 18. \bar{y} = 10/3 \quad 19. \bar{y} = 21/10$$

$$20. \bar{y} = \infty \quad 21. \bar{x} = \bar{y} = 0 \quad 22. \bar{x} = 0, \bar{y} = 1$$

$$\text{EXAMPLE 4. } \bar{x} = 8/3, \bar{y} = 0, \bar{z} = 0$$

Page 181

## EXERCISE XXXIV

2. At the distance  $a/3$  from the bounding plane.

3. On axis at distance  $a \frac{(2/3)(a^2 + r^2)^{1/2} + r}{(a^2 + r^2)^{1/2} + r}$  from vertex, where  $a$

= altitude,  $r$  = radius of base.

4. For the right half of the circle  $x^2 + y^2 = a^2$  we have (1)  $\bar{x} = \frac{3}{16}\pi a$ ,

$$\bar{y} = 0, \quad (2) \quad \bar{x} \int_{-\pi/2}^{\pi/2} \int_0^a cr \cdot r dr d\theta = \int_{-\pi/2}^{\pi/2} \int_0^a r \cos \theta \cdot cr \cdot r dr d\theta$$

$$\therefore \bar{x} = 3a/2\pi, \bar{y} = 0$$

$$5. \bar{x} = 1, \bar{y} = -5/23$$

$$6. \bar{x} = a\pi, \bar{y} = 5a/6; \bar{x} = a\pi, \bar{y} = 0$$

$$8. \bar{x} = -5a/6, \bar{y} = 16a/9\pi$$

$$10. \bar{x} = 4a/5$$

$$12. \text{Volume} = 24\pi, \text{area} = 32\sqrt{2}\pi$$

15. If the angular points be  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$ , and the lengths of the sides opposite them be  $a$ ,  $b$ ,  $c$  respectively, then

$$\bar{x} = \left[ a \frac{x_2 + x_3}{2} + b \frac{x_3 + x_1}{2} + c \frac{x_1 + x_2}{2} \right] \div (a + b + c), \text{ and } \bar{y}$$

similarly.

Page 187

### EXERCISE XXXV

$$1. I_x = 24/35, I_y = 6/35, I_0 = 6/7$$

$$2. I_x = 148/15$$

$$3. I_x = 37/2, I_y = 19/2, I_0 = 28$$

$$4. I_x = 72\pi, I_y = 117\pi$$

$$5. m \frac{a^2}{2}, (m = \pi b^2 a^2)$$

$$7. (1) \frac{5}{3}a^3, \frac{4}{3}a^3, \frac{2}{3}a^3$$

$$15. \frac{8}{15}\pi a^5 = m \frac{2}{3}a^2$$

$$19. \frac{35}{16}\pi a^4$$

$$20. k \frac{a^4}{4}$$

$$21. k \frac{4\pi a^6}{45}$$

$$22. 191\frac{2}{3}\pi^2 \text{ ft. lbs./sec.}$$

Page 191

EXAMPLE 6.  $1 k \cdot mm'/2a^2$

EXAMPLE 7.  $\frac{2kmm'}{a^2} \left[ 1 - \frac{c}{(c^2 + a^2)^{1/2}} \right]$ ,  $m$  = mass of plate,  $a$  its radius,  $c$  its distance from the particle.

EXAMPLE 8.  $\frac{2kmm'}{a^2 h} [h - \sqrt{(d+h)^2 + a^2} + \sqrt{d^2 + a^2}]$ ,  $m$  = mass of cylinder,  $a$  its radius,  $h$  its altitude,  $d$  = distance of particle from its nearer base.

EXAMPLE 9. Let  $P$  be any point on  $BC$  and set  $AP = y$  and  $AOP = \theta$ . Then  $y = a \tan \theta$ ,  $dy = a \sec^2 \theta d\theta$   $\therefore \frac{dy}{OP^2} = \frac{ad\theta}{a^2}$ .

Page 195

EXAMPLE 4. 1. Convergent 2. Divergent 3. Divergent

4. Convergent 5. Divergent 6. Divergent

EXAMPLE 5. Divergent.

Page 197

EXAMPLE 2. 1. Convergent 2. Convergent 3. Convergent

4. Convergent 5. Divergent 6. Convergent

Page 199

## EXERCISE XXXVI

1. 1. Conditionally convergent 2. Divergent 3. Absolutely convergent 4. Divergent.  
 4.  $n = 44$  when  $x = .9$ ,  $n = 16$  when  $x = -.9$

Page 204

## EXERCISE XXXVII

7. 4.12309      11. 91.140      14. 1.3863, 1.6094, 1.7918  
 15. For  $n > 1$ : Ex. 1,  $(-1)^{n+1} \frac{(2n-3)!}{2^{2n-2}(n-2)!n!}$ ;  
 Ex. 2,  $(-1)^n \frac{(2n-1)!}{2^{2n-1}(n-1)!n!}$

Page 209

EXAMPLE 2.  $\sqrt[3]{6/10}$  radians  $= 10.4^\circ$  approximately

EXAMPLE 5. .9397

Page 210

EXAMPLE 9.

1.  $\sqrt{e^x} = 1 + \frac{x}{2} + \frac{x^2}{2^2 \cdot 2!} + \frac{x^3}{2^3 \cdot 3!} + \dots$   
 2.  $e^{-x} = 1 - \frac{x}{1} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$   
 3.  $\cos^2 x = 1 - \frac{2}{2!}x^2 + \frac{2^3}{4!}x^4 - \dots$   
 4.  $\sin x \cos x = x - \frac{2^2}{3!}x^3 + \frac{2^4}{5!}x^5 - \dots$

Page 213

## EXERCISE XXXVIII

1. 1.  $1/3$       2. 1      3. 1      4.  $e^{ab}$   
 7. 1.  $1 - x - \frac{1}{2}x^2 + \frac{3}{4}x^3 + \frac{5}{8}x^4 \dots, |x| < 1$   
 2.  $1 - x - x^2 - x^3 - \frac{3}{2}x^4 \dots, |x| < \sqrt{2} - 1$   
 3.  $1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 \dots, |x| < \infty$

Page 216

EXAMPLE 2. .00000543

Page 219

EXAMPLE 2. .6932

## EXERCISE XXXIX

1. 1.1113      4.  $\Delta x < .16$  radian  $= 9^\circ$  approximately  
 6.  $y = 8 - 22x + 24x^2 - 9x^3$

## Page 227

EXAMPLE 1.  $x = 1, y = 2, z = -1$

## Page 236

## EXERCISE XLI

2.  $(2, 0, -5/2), \frac{1}{2}\sqrt{41}$   
 3.  $13; \frac{8}{1^{\frac{2}{3}}}, -\frac{4}{1^{\frac{2}{3}}}, \frac{1^{\frac{2}{3}}}{1^{\frac{2}{3}}}; 4\sqrt{10}, 3\sqrt{17}, 5.$   
 4.  $11/15$     5.  $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}; \frac{x}{1} = \frac{y-1}{1} = \frac{z+1}{-1}$     6.  $\frac{8}{7}, 2\sqrt{\frac{51}{23}}$   
 7.  $\frac{x-1}{15} = \frac{y}{1} = \frac{8+2}{-7}$

## Page 240

EXAMPLE 3.  $-\frac{y}{x^2}dx + \frac{1}{x}dy, \frac{y}{z}dx + \frac{x}{z}dy - \frac{xy}{z^2}dz,$   
 $\cos xyz(yzdx + zxdy + xydz)$

## Page 241

EXAMPLE 4.  $-.002$   
 EXAMPLE 5.  $.343 \text{ m./h.}$   
 EXAMPLE 6.  $du = \frac{1}{2}[xy \cos \theta d\theta + \sin \theta(ydx + xdy)]$

## EXERCISE XLII

1.  $\frac{\partial u}{\partial r} = y - z, \frac{\partial u}{\partial s} = x + 2y - 2z, \frac{\partial u}{\partial t} = -2x$   
 2.  $f_x a_1 + f_y a_2 + f_z a_3, f_x b_1 + f_y b_2 + f_z b_3, f_x c_1 + f_y c_2 + f_z c_3$   
 4.  $309 \text{ in./sec.}$

## Page 244

## EXERCISE XLIII

1.  $e^{xy^2}y^2, e^{xy^2}2xy$   
 2.  $-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}; 2 \cos(2x - y + 3z), -\cos(2x - y + 3z),$   
 $3 \cos(2x - y + 3z)$   
 3. 1.  $\frac{xdx + ydy}{(x^2 + y^2)^{1/2}}$     2.  $\frac{(z-y)dx + (x+z)dy - (x+y)dz}{(x+z)^2},$   
 $3. e^x \sin(y + 2z)dx + e^x \cos(y + 2z)(dy + 2dz)$   
 4.  $(32s + 40t)ds + (40s + 50t - 12t^3)dt$   
 5.  $[2\phi \cos 2\theta d\theta - \sin 2\theta d\phi]/3\phi^2.$

Page 246

## EXERCISE XLIV

4.  $\partial^2 u / \partial s^2 = 16$ ,  $\partial^2 u / dt^2 = -24$

Page 247

EXAMPLE.  $4x' + 12y' + x'^2 + 12x'y' + 12y'^2 + 3x'^2y' + 12x'y'^2 + 4y'^3 + 3x'^2y'^2 + 4x'y'^3 + x'^2y'^3 = 0$

Page 251

EXAMPLE 2.  $2(z^2 + xz)/(2z + x)^3$

EXAMPLE 3.  $-24x(3x^2 + z)^3$

Page 252

EXAMPLE 2.  $y = x - x^2 + x^3 \dots$

Page 258

## EXERCISE XLV

1. (1)  $y = \frac{1}{2}x + \frac{3}{8}x^2 \dots$ ,  $x = 2y - 3y^2 \dots$
- (2)  $y = 1 + 3(x - 1) + 27(x - 1)^2 \dots$
- (3)  $z = -x + y - x^2 + 3xy - 2y^2 \dots$
- (4)  $y = -1 - \frac{5}{7}(x - 1) + \frac{2}{3} \frac{6}{4} \frac{3}{3}(x - 1)^2 \dots$ ,  
 $z = 1 + \frac{3}{7}(x - 1) - \frac{5}{3} \frac{2}{4} \frac{3}{3}(x - 1)^2 \dots$

2.  $y, y - x/2, 0$

3.  $\frac{\partial u}{\partial x} = \frac{v - x^2}{u^2 - uv}$ ,  $\frac{\partial v}{\partial x} = \frac{u - x^2}{v^2 - uv}$       4.  $\frac{\partial z}{\partial x} = \frac{u - 2v^2}{4uv + 1}$ ,  $\frac{\partial z}{\partial y} = \frac{v + 2u^2}{4uv + 1}$

5.  $\frac{\partial u}{\partial x} = -\frac{uf_{ux}}{xf_{ux} + yf_{uy}}$ ,  $\frac{\partial u}{\partial y} = -\frac{uf_{uy}}{xf_{ux} + yf_{uy}}$

8.  $\frac{du}{dx} = -\frac{\partial f_1}{\partial v} \cdot \frac{\partial f_2}{\partial y} \cdot \frac{\partial f_3}{\partial x} \div \frac{\partial f_1}{\partial u} \cdot \frac{\partial f_2}{\partial v} \cdot \frac{\partial f_3}{\partial y}$

9.  $\frac{\partial u}{\partial x} = \frac{\partial y}{\partial v} \Big/ J$ ,  $\frac{\partial v}{\partial x} = -\frac{\partial y}{\partial u} \Big/ J$ ,  $J = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$

Page 268

EXAMPLE. Osculating plane,  $3x - 3y + z - 1 = 0$ ; tangent,  
 $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3}$ ; binomial,  $\frac{x-1}{3} = \frac{y-1}{-3} = \frac{z-1}{1}$ ;  
 principal normal,  $\frac{x-1}{11} = \frac{y-1}{8} = \frac{z-1}{-9}$ .

Page 270

## EXERCISE XLVI

1. (1)  $4x - 7y - z + 5 = 0$       (2)  $2x - y - z + 3 = 0$

2. The tangent planes to  $x^2 - y^2 + 2yz + 2x = 0$  at  $(-1, 1, 1)$ ,  $(-1, -1, -1)$  are parallel to the  $xy$ -plane.
3.  $y = x^2$       4.  $\frac{x+1}{1} = \frac{y-2}{-4} = \frac{z-3}{3}$ ;  $x - 4y + 3z = 0$
5. At the angles whose sines are  $3/\sqrt{62}$ ,  $10/\sqrt{62 \cdot 41}$ ,  $15/\sqrt{62 \cdot 766}$
10.  $\frac{2}{3}\sqrt{6} - 1 - \frac{2}{3}\log(1 + \frac{2}{3}\sqrt{6})$
12. Osculating plane,  $2x - 2y + z = 1$ ; binormal  $\frac{x-1}{11} = \frac{y-1}{7} = \frac{z-1}{-8}$
13.  $x + y + z - 6 = 0$       17.  $\frac{d\phi}{ds} = \frac{x}{(a^2 + k^2)^{1/2}}$ ,  $\frac{d\tau}{ds} = \frac{k}{(a^2 + k^2)^{1/2}}$
18.  $-48$ ;  $-4$

Page 276

## EXERCISE XLVII

1.  $y^2 - x^2 = 0$       2.  $x^2 - 4y^2 = 0$       3.  $y^2 = 4x$
4.  $(x - y)^2 = 4$       5.  $(x - y)^2 - 2(x + y) - 1 = 0$
6.  $y^2 - 4x^2 = 0$
6.  $1. 2xy = k$       2.  $x^{1/2} + y^{1/2} = k^{1/2}$
13.  $27ay^2 = 4(x - 2a)^3$

Page 281

## EXERCISE XLVIII

1. (1) Minimum  $= -29$ , at  $(-7, 5)$  (2) Minimum  $= 0$ , at  $(0, 0)$ , maximum  $= \frac{4}{27}$ , at  $(-\frac{2}{3}, \frac{2}{3})$ .
3. A cube      4.  $\sqrt[3]{6}$       5.  $8 \frac{abc}{3\sqrt{3}}$
6. The minimum distance is 2.
9.  $x = \frac{ma}{m+n+p}$ ,  $y = \frac{na}{m+n+p}$ ,  $z = \frac{pa}{m+n+p}$
10. If  $x$  = radius of base of cone and cylinder, and  $y$  and  $z$  be the altitudes of the cone and cylinder, then  $y = z = (2/\sqrt{5})x$ .
11. If  $2x$  = a side of the base,  $y$  = height to eaves, and  $z$  = height of roof, then  $y = z = (1/\sqrt{3})x$ .
14.  $(-2, -2, 4)$ .
15.  $\left[ -\frac{ad}{a^2 + b^2 + c^2}, -\frac{bd}{a^2 + b^2 + c^2}, -\frac{cd}{a^2 + b^2 + c^2} \right]$ ; point where the intersection of the given planes meets the plane  $x(bc' - b'c) + y(ca' - c'a) + z(ab' - a'b) = 0$ .



## Page 285

EXAMPLE 4. 1.  $\frac{d^3y}{dx^3} = 0$  2.  $\frac{d^2y}{dx^2} = 4y = 0$

3.  $y x^2 \frac{d^2y}{dx^2} + x^2 \left( \frac{dy}{dx} \right)^2 - 2xy \frac{dy}{dx} + y^2 = 0$  4.  $2x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

## Page 287

## EXERCISE XLIX

1. 1.  $\left( \frac{dy}{dx} \right)^2 - 4 = 0$

2.  $(x^2 + y^2) \frac{d^2y}{dx^2} + 2 \left( y - \frac{dy}{dx} x \right) \left[ \left( \frac{dy}{dx} \right)^2 + 1 \right] = 0$

2.  $y = x^3 + c_1x + c_2$ ;  $y = x^3 - 2 + c_1(x - 1)$ ;  $y = x^3 - 2$

4.  $y = -1 + 2(x - 1) - (x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{6}(x - 1)^4 \dots$

## Page 288

EXAMPLE 2. 1.  $\log(x + 1)(y - 1) + \frac{1}{2}x^2 - x + y = C$

2.  $\log \sqrt{y^2 + 1} - \tan^{-1} x = C$

3.  $y = Ce^{x^2}$

4.  $(e^x + 1)(e^y - 1) = C$

5.  $\sqrt{1 - x^2} + \sqrt{1 - y^2} = C$

## Page 289

EXAMPLE 2. 1.  $2x^2 + 2xy + y^2 = C$

2.  $\log y + x^2/y^2 = C$

3.  $\log y = y/x + C$

4.  $\log y + 2\sqrt{x/y} = C$

EXAMPLE 4. 1.  $2x^2 + 2xy + y^2 - 8x - 2y = C$

2.  $5(x - y) + 2 \log(10x + 15y - 4) = C$

## Page 291

EXAMPLE 5. 1.  $y = -(x + 1) + Ce^x$  2.  $y = -(x^2 + 2x + 2) + Ce^x$

3.  $x + y + 1 = Ce^y$  4.  $y \sin^2 x = \frac{2}{3} \sin^3 x + C$

5.  $y = x^2(1 + Ce^{1/x})$

6.  $[x + \sqrt{x^2 + 1}]y = \frac{x^3}{3} + x + \frac{1}{3}(x^2 + 1)^{3/2} + C$

7.  $\frac{1}{y} = \log^2 x + 2 \log x + 2 + Cx$  8.  $y = \frac{a}{b^2} - \frac{ax + c}{b} + Ce^{-bx}$

9.  $y \frac{x - 2}{x - 1} = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + C$  10.  $4x = y^2 + Cy^{-2}$

## Page 292

EXAMPLE 4. 1.  $\tan^{-1} \frac{y}{x} = \frac{x^2}{2} + C$  2.  $y = 2x/(C - x^2)$

3.  $\log(y + 1) - \frac{1}{xy} = C$

4.  $2Cy = C^2x^2 - 1$

## Page 293

EXAMPLE 6. 1.  $x^2 + 4xy + 3y^2 = C$  2.  $ax^2 + 2hxy + by^2 + 2gy + 2fy = C$

3.  $y x^2 - y^3 x = C$  4.  $x^2 y^3 + \log y = C$

Page 294

## EXERCISE L

1.  $x^2 + y^2 + 2xy(y - x) = Cx^2y^2$
2.  $y^3 = C(y^2 - x^2)$
3.  $\log x + \sin \frac{y}{x} = C$
4.  $ye^{\sin x} = e^{\sin x}(\sin x - 1) + C$
5.  $x^3 = -\frac{1}{a}(y + 1) - \frac{1}{a^2} + Ce^{ay}$
6.  $y - \tan^{-1}(x + y) = C$
7.  $yx(x - 1) = \frac{x^4}{4} - \frac{x^2}{2} + C$
8.  $\frac{1}{y} = \log x + 1 + Cx$
9.  $\frac{x}{y} + \frac{x^3}{3} = C$
10.  $\frac{1}{x} + \tan^{-1} \frac{y}{x} = C$
11. 1.  $x^2 + y^2 - \log x^2 = C$  2.  $x^2 + y^2 + 2Cy = 0$
3.  $mx^2 + ny^2 = C$
12.  $y^2 \pm 2xy - x^2 = C$
13.  $x^2 + y^2 + 2Cx = 0$
14.  $y = Ce^{x/y}$
15.  $y = Ce^{x/2} + \frac{1}{C}e^{-x/2}$
16.  $x^2 + y^2 = \left(\frac{x}{n} + C\right)^2$
17.  $l \log \frac{l + \sqrt{l^2 - y^2}}{y} = \sqrt{l^2 - y^2} + x + C$
21. Take opposite bank as  $x$ -axis, and opposite point as  $O$ ; also  $(0, l)$  as starting point. Then  $\frac{dx}{dt} = a - \frac{bx}{(x^2 + y^2)^{1/2}}$ ,  $\frac{dy}{dt} = -\frac{by}{(x^2 + y^2)^{1/2}}$   
 $\therefore \frac{dx}{dy} = \frac{x}{y} - r \left( \frac{x^2}{y^2} + 1 \right)^{1/2}$ , where  $r = \frac{a}{b}$ . The solution of this equation satisfied by  $(0, l)$  is  $x = \frac{1}{2}[l^r y^{1+r} - l^{-r} y^{1-r}]$

Page 297

- EXAMPLE 3. 1.  $25(xy + C)^2 = 4x^5$  2.  $x^2 = 2Cy + C^2$   
 3.  $\sin^2(\frac{1}{2}y + C) = x^2$

Page 301

## EXERCISE LI

1. Given equation and  $x = -(p + \frac{1}{2}) + C(p - 1)^{-2}$ ; singular solution,  $y = 0$ .
2.  $(y + x + C)^2 = \frac{4}{3}(x + 1)^3$ ; no singular solution.
3.  $y - Cx + C^3 = 0$ ; singular solution  $27y^2 = 4x^3$
4.  $x^2 - 2yC + C^2 = 0$ ; singular solution  $y^2 - x^2 = 0$
5.  $C^2 + Cx - y = 0$ ; singular solution  $x^2 + 4y = 0$
6.  $y + nCx^{-1/n} + aC^n = 0$ ; singular solution defined by  $1 + ax^n p^{n-1} = 0$  and the given equation.
7. Given equation and

$$x = \frac{Cp}{(p^2 - 1)^{1/2}} - \frac{ap}{(p^2 - 1)^{1/2}} \log [p + (p^2 - 1)^{1/2}];$$

no singular solution.

8.  $y^2 = 2Cx - C^2$ ; singular solution  $x^2 - y^2 = 0$   
 9.  $y + 3Cx^{-1/3} + C^3 = 0$ ; singular solution  $xy^2 + 4 = 0$   
 10.  $4y^2 = 4Cx + aC^2$ ; singular solution  $x^2 + ay^2 = 0$   
 12. The singular solution of  $(y - px)^2 = c(1 + p^2) + a^2p^2$  is  $cx^2 + (a^2 + c)y^2 = (a^2 + c)c$   
 13.  $(x + y + C)^2 - 4xy = 0$

## Page 305

- EXAMPLE 5. 1.  $y = c_1e^{4x} + c_2e^{-3x}$  2.  $y = c_1e^{x/2} + c_2e^{-2x/3}$   
 3.  $y = c_1 + e^{-x}[c_2 \cos 2x + c_3 \sin 2x]$   
 4.  $y = c_1e^{\sqrt{3}x} + c_2e^{-\sqrt{3}x} + c_3 \cos 2x + c_4 \sin 2x$   
 5.  $y = c_1e^{ax} + c_2e^{x/a}$  6.  $y = c_1e^{-x} + (c_2 + c_3x)e^{2x}$   
 7.  $y = c_1 + c_2x + c_3e^{x/2} + e^{-x}[c_4 + c_5x + c_6x^2]$   
 8.  $y = e^x[(c_1 + c_2x) \cos 3x + (c_3 + c_4x) \sin 3x]$

## Page 306

- EXAMPLE 3. 1.  $y = c_1e^{3x} + c_2e^{-2x} - \frac{1}{4}e^{2x} + \frac{1}{6}e^{4x}$   
 2.  $y = (c_1 + c_2x + c_3x^2 + \frac{1}{6}x^3)e^{5x}$   
 3.  $y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4}x$  4.  $y = c_1e^{2x} + c_2e^{3x} + c_3 + \frac{1}{2}e^x$

## Page 308

§ 263, EXAMPLE 2.  $Y = -(2x^4 + 8x^3 + 24x^2 + 48x)$

EXAMPLE 3. 1.  $y = c_1e^{2x} + e^{-x}[c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x] - \frac{1}{8}x^2$

2.  $y = e^x(c_1 + c_2x) + x^2 + 5x + 8$

§ 264, EXAMPLE 3.

1.  $y = c_1e^{2x} + c_2e^{3x} + \frac{1}{7}(5 \cos 3x - \sin 3x) - \frac{1}{5}(2 \sin 4x + \cos 4x)$

\*2.  $y = c_1 + c_2x + c_3e^x + c_4e^{-x} + \frac{1}{2} \sin x$

## Page 309

EXAMPLE 2. 1.  $y = c_1e^{-x} + c_2e^{-4x} + \frac{1}{2}e^{-x}(3x^3 - 3x^2 + 2x)$

2.  $y = c_1 + c_2x + c_3x^2 - \frac{1}{2}e^x \cos x$

## Page 310

## EXERCISE LII

1.  $y = c_1 + c_2x + c_3x^2 + c_4e^{5x/2}$

$$+ e^{-x/2} \left[ (c_5 + c_6x) \cos \frac{\sqrt{3}}{2}x + (c_7 + c_8x) \sin \frac{\sqrt{3}}{2}x \right]$$

2.  $y = c_1e^{2x} + c_2e^{-3x/2} + \frac{1}{3}e^{3x}$  3.  $y = [c_1 + c_2x + \frac{1}{2}x^2]e^{2x} + 2e^x$

4.  $y = c_1 + c_2e^{-x} + \frac{1}{2}e^x + 3x$  5.  $y = c_1e^{-3x} + c_2e^{-4x} + \frac{1}{2}x + \frac{1}{2}\frac{3}{2}$

6.  $y = e^{-x}(c_1 \cos x + c_2 \sin x) + \frac{1}{2}x^2 - x + \frac{1}{2}$

7.  $y = c_1 + c_2e^{-2x} + \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{4}x$

8.  $y = c_1e^{-4x} + 2x^3 - x^2 + \frac{1}{2}x + \frac{3}{8}$

9.  $y = c_1 + c_2e^{-x} - \frac{1}{15}[\cos 2x + 2 \sin 2x] + \frac{1}{2}[\sin x - \cos x]$

10.  $y = c_1e^x + (c_2 + c_3x)e^{-x} + \frac{1}{4}(\cos x - \sin x)$

11.  $y = c_1 \cos x + c_2 \sin x - \frac{1}{2}x \cos x$

12.  $y = c_1 e^{2x} + c_2 e^{-3x} - e^x \left[ \frac{1}{4} x^2 + \frac{3}{8} x + \frac{1}{8} \right]$   
 13.  $y = c_1 + c_2 x + \frac{1}{16} e^{3x} [12 \sin 2x + 5 \cos 2x]$   
 14.  $y = c_1 \sin x + c_2 \cos x + x \sin x + \cos x \log \cos x$   
 15.  $y = \frac{1}{5} e^x [3 \sin x - \cos x] + \frac{1}{5} [\cos x - 2 \sin x]$

Page 315

## EXERCISE LIII

1.  $y = -\log [\cos (x + c_1)] + c_2$     2.  $y = c_1 \tan (c_1 x + c_2)$     6.  $a^2/\sqrt{k}$   
 7. The equation of motion is  $m \frac{d^2 x}{dt^2} + k \frac{dx}{dt} = 0$  or  $\frac{d^2 x}{dt^2} + \mu \frac{dx}{dt} = 0$ ,  
 where  $\mu = k/m > 0$ . The solution for which  $x = 0$  and  $v = v_0$   
 when  $t = 0$  is  $x = (v_0/\mu)(1 - e^{-\mu t})$ . Hence, as  $t \rightarrow \infty$ , the particle  
 approaches the distance  $v_0/\mu$  from  $O$  as limit.  
 9. We have  $hT = \int_0^{2\pi} r^2 d\theta = 2\pi ab \therefore T = \frac{2\pi ab}{h} = 2\pi a \sqrt{\frac{a}{\mu}}$   
 $= 2\pi a^{3/2}/\mu^{1/2}$   
 10. When  $\mu < h^2$ , then  $\frac{1}{r} = a \cos (k\theta - \beta)$ , where  $k = \left[ 1 - \frac{\mu}{h^2} \right]^{1/2}$   
 When  $\mu = h^2$ , then  $\frac{1}{r} = a\theta + h$   
 When  $\mu > h^2$ , then  $\frac{1}{r} = c_1 e^{k\theta} + c_2 e^{-k\theta}$ , where  $k = \left[ \frac{\mu}{h^2} - 1 \right]^{1/2}$   
 11. The general solutions of the two differential equations  
 $d^2 x/dt^2 + k^2 x = 0$ ,     $d^2 y/dt^2 + k^2 y = 0$   
 are  $x = A \cos kt + B \sin kt$  and  $y = A' \cos kt + B' \sin kt$ . Hence,  
 eliminating  $\cos kt, \sin kt$ ,  
 $(A'x - Ay)^2 + (B'x - By)^2 = (A'B - AB')^2$

Page 321

## EXERCISE LIV

1.  $yz^2 = Cx$     2.  $x^2 + y^2 - 2z - 2 \log z = C$   
 3.  $x^2 + z^2 - a^2 = Ce^{-2y}$     4.  $2(x+y)e^z + 1 = Ce^{2z}$   
 5.  $y + z^2 + 1 = Ce^{x^2}$     6.  $x^2 + z^2 + (y+C)^2 = 1$   
 7.  $2x^2 + 4y^2 + 4z^2 - x + \frac{1}{4} = Ce^{-4x}$     8.  $yz + zx + xy = C$   
 9.  $(ax - cz) = C(ay - bz)$     10.  $(y - z)(x + y)^2 = C$

Page 324

## EXERCISE LV

1.  $(x + y)/z = c_1, x^2 - y^2 = c_2$   
 2.  $y^2 + z^2 = c_1, \log x + \tan^{-1} (z/y) = c_2$

3.  $xy = c_1$ ,  $x + z = c_2xz$     4.  $ax + by + cz = c_1$ ,  $x^2 + y^2 + z^2 = c_2$   
 5.  $y = c_1z$ ,  $x^2 + y^2 + z^2 = c_2z$     6.  $(x + y + z)^{1/2} = \frac{c_1}{y - x} = \frac{c_2}{z - x}$   
 7.  $z = c_1e^{(1+2\sqrt{2})x} + c_2e^{(1-2\sqrt{2})x}$ ,  
 $y = c_1(-2 + \sqrt{2})e^{(1+2\sqrt{2})x} - c_2(2 + \sqrt{2})e^{(1-2\sqrt{2})x}$   
 8.  $x = c_1e^{mt} + c_2e^{-mt} + c_3 \cos mt + c_4 \sin mt$ ,  
 $y = c_1e^{mt} + c_2e^{-mt} - c_3 \cos mt - c_4 \sin mt$   
 9. Solution of  $\frac{dy}{dx} - \frac{dz}{dx} + y = \cos 2x$ ,  $\frac{d^2y}{dx^2} - \frac{dz}{dx} + 3y - z = e^{2x}$  is  
 $z = c_1e^x + c_2 \cos x + c_3 \sin x + \frac{3}{5}e^{2x} + \frac{1}{15}(\cos 2x - 2 \sin 2x)$   
 $y = \frac{1}{2}c_1e^x + \frac{1}{2}(c_3 + c_2) \cos x + \frac{1}{2}(c_3 - c_2) \sin x + \frac{2}{5}e^{2x}$   
 $+ \frac{1}{15}(3 \cos 2x + 4 \sin 2x)$   
 10. 1.  $y = c_1x^2$ ,  $z^2 = c_2x$     2.  $x^2 - y^2 = c_1$ ,  $x + y = c_2e^x$

Page 326

## EXERCISE LVI

1.  $\phi[(x + y + z)(z - x)^2, (z - x)/(y - x)] = 0$     2.  $\phi(xz, y) = 0$   
 3.  $\phi(x^2 + y^2 + z^2, ax + by + cz) = 0$     4.  $\phi[x^2 - y^2, (x + y)/z] = 0$   
 5.  $\phi(xy - z^2, y/x) = 0$     6.  $\phi[x^2 - z^2, (x + y + z)/y] = 0$   
 7.  $\phi\left[\frac{y}{x}, \frac{z + xy}{x}\right] = 0$     8.  $\phi\left[xy, \frac{3xz - y^2}{3x}\right] = 0$   
 9.  $\phi\left[\frac{x^2 + y^2 + z}{x}, \frac{y}{x}\right] = 0$     10.  $\phi\left[\frac{x^2 + y^2}{y}, z\right] = 0$   
 14.  $y = x^2z$     15.  $lp + mq = 1$

Page 329

## EXERCISE LVII

2. 1.  $z = \frac{x^2y}{2} + \frac{xy^2}{2} + \frac{y^2 \log x}{2} + \phi(x) + \psi(y)$   
 2.  $z = \frac{x^4y}{12} + \phi(y)x + \psi(y)$   
 7. General integral:  $z = ax + \phi(a)y + a\phi(a)$ ,  $x + \phi'(a)y + [\phi(a) + a\phi'(a)] = 0$ . Singular integral:  $z + xy = 0$ .  
 10.  $z^2 = x^2 + y^2$ ,  $x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0$

Page 343

## EXERCISE LVIII

1.  $a^3\pi$     2. 1    3. 76, 52  
 4.  $7/30, 3/40$     5. 99    6.  $\frac{x^2 + y^2}{xy} - \frac{x_0^2 + y_0^2}{x_0y_0}$

8.  $\frac{4}{1215}(169\sqrt{13} - 212)$

9. The area of the surface bounded by  $y = x$ ,  $y = 0$ ,  $x = a$  is

$$\frac{32}{1215} \left[ \left( \frac{9}{2}a + 1 \right)^{5/2} - 2 \left( \frac{9}{4}a + 1 \right)^{5/2} + 1 \right]$$

11.  $2\sqrt{2}\pi$

13. The surface got by revolving  $y = f(x)$  about  $Ox$  is  $(y^2 + z^2)^{1/2} = f(x)$ , or  $z^2 = f^2(x) - y^2$ . Hence, by § 290 (1), the area between  $x = a$  and  $x = b$  is

$$\begin{aligned} 4 \int_a^b \int_0^{f(x)} \frac{f(x)[1 + f'^2(x)]^{1/2}}{[f^2(x) - y^2]^{1/2}} dy dx &= 4 \int_a^b f(x)[1 + f'^2(x)]^{1/2} \\ &\quad \times \left[ \sin^{-1} \frac{y}{f(x)} \right]_0^{f(x)} dx \\ &= 2\pi \int_a^b f(x)[1 + f'^2(x)]^{1/2} dx \end{aligned}$$

Page 350

### EXERCISE LIX

1. By § 163 and § 295 (2),

$$W = -km' \int_{AB} \frac{x dx + y dy + z dz}{(x^2 + y^2 + z^2)^{3/2}} = km' \left( \frac{1}{OB} - \frac{1}{OA} \right)$$

2. If the mass of the particle be  $m'$ , the components of  $f$  are 0,  $2m'$ ,  $6m't$ ; and  $W = 1177\frac{1}{2}m'$ .

5. 1.  $\frac{1}{4}a^4\pi$       2.  $a^2b \left( \frac{3}{8}a + \frac{\pi}{8}b \right)$

8. By § 293 (5),  $V = \frac{1}{3} \iint_S (x \cos X + y \cos Y + z \cos Z) dS$   
 $= \frac{1}{3} \iint_S r \cos(r, n) dS$

Page 354

### EXERCISE LX

4.  $\frac{1}{12}(\sqrt{2} - 1)a^2\pi$ ,  $a^2$

5. The cone  $k^2(x^2 + y^2) = z^2$  got by revolving the line  $z = kx$ ,  $y = 0$  about  $Oz$ ;  $u$  is the radius of any section of the cone perpendicular to  $Oz$  and  $v$  is the angle which this radius makes with  $Ox$ .

$$ds^2 = (k^2 + 1)du^2 + u^2dv^2 \quad dS = (k^2 + 1)^{1/2}u du dv$$

6. It is the surface  $y^2 + z^2 = f^2(x)$  got by revolving the curve  $y = f(x)$ ,  $z = 0$  about  $Ox$ .

$$S = \int_a^b \int_0^{2\pi} f(u)[1 + f'^2(u)]^{1/2} dv du = 2\pi \int_a^b f(x)[1 + f'^2(x)]^{1/2} dx$$

$$7. \frac{x}{x'} + \frac{y}{y'} + \frac{z}{z'} = 3$$

## Page 364

EXAMPLE 3.  $y = \frac{1}{4}x + \frac{1}{4}x^2 + \frac{8}{16}x^3 + \frac{1}{16}x^4$

EXAMPLE 4.  $y = -\frac{1}{2}x - \frac{3}{8}x^2 \dots$ ; hence the tangent at  $O$  is  $y = -\frac{1}{2}x$  and near  $O$  the curve lies below this tangent.

## Page 368

EXAMPLE 3.  $y = x + x^{3/2} + \frac{3}{2}x^2 + \frac{3}{8}x^{5/2}$

EXAMPLE 4.  $y = 2x + 4x^2$ ,  $y = -2x - 4x^2$ ,  $y = x^{1/2} - 2x^{3/2}$ ,  
 $y = -x^{1/2} + 2x^{3/2}$

## Page 385

## EXERCISE LXII

1.  $1. \frac{\pi^2}{3} + 4 \sum_{m=1}^{\infty} \frac{\cos m\pi}{m^2} \cos mx - 2 \sum_{m=1}^{\infty} \frac{\cos m\pi}{m} \sin mx$

2.  $1 + \sin x$

3.  $1 - \frac{1}{2} \cos x + \sum_{m=2}^{\infty} \frac{2}{m^2 - 1} \cos(m-1)\pi \cdot \cos mx$

2.  $\frac{\pi}{4} = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots$ . Cosine series is simply the term  $\frac{\pi}{4}$ .

3.  $\sin x = \frac{4}{\pi} \left\{ \frac{1}{2} - \frac{\cos 2x}{1 \cdot 3} - \frac{\cos 4x}{3 \cdot 5} - \frac{\cos 6x}{5 \cdot 7} - \dots \right\}$

4.  $f(x) = -\frac{4}{\pi} \sum_{p=1}^{\infty} \frac{1}{2p-1} \sin 2(2p-1)x$   
 $= -\frac{4}{\pi} \left\{ \frac{\sin 2x}{1} + \frac{\sin 6x}{3} + \frac{\sin 10x}{5} + \dots \right\}$

5.  $\cos ax = \frac{\sin a\pi}{\pi} \left\{ \frac{1}{a} + 2 \sum_{m=1}^{\infty} (-1)^{m+1} \frac{a}{m^2 - a^2} \cos mx \right\}$

## Page 401

## EXERCISE LXIII

1. The square roots are  $\pm [\frac{1}{2}3\sqrt{2} + \frac{1}{2}\sqrt{2}i]$ ; the points are  $(\frac{1}{2} \tan^{-1} \frac{1}{3}, 5)$ ,  $(\frac{1}{2} \tan^{-1} \frac{1}{3} + \pi, 5)$

2.  $(\frac{1}{6}\pi, 1)$ ,  $(\frac{5}{6}\pi, 1)$ ,  $(\frac{3}{2}\pi, 1)$ ;  $(\frac{1}{5}\pi, 2)$ ,  $(\frac{3}{5}\pi, 2)$ ,  $(\pi, 2)$ ,  $(\frac{7}{5}\pi, 2)$ ,  $(\frac{9}{5}\pi, 2)$

## Page 407



## EXERCISE LXIV

$$4. \quad (1) \ e \cdot e^{5i} \qquad (2) \ e^{-2k\pi} \cdot e^{i \log 2} \qquad (3) \ e^{-(\pi/4+2k\pi)} e^{i \log \sqrt{2}}$$

$$(4) \ e^{\log 2 - (\pi/3+2k\pi)} e^{i \log 2 + \pi/3+2k\pi}$$

$$6. \quad \frac{dz}{dt} = \frac{dr}{dt} e^{i\theta} + r \frac{d\theta}{dt} i e^{i\theta};$$

$$\frac{d^2 z}{dt^2} = \left[ \frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right] e^{i\theta} + \left[ 2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2 \theta}{dt^2} \right] i e^{i\theta}$$

Page 417

## EXERCISE LXV

$$5. \ v = y^2 - x^2 - x$$

$$9. \ 2\pi i$$

$$11. \quad \frac{\pi}{\sqrt{3}}, \quad \frac{\pi}{ab(a+b)}, \quad \frac{\pi}{2}$$









C0-APY-005

